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# Linear maps over abelian group algebras

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**Abstract**

Let $G$ be a finite abelian $p$-group, and $K$ a field of characteristic $p$. In this paper we show that if $F$ is a free $KG$ module and $\Psi \in \mathrm{End}_{KG}(F)$ is not an isomorphism, then $\dim_K \mathrm{Ker}\, \Psi \geq |G|/\exp(G)$. This result sheds some light on questions related to finite, free $G$-CW complexes as well as having applications to modular representation theory.

## 0. Introduction

Let $K$ denote a field of a characteristic $p > 0$, and $G$ a finite abelian $p$-group. Then $KG$ is a local $K$-algebra of finite dimension over $K$. In this paper we address a very simple but fundamental question: given a free $KG$-module $F$, and $\Psi \in \mathrm{End}_{KG}(F)$, does there exist an integer $d > 0$ (depending on $G$) such that

$$\dim_K \ker \Psi \geq d$$

if $\Psi$ is not an isomorphism?

The following is an affirmative answer to this question:

**Theorem.** *If $F$ is a finitely generated, free $KG$-module, and $\Psi : F \to F$ is a non-injective $KG$-map, then*

$$\dim_K \ker \Psi \geq |G|/\exp(G).$$

As an immediate consequence of this we have:

**Corollary.** *If $M$ is a finitely generated $KG$-module with $\hat{H}^0(G, M) \cong \hat{H}^1(G, M)$, then*

$$\dim_K M \geq |G|/\exp G.$$

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Denote by $\Omega^i(\ )$ the $i$th Heller operator on $KG$-modules (see Section 2 for its definition). Then we have:

**Corollary.** *For any $KG$-module $M$,*

$$\dim_K M + \dim_K \Omega^2(M) \geq 2(|G|/\exp G).$$

The methods used are elementary, but the results are part of much more general questions on chain complexes of free $KG$-modules which are of considerable interest.

In the final section we state a general result for local $K$-algebras. However, as our main motivation has been from topological group actions, we have chosen to concentrate on the case of finite group algebras. We are grateful to the referee for pointing out this algebraic statement.

## 1. Endomorphisms of free $KG$-modules

Denote by $K$ a field of characteristic $p > 0$, and let $G$ be a finite abelian $p$-group. There exist integers $r_1 \leq r_2 \leq \cdots \leq r_n$ such that

$$G \cong \mathbb{Z}/p^{r_1} \times \mathbb{Z}/p^{r_2} \times \cdots \times \mathbb{Z}/p^{r_n}.$$

If $x_1, \ldots, x_n$ are generators corresponding to this decomposition, then the group algebra $KG$ can be expressed as

$$KG \cong K[x_1, \ldots, x_n]/x_1^{p^{r_1}} - 1, \ldots, x_n^{p^{r_n}} - 1.$$

Now let $u_i = x_i - 1$; then, as $\operatorname{char} K = p$,

$$u_i^{p^{r_i}} = (x_i - 1)^{p^{r_i}} = x_i^{p^{r_i}} + (-1)^{p^{r_i}} = 0.$$

Therefore we may write

$$KG \cong K[u_1, \ldots, u_n]/u_1^{p^{r_1}}, \ldots, u_n^{p^{r_n}}. \tag{1.1}$$

Note that as a consequence of this decomposition, we have that if $\alpha \in I$ (the augmentation ideal of $KG$), then

$$\alpha^{p^{r_n}} = 0. \tag{1.2}$$

We remark here that $p^{r_n}$ is precisely the *exponent* of $G$, which we will denote by $\exp(G)$.

We start with the following lemma.

**Lemma 1.1.** *Let $F$ be a free $KG$-module of finite rank, and $\Psi \in \operatorname{End}_{KG}(F)$. Then there exists another free module $F'$, $\operatorname{rk} F' \leq \operatorname{rk} F$ and $C \in \operatorname{End}_{KG}(F')$ such that* $\operatorname{coker} C$ *and* $\operatorname{coker} \Psi$ *have the same dimension over $K$, and* $\operatorname{Im} C \subseteq IF'$.

**Proof.** We can identify $\Psi$ with a matrix $A \in M_{m\times m}(KG)$, where $F \cong (KG)^m$. Suppose that some entry, say, $a_{ij}$, is not in $I$. As $KG$ is local, with maximal ideal $I$, this means $(a_{ij}) = KG$, and hence, $KG$ splits off Im $\Psi$, and we obtain a new map

$$(KG)^{m-1} \to (KG)^{m-1},$$

whose cokernel has the same dimension as that of the original map. We proceed like this until we obtain the desired $F'$. In particular, $\Psi$ is injective if and only if $F' = 0$. □

We are now ready to prove the following result.

**Theorem 1.2.** *Let $G$ be a finite abelian $p$-group, $K$ a field of characteristic $p$, and $F$ a free $KG$-module of finite rank. Then, if $\Psi \in \mathrm{End}_{KG}(F)$ is not injective,*

$$\dim_K \ker \Psi \geq \frac{|G|}{\exp(G)}.$$

**Proof.** Note that $\dim_K \ker \Psi = \dim_K \operatorname{coker} \Psi \neq 0$, hence, by the preceding lemma, we can assume that Im $\Psi \subseteq IF$.

Again, identifying $\Psi$ with an $m \times m$-matrix $A$ over $KG$, we can construct its characteristic polynomial

$$p(t) = t^m + a_{m-1}t^{m-1} + \cdots + a_1 t + a_0,$$

where, by the above, $a_0, \ldots, a_{m-1} \in I$.

By the Cayley–Hamilton theorem, we have an identity

$$\Psi^m = -a_{m-1}\Psi^{m-1} - \cdots - a_1\Psi - a_0. \tag{1.3}$$

Take $\exp(G) = p^{r_n}$ power on both sides:

$$\Psi^{\exp(G)m} = (-a_{m-1}\Psi^{m-1} - \cdots - a_1\Psi - a_0)^{\exp(G)} = 0$$

as $z \mapsto z^{\exp(G)}$ is zero on $I$, and each coefficient $a_i \in I$.

From this we obtain a filtration

$$F \supset \Psi F \supset \Psi^2 F \supset \cdots \supset \Psi^{\exp(G)m} F = 0$$

and surjections

$$F/\Psi F \twoheadrightarrow \Psi F/\Psi^2 F \twoheadrightarrow \Psi^2 F/\Psi^3 F \twoheadrightarrow \cdots.$$

We have the equality

$$m|G| = \dim_K F = \sum_{i=0}^{\exp(G)m-1} \dim_K \Psi^i F/\Psi^{i+1} F.$$

As clearly,

$$\dim_K \Psi^i F/\Psi^{i+1} F \leq \dim_K F/\Psi F,$$

we obtain

$$(\exp(G)\cdot m)\dim_K \operatorname{coker} \Psi \geq m\cdot|G|,$$

from which we deduce

$$\dim_K \operatorname{coker} \Psi \geq \frac{|G|}{\exp(G)}. \qquad \square$$

## 2. Applications

Let $G = (\mathbb{Z}/p)^k$, $K$ as before, and $C_*$ a finite $KG$-chain complex of free $KG$-modules. Carlsson [2] has conjectured that under these conditions, if $H_*(C) \neq 0$, then

$$\sum \dim_K H_i(C) \geq 2^k.$$

A simple consequence of Theorem 1.2 is the verification of this for one-dimensional chain complexes.

**Corollary 2.1.** *If $F_1$, $F_2$ are free $K(\mathbb{Z}/p)^k$-modules, and* $\operatorname{char}(K) = p$, *then, for any equivariant map which is not an isomorphism,*

$$\Psi : F_1 \to F_2,$$

*we have*

$$\dim_K \ker \Psi + \dim_K \operatorname{coker} \Psi \geq 2^k.$$

Perhaps Corollary 2.1 will be a useful initial step in analyzing this problem. However, it does have interesting consequences in representation theory. We recall the Heller operator for $KG$-modules.

**Definition 2.2.** If $K$ is a field of characteristic $p$, and $M$ is a finitely generated $KG$-module, then $\Omega^i(M)$ denotes the unique $KG$-module without projective summands such that

$$\hat{H}^*(G, \Omega^i(M)) \cong \hat{H}^{*-i}(G, M)$$

for all $* \in \mathbb{Z}$.

Note that $\Omega^2(M)$ arises from an exact sequence

$$0 \to \Omega^2(M) \to F_1 \to F_2 \xrightarrow{\rho} M \to 0 \tag{2.1}$$

where $F_2$ is the projective cover of $M$ and $F_1$ the projective cover of $\ker \rho$.

Applying Theorem 1.2 to this, we obtain:

**Theorem 2.3.** *If $G$ is a non-trivial finite abelian $p$-group, and $M$ a finitely generated $KG$-module* (char $K = p$), *then*

$$\dim_K M + \dim_K \Omega^2(M) \geq 2\left(\frac{|G|}{\exp(G)}\right).$$

**Proof.** If $F_1 \neq F_2$, then we get $|G|$ as a lower bound. Hence, we are reduced to Theorem 1.2, which immediately yields this result. □

**Remarks.** Note that for $G = (\mathbb{Z}/2)^k$ we obtain $|G|$ as a lower bound in Theorem 2.3. Also, it might be possible to prove a similar formula for $M$ and $\Omega^{2i}(M)$.

**Theorem 2.4.** *Let $G$ be a finite abelian $p$-group, $K$ a field of characteristic $p$, and $M$ a finitely generated $KG$-module without projective summands such that $\hat{H}^i(G,M) \cong \hat{H}^{i+1}(G,M)$, then*

$$\dim_K \Omega^{-i}(M) \geq \frac{|G|}{\exp(G)}.$$

**Proof.** We will do the case when $i \geq 0$, the situation when $i < 0$ can be reduced to this by using the dual module $M^*$.

Consider a minimal projective resolution for $M^*$ over $KG$, truncated at the $i$th stage:

$$0 \to \Omega^{i+2}(M^*) \to F_{i+1} \to F_i \to \cdots \to F_0 \to M^* \to 0.$$

By minimality it is direct to verify that

$$\dim_K F_s = |G| \dim H^s(G,M).$$

Moreover, as $M$ has no projective summands, then

$$H^0(G,M) \cong \hat{H}^0(G,M).$$

Hence, $F_i \cong F_{i+1}$, and by Theorem 1.2 we infer that

$$\dim_K \Omega^i(M^*) \geq |G|/\exp(G).$$

By dualizing one can verify that

$$\Omega^i(M^*)^* \cong \Omega^{-i}(M)$$

from which the result follows. □

**Corollary 2.5.** *If* $\hat{H}^0(G,M) \cong \hat{H}^1(G,M)$, *then*

$$\dim_K M \geq |G|/\exp(G).$$

It is interesting to compare Corollary 2.5 with a result due to Carlson [1]: if $M$ is any periodic $KG$-module, then $|G|/\exp(G)$ divides $\dim_K M$. The proof of this involves a characterization of periodic modules using the theory of complexity.

## 3. A Generalization

In this section we prove a general result for local $K$-algebras which involves only a slight variation on our previous methods.

We must first introduce the notion of *Loewy length*. We will assume that $R$ is a local $K$-algebra of finite dimension $d(R)$ over $K$.

**Definition 3.1.** *The Loewy length of a subring* $S \subseteq R$ *is the minimum integer* $q \geq 0$ *such that* $(\mathrm{Rad}(S))^q = 0$.

Next, we define the integer $L_k(R)$ as the maximum of the Loewy lengths of all the subrings of $R$ generated by 1 and $k$ elements $a_1, a_2, \ldots, a_k$ in $\mathrm{Rad}(R)$. We denote the Loewy length of $R$ by $L(R)$. We can now state a general theorem.

**Theorem 3.2.** *If* $\Psi \in \mathrm{End}_R(R^m, R^m)$ *is such that its image is contained in* $\mathrm{Rad}(R)\cdot R^m$, *then*

$$\dim_K \ker \Psi \geq m\frac{d(R)}{L_{m^2}(R)}$$

**Proof.** As in the proof of Theorem 1.2, we use the associated $m \times m$ matrix $A = (a_{ij})$, noting that its $m^2$ entries must lie in $\mathrm{Rad}(R)$. Hence $(\Psi)^{L_{m^2}(R)} = 0$, and we can filter $F = R^m$ as before, to obtain the stated inequality. □

Finally, it is worthwhile to note that the inequality

$$\dim_K \ker \Psi \geq m\frac{d(R)}{L(R)}$$

will always hold.

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## References

[1] J. Carlson, The dimensions of periodic modules over modular group algebras, Illinois J. Math. 23 (1979) 295–306.

[2] G. Carlsson, Free $(\mathbb{Z}/2)^k$-actions and a problem in commutative algebra, Lecture Notes in Mathematics, Vol. 1217 (Springer, Berlin, 1986) 79–83.