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The free rank of symmetry of $(S^n)^k$ *

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§ 0. Introduction

P. Smith proved a long time ago that if a group acts freely on a sphere, then its abelian subgroups are cyclic. In this paper we will prove the natural generalization of this result to products of spheres:

Theorem 4.2. *Let $G=(\mathbf{Z}/p)^r$, p odd, act freely on $(S^n)^k$, then*

$$r \leqq k. \quad \square$$

For $p=2$ we prove an analogous theorem, provided $n \neq 1, 3, 7$.

Although this inequality was conjectured several years ago, the first real progress was made by G. Carlsson [5, 3]. He proved this result when the action is homologically trivial, using powerful homological tools which actually yield a lot more information than the inequality alone. Later, Browder [1, 2] introduced the idea of using degree to analyze free group actions, and the homologically trivial case follows as a corollary of his techniques.

In this paper we analyze the problem by separating the homologically trivial part from the representation-theoretic difficulties, and dealing with each of them separately.

We first prove a refinement of Carlsson's result

Theorem 1.1. *Let $G=(\mathbf{Z}/p)^r$, p odd, act freely on an orientable $\mathbf{Z}_{(p)}$-homology manifold X with $H^*(X, \mathbf{Z}_{(p)}) \cong H^*((S^n)^k, \mathbf{Z}_{(p)})$; then*

$$\dim H_n(X, \mathbf{F}_p)^G \geqq r k H$$

where $H \subset G$ is the subgroup of elements in G acting trivially on $H_*(X, \mathbf{Z}_{(p)})$. $\square$

Then, using rational representation theory, we prove the following theorem about $\mathbf{Z}_{(p)}G$-lattices:

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Theorem 3.1. *Let $G=(\mathbf{Z}/p)^r$, p odd and M a finitely generated $\mathbf{Z}_{(p)}G$-lattice. Then*

$$\dim_{\mathbf{F}_p} M\otimes \mathbf{F}_p-\dim_{\mathbf{F}_p}(M\otimes \mathbf{F}_p)^G \geqq \left(\frac{p-2}{p-1}\right)(rk_{\mathbf{Z}_{(p)}}M-rk_{\mathbf{Z}_{(p)}}M^G). \quad \square$$

A combination of these two inequalities leads to the following theorem, which implies Theorem 4.2:

Theorem 4.1. *Let $G=(\mathbf{Z}/p)^r$, p odd, act freely on an orientable $\mathbf{Z}_{(p)}$-homology manifold X with $H^*(X, \mathbf{Z}_{(p)})\cong H^*(S^n)^k, \mathbf{Z}_{(p)})$. Then*

$$r\leqq \dim_{\mathbf{F}_p} H_n(X, \mathbf{F}_p)^G+\left(\frac{1}{p-2}\right)(\dim H_n(X, \mathbf{F}_p)-\dim H_n(X, \mathbf{F}_p)^G). \quad \square$$

For $p=2$ this method fails, but instead we work over $\mathbf{F}_2$, and assume that X is a finitistic space homotopy equivalent to $(S^n)^k$. For $n\neq 1, 3, 7$, homotopy theoretic considerations force the homology representations over $\mathbf{F}_2$ to be particularly simple, and we can recover an analogue of 4.2. Hence the only remaining situations are $p=2$ with $n=1, 3, 7$.

The results here are stated for free actions, but they can be generalized to arbitrary ones; in Sect. 6 we describe how this can be done.

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§ 1. The homologically trivial part

In this section we will apply the usual cohomological methods to study free $(\mathbf{Z}/p)^r$ actions on $(S^n)^k$.

Let G be a finite group and X a finitistic space on which G acts. The Borel construction on X is defined as

$$X\times_G EG=X\times EG/G$$

where EG is the universal contractible free G-space and G acts diagonally on $X\times EG$. We have a fibration

$$\begin{array}{ccc} X & \xrightarrow{i} & X\times_G EG \\ & & \downarrow \\ & & BG \end{array}$$

and the induced spectral sequence in cohomology has E_2 term

$$E_2^{p,q}=H^p(G, H^q(X, R))\Rightarrow H^{p+q}(X\times_G EG, R)$$

where G acts possibly non-trivially on $H^*(X, R)$.

The following theorem is a consequence of an analysis of this spectral sequence in our situation.

Theorem 1.1. *Let $G=(\mathbf{Z}/p)^r$, p odd, act freely on an orientable $\mathbf{Z}_{(p)}$-homology manifold X with $H^*(X, \mathbf{Z}_{(p)})\cong H^*((S^n)^k, \mathbf{Z}_{(p)})$ and denote by H the subgroup of elements in G which act homologically trivially on $H_*(X, \mathbf{F}_p)$. Then*

$$rkH\leqq \dim_{\mathbf{F}_p} H_n(X, \mathbf{F}_p)^G.$$

Proof. Let $R=\mathbf{Z}_{(p)}$ and consider the spectral sequence in cohomology with R-coefficients associated to

$$\begin{array}{ccc} X & \xrightarrow{\;i\;} & X \times_H EH \\ & & \downarrow \\ & & BH. \end{array}$$

Look at the differential

$$d_{n+1}\colon\ H^n(X,R)\to H^{n+1}(H,R).$$

We can choose an R-basis $v_1, \ldots, v_k$ of $H^n(X, R)$ such that

$$\{d_{n+1}(v_1), \ldots, d_{n+1}(v_s)\}$$

is an $\mathbf{F}_p$ basis for the vector space $\operatorname{im} d_{n+1}\subset H^{n+1}(H, R)$ (H is elementary abelian) and

$$v_{s+1}, \ldots, v_k\in\ker d_{n+1}.$$

Then clearly $pv_1, \ldots, pv_s, v_{s+1}, \ldots, v_k$ are all in $\ker d_{n+1}$.

Now d_{n+1} is the only differential on $E_r^{0,n}$, any r. On $E_{n+1}^{0,n}$, $\ker d_{n+1}$ consists of permanent cocycles. This means that

$$\ker d_{n+1}=\operatorname{im} i^*$$
$$i^*\colon\ H^*(X\times_H EH, R)\to H^*(X, R).$$

From this it follows that

$$p^s v_1 \ldots v_k\in\operatorname{im} i^*.$$

We also have a commutative diagram, with vertical arrow a homotopy equivalence

$$\begin{array}{ccc} X & \xrightarrow{\;i\;} & X \times_H EH \\ & \searrow^{\pi} & \downarrow \\ & & X/H \end{array}$$

The map π is a covering of homology manifolds of degree $|H|$, hence in the top cohomology group $\operatorname{im} i^*=(|H|\mu_X)$, where μ_X is a generator. However, this class can be chosen so that

$$\mu_X=v_1\ldots v_k.$$

We conclude that $|H|\,|p^s$, so that
$$rkH \leqq s.$$

As G is abelian, its action commutes with that of H on X, and so it acts on the whole spectral sequence, trivially on the horizontal edge. Hence d_{n+1} is G-equivariant, and we have a short exact sequence of G-modules

$$0 \to \operatorname{im} i^* \to H^n(X, R) \to \operatorname{im} d_{n+1} \to 0.$$

The module on the right is trivial, and of dimension at least rkH. From this it follows that
$$rkH \leqq \dim_{\mathbf{F}_p} H^n(X, \mathbf{F}_p)_G \quad \text{(coinvariants)}.$$

Dualizing we obtain
$$rkH \leqq \dim_{\mathbf{F}_p} H_n(X, \mathbf{F}_p)^G. \quad \square$$

Remarks. This proof works just as well for homology near-manifolds $M_{(p)}^{\sim}(S^n)^k$ (see [1]). Using Carlsson's original proof of the homologically trivial case ([3]), it can also be extended to finitistic spaces $X \simeq (S^n)^k$. Another advantage of the proof we use is that it can easily by adapted to yield a generalization of 1.1 to the non-free case (this will be discussed in Sect. 6).

In the situation of Theorem 1.1, it is plain that $H_n(X, \mathbf{F}_p)$ is a faithful G/H-module, and so is $H_n(X, \mathbf{Z}_{(p)})$. The next step in our proof of the main theorem will be to find a suitable lower bound on

$$\dim_{\mathbf{F}_p} H_n(X, \mathbf{F}_p) - \dim_{\mathbf{F}_p} H_n(X, \mathbf{F}_p)^G.$$

For this we require some facts about rational representations of $(\mathbf{Z}/p)^r$.

§ 2. Rational representations of $(\mathbf{Z}/p)^r$

In this section we will describe the complete collection of irreducible rational representations of $(\mathbf{Z}/p)^r$ in a particularly simple way.

We recall a fundamental theorem about $\mathbf{Q}G$-modules (see [7]):

Theorem 2.1. *Let G be a finite group. Then $\mathbf{Q}G$ is a direct sum $\mathbf{Q}G = \bigoplus_{i=1}^{r} S_i$ of irreducible $\mathbf{Q}G$-modules. All irreducibles appear in this decomposition and any other $\mathbf{Q}G$-module can be expressed uniquely as a direct sum of copies of these simple summands.* $\square$

Now let $G = (\mathbf{Z}/p)^r$: we construct a collection of irreducibles as follows. Let $H \subset G$ be a subgroup of index p and take the reduced regular representation of G/H: $\mathbf{Q}[G/H]/(N_{G/H})$, where $N_{G/H}$ is the norm. Then G acts on this module through $G \to G/H$. As a G-module we will denote it by $M(H)$.

Lemma 2.2. *Each $M(H)$ is a simple $\mathbf{Q}G$-module and $M(H) \simeq M(H')$ if and only if $H = H'$.*

Proof. Clearly $\mathbf{Q}[G/H]/(N_{G/H})$ is a simple G/H-module, hence $M(H)$ is a simple $\mathbf{Q}G$-module.

For the second part, look at the rational character which $M(H)$ affords

$$\mu_H(g)=\begin{cases}p-1 & \text{if } g\in H\\ -1 & \text{if } g\notin H.\end{cases}$$

Clearly $\mu_H=\mu_{H'}$ if and only if $H=H'$. □

Proposition 2.3. *The* $\{M(H)|H\subset G,\ [G:H]=p\}$ *and the trivial one-dimensional representation* $\mathbf{Q}$ *are a complete collection of irreducible* $\mathbf{Q}G$*-modules.*

Proof.

$$\dim_{\mathbf{Q}}\left(\mathbf{Q}\oplus\left(\bigoplus_H M(H)\right)\right)=1+\left(\frac{p^r-1}{p-1}\right)(p-1)=p^r$$

as there are $\dfrac{p^r-1}{p-1}$ subgroups of index p in G. The result follows from Theorem 2.1. □

Therefore we have shown that any finitely generated $\mathbf{Q}G$-module M can be expressed uniquely as

$$M\cong\mathbf{Q}^m\oplus\left(\bigoplus_{i=1}^t M(H_i)\right).$$

§ 3. Integral representations of $(\mathbf{Z}/p)^r$

As before, let $R=\mathbf{Z}_{(p)}$. Then, if G is a finite group, an RG-lattice is a finitely generated R-torsion-free RG-module. We use the results in § 2 to prove

Theorem 3.1. *Let* $G=(\mathbf{Z}/p)^r$, *p odd and* M *an* RG*-lattice. Then*

$$\dim_{\mathbf{F}_p} M\otimes\mathbf{F}_p-\dim_{\mathbf{F}_p}(M\otimes\mathbf{F}_p)^G\geqq\left(\frac{p-2}{p-1}\right)(rk_R M-rk_R M^G).$$

Proof. For an RG-lattice M denote

$$\gamma_p(M)=\dim_{\mathbf{F}_p} M\otimes\mathbf{F}_p-\dim_{\mathbf{F}_p}(M\otimes\mathbf{F}_p)^G.$$

For short exact sequences of RG-lattices $0\to M'\to M\to M''\to 0$ it is easy to verify that

$$\gamma_p(M)\geqq\gamma_p(M')+\gamma_p(M''). \tag{3.2}$$

Notice that we have a short exact sequence of RG-lattices:

$$0\to M^G\to M\to M/M^G\to 0.$$

From this we have $\gamma_p(M)\geqq\gamma_p(M/M^G)$, hence it suffices to prove the theorem for M/M^G $((M/M^G)^G=0)$.

Denote $\bar{M}=M/M^G$. We will use induction on the simple summands in $\mathbf{Q}\bar{M}=\mathbf{Q}\otimes\bar{M}$.

First assume

$$\mathbf{Q}\bar{M}\cong M(H)\qquad\text{for some } H\subseteq G.$$

Then, as a G/H-module

$$\mathbf{Q}\bar{M} \cong \mathbf{Q}[G/H]/(N_{G/H}).$$

From the Diederichsen-Reiner Theorem on $\mathbf{Z}/p$-lattices (see [6]) we deduce that $\bar{M}$ is isomorphic to $R[\xi]$, where ξ is a primitive p-th root of unity and a generator of G/H acts through ξ. Then, reducing mod p, we necessarily have

$$\bar{M} \otimes \mathbf{F}_p \cong \mathbf{F}_p[G/H]/(N_{G/H}).$$

From this it follows that

$$\gamma_p(\bar{M}) = \left(\frac{p-2}{p-1}\right) rk_R \bar{M} = p-2$$

and the result holds in this case.

Now suppose the result holds for modules which over $\mathbf{Q}$ are a sum of $k-1$ simple summands, and assume

$$\mathbf{Q}\bar{M} \cong \bigoplus_{i=1}^{k} M(H_i) \cong M(H_1) \oplus \left(\bigoplus_{i=2}^{k} M(H_i) \right).$$

This rational splitting gives us a projection

$$\mathbf{Q}\bar{M} \xrightarrow{\pi} M(H_1).$$

Let $N = \pi(\bar{M})$; then we have a short exact sequence of RG-lattices

$$0 \to N' \to \bar{M} \to N \to 0$$

where

$$\mathbf{Q}N' \cong \bigoplus_{i=2}^{k} M(H_i), \qquad \mathbf{Q}N \cong M(H_1).$$

By 3.2

$$\gamma_p(\bar{M}) \geqq \gamma_p(N') + \gamma_p(N) \geqq k(p-2).$$

The second inequality follows from our induction hypothesis, and the proof is complete. □

Corollary 3.3. *Under the condition of Theorem 3.1, if M is a faithful representation, then*

$$\dim_{\mathbf{F}_p} M \otimes \mathbf{F}_p - \dim_{\mathbf{F}_p} (M \otimes \mathbf{F}_p)^G \geqq (p-2)\, rk(G).$$

Proof. Let $\mathbf{Q}M \cong \mathbf{Q}^T \oplus \left(\bigoplus_{i=1}^{s} M(H_i) \right)$, then in particular it is a faithful $\mathbf{Q}G$-module. Therefore we must have

$$\bigcap_{i=1}^{s} H_i = 0 \quad \text{in } G = (\mathbf{Z}/p)^r.$$

The H_i are hyperplanes in G, so that there must be at least r distinct ones among them, i.e. $s \geqq r$. Applying the theorem,

$$\gamma_p(M) \geqq \binom{p-2}{p-1}(s(p-1)) \geqq r(p-2). \quad \square$$

The inequality can also be rearranged to show

Corollary 3.4. *Under the conditions of Theorem* 3.1,

$$\dim_{\mathbf{F}_p} H^1(G, M) \otimes \mathbf{F}_p \leqq \frac{1}{p-1}(rk_R\, M - rk_R\, M^G). \quad \square$$

Recently G. Janusz [9] has proved a beautiful generalization of 3.2 for *all* abelian p-groups.

§ 4. The main theorem

We will use the preceding results to prove

Theorem 4.1. *Let* $(\mathbf{Z}/p)^r$, *p odd, act freely on a* $\mathbf{Z}_{(p)}$*-homology manifold X with* $H^*(X, \mathbf{Z}_{(p)}) \cong H^*((S^n)^k, \mathbf{Z}_{(p)})$. *Then*

$$r \leqq \dim H_n(X, \mathbf{F}_p)^G + \frac{1}{p-2}(\dim H_n(X, \mathbf{F}_p) - \dim H_n(X, \mathbf{F}_p)^G).$$

Proof. By Theorem 1.1, if $H \subset G$ is the subgroup of elements of G which act trivially on $H_*(X, \mathbf{F}_p)$, then

$$rkH \leqq \dim_{\mathbf{F}_p} H_n(X, \mathbf{F}_p)^G.$$

On the other hand, by Corollary 3.2, as G/H acts faithfully on $H_n(X, \mathbf{F}_p)$, we have

$$rkG/H \leqq \frac{1}{p-2}(\dim H_n(X, \mathbf{F}_p) - \dim H_n(X, \mathbf{F}_p)^G).$$

Combining these two inequalities completes the proof. $\square$

As a corollary, we obtain the generalization of Smith's theorem:

Theorem 4.2. *Let* $G = (\mathbf{Z}/p)^r$, *p odd, act freely on* $(S^n)^k$; *then*

$$r \leqq k. \quad \square$$

This proof will work as long as 1.1 holds, hence it can be extended to the other cases mentioned.

For an arbitrary finite, connected free $(\mathbf{Z}/p)^r$-complex X we can use the same methods to find an interesting "global" bound on r, provided X has torsion-free $\mathbf{Z}_{(p)}$-homology. Denote by $N(X)$ the number of such non-zero reduced homology groups of X and let $J \subset \mathbf{F}_p G$ be the augmentation ideal.

Proposition 4.3. (Compare [4].) *If $G=(\mathbf{Z}/p)^r$, p odd, and X is a finite, free, connected $G-CW$ complex with $\mathbf{Z}_{(p)}$ torsion free homology, then*

$$r \leqq N(X) + \left(\frac{1}{p-2}\right)(\dim_{\mathbf{F}_p} JH^*(X, \mathbf{F}_p)).$$

Proof. Once again if $H \subseteq G$ acts homologically trivially on X, then

$$rkH \leqq N(X).$$

This is a theorem due to Browder [2]. Now G/H acts faithfully on $H_*(X, \mathbf{F}_p)$, hence

$$\dim H_*(X, \mathbf{F}_p) - \dim H_*(X, \mathbf{F}_p)^G \geqq (p-2)\, rkG/H.$$

By duality, the term on the left is just $\dim JH^*(X, \mathbf{F}_p)$ and the two inequalities imply the result. □

In the next section we deal with the case $p=2$.

§ 5. The case $p=2$

In this case the rational representation theory is of no use. We shall work only over $\mathbf{F}_2$. We recover 1.1 for $p=2$ by using Carlsson's proof when the action is homologically trivial [5]. This proof uses exclusively arguments over $\mathbf{F}_2$, and works for finitistic spaces $X \simeq (S^n)^k$ with a free action of $G=(\mathbf{Z}/2)^r$.

Now the problem is to find a suitable lower bound for $\gamma_2(H_n(X, \mathbf{Z}_{(2)}))$ (notation as before). The following lemma was first proved by Schultz [10] using different methods:

Lemma 5.1. *Let G be a finite group acting on $X \simeq (S^n)^k$ for $n \neq 1, 3, 7$. Then*

$$H_n(X, \mathbf{F}_2) \cong \bigoplus_{i=1}^{s} \mathbf{F}_2[G/H_i],$$

a permutation module.

Proof. Let $g \in G$, then up to homotopy we can express its action g_* on $H_n(X, \mathbf{F}_2)$ in terms of the basis given by $S^n \hookrightarrow (S^n)^k$. In this basis, no two elements on the same row in the matrix for g_* can be non-zero, otherwise we would have a map $S^n \times S^n \to S^n$ of bidegree (odd, odd) and this is impossible for $n \neq 1, 3, 7$.

We conclude that in this basis g_* is represented by a permutation matrix. This holds for all $g \in G$ with the same basis, implying the result. □

Lemma 5.2. *Let $G=(\mathbf{Z}/2)^r$ and M a faithful $\mathbf{F}_2 G$ permutation module. Then*

$$\dim_{\mathbf{F}_2} M - \dim_{\mathbf{F}_2} M^G \geqq r.$$

Proof. As easy inductive argument. □

We are ready to prove an approximation to Theorem 4.2.

Theorem 5.3. *Let $G=(\mathbf{Z}/2)^r$ act freely on a finitistic space $X\cong(S^n)^k$, $n\neq 1, 3, 7$. Then*

$$r\leqq k.$$

Proof. Let $H\subset G$ be the subgroup of elements acting trivially on $H_n(X, \mathbf{F}_2)$. As we said before

$$rkH\leqq\dim H_n(X,\mathbf{F}_2)^G.$$

Now $H_n(X, \mathbf{F}_2)$ is a faithful $\mathbf{F}_2[G/H]$-permutation module, so by Lemma 5.2,

$$rkG/H\leqq\dim H_n(X,\mathbf{F}_2)-\dim H_n(X,\mathbf{F}_2)^G.$$

As before, we combine these inequalities to complete the proof. □

The remaining cases $p=2$ with $n=1, 3, 7$ are rather different, as the homology representations are quite arbitrary. On the other hand a richer geometric structure is available, and perhaps this can be exploited to settle them.

§6. General $(\mathbf{Z}/p)^r$-actions

For this situation we recall a theorem due to Browder [1]:

Theorem 6.1. *Let $G=(\mathbf{Z}/p)^r$ act on an orientable $\mathbf{Z}_{(p)}$-homology near-manifold M^n, preserving orientation. Then if $i\colon M\to M\times_G EG$ is the fiber inclusion*

$$|H^n(M,\mathbf{Z})/i^*H^n(M\times_G EG,\mathbf{Z})|\qquad \text{is divisible by } |G:G_\sigma|,$$

where G_σ is an isotropy subgroup of maximal rank in G and n is the top dimension. □

This leads to an immediate extension of 1.1 and hence 4.1, which we now state in full generality:

Theorem 6.2. *Let $G=(\mathbf{Z}/p)^r$, p odd act on an orientable $\mathbf{Z}_{(p)}$-homology near manifold X with $H^*(X, \mathbf{Z}_{(p)})\cong H^*((S^n)^k, \mathbf{Z}_{(p)})$ and let $H_0\subset G$ be an isotropy subgroup of maximal rank acting homologically trivially on X. Then*

$$rkG\leqq\dim H_n(X,\mathbf{F}_p)^G+\frac{1}{p-2}(\dim H_n(X,\mathbf{F}_p)-\dim H_n(X,\mathbf{F}_p)^G)+rkH_0. \quad \square$$

For $p=2$ Carlsson's proof for the homologically trivial case [5] can be generalized to show that the corank of an isotropy subgroup of maximal rank is bounded by the number of spheres. As before, this can be used to prove

Theorem 6.3. *If $G=(\mathbf{Z}/2)^r$ acts on a finitistic space $X\cong(S^n)^k$ and H_0 is an isotropy subgroup of maximal rank acting homologically trivially on X, then*

$$rkG\leqq\dim H_n(X,\mathbf{F}_2)+rkH_0. \quad \square$$

§ 7. Questions

To conclude we mention three related questions.

7.1 With the hypothesis of 4.1, does the stronger inequality

$$rk G \leqq \dim_{\mathbf{F}_p} H_n(X, \mathbf{F}_p)^G$$

hold (same for $p=2$)?

7.2 If $(\mathbf{Z}/p)^r$ acts freely on $S^{n_1} \times S^{n_2} \times \ldots S^{n_k}$, is $r \leqq k$? (Heller [8] has proved this for $k=2$.)

7.3 (G. Carlsson) If $G=(\mathbf{Z}/p)^r$ and C_* is a finite, free, connected $\mathbf{F}_p G$-chain complex, then is

$$\sum_{i=0} \dim H_i(C_*) \geqq 2^r?$$

(Carlsson [6] has proved this for $rkG \leqq 3$, $p=2$.)

References

1. Browder, W.: Actions of elementary abelian groups (To appear in Topology)
2. Browder, W.: Cohomology and group actions. Invent. Math. **71**, 599–607 (1983)
3. Carlsson, G.: On the rank of abelian groups acting freely on $(S^n)^k$. Invent. Math. **69**, 393–404 (1982)
4. Carlsson, G.: On the homology of finite free $(\mathbf{Z}/2)^n$-Complexes. Invent. Math. **74**, 139–147 (1983)
5. Carlsson, G.: On the Non-existence of Free Actions of Elementary Abelian Groups on Products of Spheres. Am. J. Math. **102**, 1147–1157 (1980)
6. Carlsson, G.: $(\mathbf{Z}/2)^3$ Actions on Finite Complexes. Algebraic Topology and Algebraic K-Theory. Ann. Math. Stud. 113 (1987)
7. Curtis, G., Reiner, I.: Methods of Representation Theory. New York: J. Wiley & Sons, 1981
8. Heller, A.: A note on Spaces with Operators. Ill. J. Math. **3**, 98–100 (1959)
9. Janusz, G.J.: An Inequality in Integral Representation Theory (To appear)
10. Schultz, R.: On the Inertia Group of a Product of Spheres. Trans. Am. Math. Soc. **156**, 137–153 (1971)

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