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$\mathbf{Z}/p\mathbf{Z}$ ACTIONS ON $(S^n)^k$

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ABSTRACT. Let $\mathbf{Z}/p$ act on a finitistic space X with integral cohomology isomorphic to that of $(S^n)^k$ as a ring. We show a direct relationship between the $\mathbf{Z}/p$-module structure of $H^n(X;\mathbf{Z})$ and the nature of the fixed-point set. In particular, we obtain a significant restriction on $H^n(X;\mathbf{Z})$ for free actions.

Introduction. Let $\mathbf{Z}/p$, p prime, act on a space X with integral cohomology isomorphic to that of $(S^n)^k$ as a ring, which we abbreviate as $X \sim (S^n)^k$. Then $H^*(X;\mathbf{Z})$ is a graded $\mathbf{Z}/p$-module whose structure is determined by $H^n(X;\mathbf{Z})$ and the cup product.

The indecomposable integral representations of $\mathbf{Z}/p$ have been completely described. Therefore it is natural to inquire whether there is any relationship between the $\mathbf{Z}/p$-module structure of $H^n(X;\mathbf{Z})$ and the nature of the fixed-point set. In this paper, we answer this question affirmatively: the following is an outline of this relationship.

A theorem due to Diederichsen and Reiner (see [**C-R**]) states that there are a finite number of isomorphism classes of indecomposable integral representations of $\mathbf{Z}/p$, and that they fall into three different types:

(1) $\mathbf{Z}$, the trivial $\mathbf{Z}/p$-module.

(2) A_i, of rank $p-1$, corresponding to elements of the ideal class group.

(3) P_i, of rank p, one for each A_i and constructed from them. These are the projective indecomposables.

Every integral representation can be expressed as a direct sum of these modules. We shall say that M is of type (r,s,t) if $M \cong (\bigoplus^r A_i) \oplus (\bigoplus^s P_j) \oplus (\bigoplus^t \mathbf{Z})$.

For the following statements we will suppose that $\mathbf{Z}/p$ acts on a finitistic space $X \sim (S^n)^k$ with fixed-point set F.

THEOREM 4.5. *If $H^n(X;\mathbf{Z})$ is of type $(0,s,0)$, then $F \neq \emptyset$ and has the cohomology ring of $(S^n)^s$ with $\mathbf{Z}/p$ coefficients.*

THEOREM 4.6. *If p is odd and $H^n(X;\mathbf{Z})$ is of type $(r,0,0)$, then $F \neq \emptyset$, $H^*(F;\mathbf{Z}_{(p)})$ is torsion-free, zero in odd dimensions, and of rank p^r.*

THEOREM 4.7. *If p is odd and $H^n(X;\mathbf{Z})$ is of type $(r,s,0)$, then $F \neq \emptyset$, $\operatorname{rk} H^*(F;\mathbf{Z}/p) \geq 2^{r+s}$, and $H^*(F;\mathbf{Z}/p)$ contains an exterior algebra on s n-dimensional generators.*

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As a corollary we obtain the following restriction for free $\mathbf{Z}/p$-actions:

COROLLARY 4.8. *Let $\mathbf{Z}/p$, p an odd prime, act freely on a finitistic space $X \sim (S^n)^k$. Then $H^n(X;\mathbf{Z})$ splits off a trivial direct summand as a $\mathbf{Z}/p$-module.*

Using some algebraic K-theory and 4.8 yield

THEOREM 4.11. *Let X be a finitely dominated complex such that $\pi_1(X) \cong \mathbf{Z}/p$ and $\tilde{X} \sim (S^n)^k$, where $\tilde{X}$ is the universal covering space of X $(n > 1)$. Then X is equivalent to a finite complex.*

The proof of these results requires using the cohomological description of $H^n(X;\mathbf{Z})$ and its exterior powers in the spectral sequence associated to the Borel construction $X \times_{\mathbf{Z}/p} E\mathbf{Z}/p$. We extend results due to Bredon [**Bre1**], although he made no use of the representation theory as we do. Our work provides information about $(\mathbf{Z}/p)^r$-actions on $(S^n)^k$, and is a step towards extending work of Carlsson (see [**Ca1**]) to homologically nontrivial actions. For example, Corollary 4.8 can be placed as part of a more general conjecture about free $(\mathbf{Z}/p)^r$-actions for p odd (if $p = 2$, use $\mathbf{F}_2$ coefficients):

CONJECTURE. If $(\mathbf{Z}/p)^r$ acts freely on $X \sim (S^n)^k$, then $r \leq \operatorname{rk}(H^n(X;\mathbf{Z}))^G$.

The paper is organized as follows: in §1 we describe the integral representations of $\mathbf{Z}/p$ and their relevant cohomological properties; in §2 we discuss the (co)homology automorphisms induced by $\mathbf{Z}/p$-actions on $(S^n)^k$; in §3 we apply the Lefschetz fixed-point theorem to our problem; and finally in §4 we prove the main results in the paper by using cohomological methods.

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1. Integral representations of $\mathbf{Z}/p$. In this section we will give a complete description of the indecomposable integral representations of $\mathbf{Z}/p$. This is a theorem due to Reiner and Diederichsen; a reference for this section is [**C-R**, §74].

Let Θ be a primitive pth root of unity over Q,

$K = Q(\Theta)$, $\operatorname{Irr}(\Theta, Q) = 1 + X + \cdots + X^{p-1} = \Phi_p(X)$,

$R =$ ring of algebraic integers in K.

DEFINITION 1.1. An ideal in K is a finitely generated nonzero R-submodule of K. □

One can show that $\operatorname{rk}_{\mathbf{Z}}(R) = (K : Q) = p - 1$ and hence all the ideals A have rank $p - 1$. Let $G \cong \mathbf{Z}/p$, g a generator; we define a G-action on A as follows.

$$g \cdot a = \Theta \cdot a \quad \forall a \in A.$$

Therefore every ideal is a $\mathbf{Z}/p$-module of rank $p - 1$ and they are related in the following way.

PROPOSITION 1.2 [**C-R**]. *$A_1 \cong A_2$ as G-modules if and only if they are in the same ideal class (i.e. there exists a $\gamma \neq 0$ in K such that $A_2 = \gamma A_1$).* □

A result from number theory is that there are only a finite number of ideal classes for any prime p. Therefore, by 1.2, there are only a finite number of isomorphism classes of $\mathbf{Z}/p$-modules amongst the ideals for any prime p.

Let A be an ideal: take the abelian group $A \oplus \mathbf{Z}y$ and fix $a_0 \in A$. Define a G-action:

$$g \cdot a = \Theta \cdot a, \qquad a \in A,$$
$$g \cdot y = y + a_0.$$

This defines a $\mathbf{Z}/p$-module of rank p, which is denoted by $A(a_0)$.

PROPOSITION 1.3. *Let A be an ideal.*
(1) *If $c \in \mathbf{Z}$, $p \nmid c$, then $A(a_0) \cong A(ca_0)$, $a_0 \in A$.*
(2) *If $a \equiv a' \bmod (\Theta - 1)A$, then $A(a) \cong A(a')$, $a, a' \in A$.*
(3) *$A/(\Theta - 1)A \cong \mathbf{Z}/p$.*
(4) *If $a_0 \notin (\Theta - 1)$ the following is exact (over $\mathbf{Z}G$):*

$$0 \to \mathbf{Z} \to A(a_0) \overset{g-1}{\to} A \to 0.$$

PROOF. (1), (2) can be easily verified; (3) follows from the fact that $(\Theta - 1)R$ is a prime ideal of norm p in R and $A/(\Theta - 1)A \cong R/(\Theta - 1)R$.

For (4) we need $g - 1$ to be onto A. Now

$$(g - 1)A(a_0) = (\Theta - 1)A + \mathbf{Z}a_0.$$

This is an ideal, properly containing $(\Theta - 1)A$, as $a_0 \notin (\Theta - 1)A$. This is a maximal ideal, hence $(g - 1)A(a_0) = A$. □

From the above it follows that

$$A(a) \cong A(a'), \qquad a, a' \notin (\Theta - 1)A,$$

and

$$A(a) \cong A \oplus \mathbf{Z} \quad \text{if } a \in (\Theta - 1)A.$$

THEOREM 1.4 (REINER-DIEDERICHSEN). *Every $\mathbf{Z}/p$-module with finite $\mathbf{Z}$-basis is isomorphic to a direct sum of indecomposables, all of which fall into three types*: (i) $\mathbf{Z}$, (ii) A_i *ideals in* K; (iii) $A_i(a)$, $a \notin (\Theta - 1)A_i$.

Furthermore if $M = (\bigoplus^r A_i) \oplus (\bigoplus^s A_j(a)) \oplus (\bigoplus^t \mathbf{Z})$, the isomorphism class of M is determined by r, s, t and the ideal class $(\prod^r A_i)(\prod^s A_j)$.

PROOF. See [**C-R**, §74]. □

Up to G-isomorphism, there are only a finite number of ideals A_i, say h. By the correspondence $A_i \to A_i(a)$, $a \notin (\Theta - 1)A$ it follows that up to isomorphism there are only h indecomposables of type (iii) and therefore in total there are $2h + 1$.

NOTATION. Let M be a $\mathbf{Z}/p$-module with finite $\mathbf{Z}$-basis. We shall say that M is of type (r, s, t) if there is a decomposition

$$M \cong \left(\bigoplus_1^r A_i\right) \oplus \left(\bigoplus_1^s A_j(a)\right) \oplus \left(\bigoplus_1^t \mathbf{Z}\right).$$

This decomposition theorem allows us to have a good hold on $\mathbf{Z}/p$-modules, simplifying the computation of their cohomological invariants.

We recall that if $G \cong \mathbf{Z}/p$ with generator g and M is a G-module

$$H^{2i}(G; M) \cong M^G/NM, \quad H^{2i+1}(G; M) \cong \ker N/(g - 1)M,$$

where $N = 1 + g + \cdots + g^{p-1}$ is the norm map.

PROPOSITION 1.5. *$H^*(G; A(a)) = 0$ $\forall A$ ideal, $a \notin (\Theta - 1)A$.*

PROOF. Recall that as an abelian group,

$$(*) \qquad A(a_0) \cong A \oplus \mathbf{Z}y.$$

Let $x \in A(a_0)$; then it is of the form $a + \alpha y$

$$(g-1)x = (g-1)(a+\alpha y) = (\Theta - 1)a + \alpha(g-1)y = (\Theta - 1)a + \alpha a_0.$$

Suppose now that $(g-1)x = 0$; then $\alpha a_0 \in (\Theta - 1)A$ and hence $p|\alpha$. Using the fact that $NA = 0$, we can express

$$-pa_0 = (\Theta - 1)(1 + (1+\Theta) + \cdots + (1 + \cdots + \Theta^{p-2}))a_0.$$

Therefore if $\alpha = mp$, we obtain

$$(\Theta - 1)a = (\Theta - 1)(1 + (1+\Theta) + \cdots + (1 + \cdots + \Theta^{p-2}))ma_0.$$

By 1.3(3), $\Theta - 1$ is injective on A, so that

$$a = (1 + (1+\Theta) + \cdots + (1 + \Theta + \cdots + \Theta^{p-2}))ma_0.$$

It follows that

$$\begin{aligned} x = a + \alpha y &= m((1 + (1+\Theta) + \cdots + (1 + \cdots + \Theta^{p-2}))a_0 + py) \\ &= m(1 + g + \cdots + g^{p-1})(y) \in \operatorname{im} N. \end{aligned}$$

This implies that $\ker g - 1 = \operatorname{im} N$ and so $H^{2i}(G; A(a_0)) = 0$.

From the above we have that

$$Ny = (1 + (1+\Theta) + \cdots + (1 + \cdots + \Theta^{p-2}))a_0 + py.$$

By the decomposition $(*)$ we obtain $Ny \neq 0$ and as $NA = 0$, $\ker N = A \subset A(a_0)$. From 1.3(4) $\operatorname{im} g - 1 = A \subset A(a_0)$ so that

$$H^{2i+1}(G; A(a_0)) \cong \ker N / \operatorname{im} g - 1 = 0. \quad \square$$

COROLLARY 1.6. *The $A(a_0)$ are projective $\mathbf{Z}[G]$-modules.*

PROOF. These modules are torsion free and cohomologically trivial in positive dimensions. By Rim's theorem (see [**Brown**, p. 152]), they must be projective.

COROLLARY 1.7. *If A is an ideal, then*

$$H^k(G; A) \cong H^{k+1}(G; \mathbf{Z}), \qquad k > 0.$$

PROOF. By 1.3(4), there is a short exact sequence $0 \to \mathbf{Z} \to A(a_0) \to A \to 0$ with $A(a_0)$ G-cohomologically trivial. $\square$

Let G be a p-group and M a torsion-free $\mathbf{Z}G$-module. Then M being projective as a G-module is equivalent to $M \otimes \mathbf{F}_p$ being free as an $\mathbf{F}_pG$-module.

Therefore for every $A(a_0)$, $a_0 \notin (\Theta - 1)A$, we have

$$A(a_0) \otimes \mathbf{F}_p \cong \mathbf{F}_p[\mathbf{Z}/p].$$

Given the topological problem that interests us, we require a certain understanding of the exterior powers of a module.

Let M be a RG-module with a finite R-basis. Then clearly G acts diagonally on $\bigotimes^n M$ and therefore on its quotient module $\bigwedge_R^n(M)$. If $e_1, \ldots, e_K$ is an R-basis for M, then the G-action on $\bigwedge_R^n(M)$ is determined by

$$e_{i_1} \wedge \cdots \wedge e_{i_n} \xrightarrow{g} ge_{i_1} \wedge \cdots \wedge ge_{i_n}.$$

In this manner $\bigwedge_R^*(M)$ may be considered as a graded RG-module with a basis of $\binom{k}{i}$ elements in dimension i.

We now return to the case $G = \mathbf{Z}/p$.

LEMMA 1.8. *Let $G = \mathbf{Z}/p$. $\bigwedge_{\mathbf{F}_p}^i(\mathbf{F}_p[G])$ is a free G-module for $0 < i < p$.*

PROOF. The elements $1, g, \ldots, g^{p-1}$ are a basis for $\mathbf{F}_p[G]$ freely permuted by G. Their products $g^{i_1} \wedge \cdots \wedge g^{i_k}$ also have full G-orbits and hence provide a G-free basis for $\bigwedge_{\mathbf{F}_p}^k(\mathbf{F}_pG)$, $0 < k < p$. □

PROPOSITION 1.9. *If P is a projective $\mathbf{Z}G$-module, then $\bigwedge_{\mathbf{Z}}^j(P)$ is projective for $0 < j < p$.*

PROOF.

$$P \otimes_{\mathbf{Z}} \mathbf{F}_p \cong \bigoplus_1^m \mathbf{F}_p[G].$$

Hence

$$\bigwedge_{\mathbf{F}_p}^j (P \oplus \mathbf{F}_p) \cong_G \bigoplus_{i_1+\cdots+i_m=j} \bigwedge^{i_1}(\mathbf{F}_pG) \otimes \cdots \otimes \bigwedge^{i_m}(\mathbf{F}_pG).$$

By hypothesis, at least one i_r satisfies $0 < i_r < p$ in each summand, implying that $\bigwedge^{i_r}(\mathbf{F}_pG)$ is G-free and hence the summand $\bigwedge_{\mathbf{F}_p}^j(P \otimes \mathbf{F}_p)$ is G-free for $0 < j < p$. But

$$\bigwedge_{\mathbf{F}_p}^j (P \otimes \mathbf{F}_p) \cong \bigwedge_{\mathbf{Z}}^j(P) \otimes \mathbf{F}_p$$

$$\Rightarrow \bigwedge_{\mathbf{Z}}^j(P) \text{ is } \mathbf{Z}G\text{-projective for } 0 < j < p. \qquad \square$$

Let A be an indecomposable of rank $p-1$; there is a short exact sequence (1.3(4)) $0 \to \mathbf{Z} \to P \to A \to 0$ where P is projective.

This induces another exact sequence

$$0 \to \bigwedge_{\mathbf{Z}}^{i-1}(A) \to \bigwedge_{\mathbf{Z}}^{i}(P) \to \bigwedge_{\mathbf{Z}}^{i}(A) \to 0.$$

Apply Corollary 1.7 and induction; we obtain

$$H^k\left(G; \bigwedge_{\mathbf{Z}}^i(A)\right) \cong H^{k+i}(G; \mathbf{Z}), \qquad k > 0,\ 0 \le i \le p-1,$$

because $\bigwedge_{\mathbf{Z}}^0(A) \cong \mathbf{Z}$, trivial G-module.

If A' is another ideal, then

$$0 \to \bigwedge^{i-1}(A) \otimes \bigwedge^{j}(A') \to \bigwedge^{i}(P) \otimes \bigwedge^{j}(A') \to \bigwedge^{i}(A) \otimes \bigwedge^{j}(A') \to 0$$

is exact because all the modules involved are $\mathbf{Z}$-free. Now if M is G-cohomologically trivial, $M \otimes_{\mathbf{Z}} N$ is G-cohomologically trivial. Hence we obtain the following proposition inductively:

PROPOSITION 1.10. *Let $A_1, \ldots, A_r$ be ideals of rank $p-1$. Then*

$$H^k\left(G; \bigwedge^{N}(A_1 \oplus \cdots \oplus A_r)\right) \cong \sum_{\substack{l_1 + \cdots + l_r = N \\ 0 \leq l_q \leq p-1}} H^{k+N}(G; \mathbf{Z}). \qquad \square$$

The above results will be used in §4.

2. Automorphisms induced by $\mathbf{Z}/p\mathbf{Z}$-actions. Let $X \sim (S^n)^k$; then $H^*(X; \mathbf{Z})$ is an exterior algebra on k n-dimensional generators $\bigwedge_{\mathbf{Z}}^*(e_1, \ldots, e_k)$.

If $G \cong \mathbf{Z}/p$ acts on X, then $H^n(X; \mathbf{Z}) = M$ has a natural G-structure induced on it and so do its exterior powers, i.e. as a graded $\mathbf{Z}G$-module $H^*(X; \mathbf{Z}) \cong_G \bigwedge_{\mathbf{Z}}^*(M)$.

EXAMPLES. (1) $p = 3$, $X = S^3 \times S^3$, $T(x, y) = (y, y^{-1}x^{-1})$, $T_* = \left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)$ in the usual basis, $X^G = \{(x, x) | x^3 = 1\}$, $H^3(X; \mathbf{Z})$ indecomposable $\mathbf{Z}[\mathbf{Z}/3]$-module of rank 2 ($X^G = S^2 +$ point).

This example is due to Bredon [**Bre1**].

(2) Consider $S^{2n+1} \subset \mathbf{C}^n$. Let $\rho = e^{2\pi i/p}$; then ρ acts on S^{2n+1} by complex multiplication. This action is trivial in homology and free, and using it we can find free and homologically trivial actions on any $(S^{2n+1})^k$.

(3) On S^{2n} the antipodal map defines a $\mathbf{Z}/2$-action which is free, and changes orientation: as before we can use this to produce free involutions on $(S^{2n})^k$.

(4) $T\colon (S^n)^p \to (S^n)^p$, $T(X_1, \ldots, X_p) = (X_2, \ldots, X_p, X_1)$,

$$T^* = \begin{pmatrix} 0 & 1 & & & 0 \\ \cdot & 0 & & & \cdot \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & \cdot \\ 0 & \cdot & & 1 & 0 \\ 1 & 0 & \cdots & 0 & 1 \end{pmatrix}, \qquad H^n(X; \mathbf{Z}) \cong \mathbf{Z}[\mathbf{Z}/p].$$

(5) Select $\alpha \in \mathrm{GL}(k; \mathbf{Z})$; then α can be considered as a map of the k-torus $\overline{\alpha}\colon \mathbf{R}^k/\mathbf{Z}^k \to \mathbf{R}^k/\mathbf{Z}^k$. In this example

$$\overline{\alpha}_* = \alpha \in \mathrm{Aut}(H_1(\mathbf{R}^k/\mathbf{Z}^k; \mathbf{Z})) = \mathrm{GL}(k; \mathbf{Z}).$$

Hence any torsion-free $\mathbf{Z}/p$-module can be "realized" on $H_1((S^1)^k; \mathbf{Z})$ via an action on $(S^1)^k$.

If X is homotopy equivalent to a product of n-spheres, one may ask whether the Hopf invariant theorem imposes some restriction on the possible automorphisms of X. The following proposition sheds light on this issue.

PROPOSITION 2.1. *Let* $T\colon (S^n)^k \to (S^n)^k$, $n \neq 1,3,7$, *such that* $T_n^* \in \mathrm{GL}(k;\mathbf{Z})$. *Then its reduction* mod 2, $T_n^*(2) \in \mathrm{SL}(k;\mathbf{F}_2)$, *is a permutation matrix in the usual basis.*

PROOF. Express T_n^* in the usual basis $S^n \to (S^n)^k$. Reducing mod 2, no two entries in the same row can be nonzero, because otherwise we could find a map $S^n \times S^n \to S^n$ of bidegree (odd, odd), which is impossible for $n \neq 1,3,7$. As $\det T_n^* \not\equiv 0 \bmod 2$, then there must be exactly one nonzero entry in each row, and they must be in different columns.

$$\Rightarrow T_n^*(2) \text{ is a permutation matrix.} \qquad \square$$

Consider the case when $(T_n^*)^p = 1$, p an odd prime. Then $T_n^*(2)$ permutes the basis mod 2, and this decomposes into orbits of length p or 1. In other words, by reordering the basis, we can express $T_n^*(2)$ as

$$\begin{bmatrix} P_1 & & & \\ & \ddots & & \\ & & P_r & \\ & & & I_s \end{bmatrix}, \quad P_i = \begin{pmatrix} 0 & \cdot & \cdot\cdot & 0 & 1 \\ 1 & & & & 0 \\ 0 & 1 & & & \vdots \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & & & 1 & 0 \end{pmatrix}, \quad I_s = \mathrm{id}_{s\times s}.$$

This is a matrix in rational form over $\mathbf{F}_2$, with elementary divisors (X^p+1), r times, and $(X+1)$, s times.

We derive-

COROLLARY 2.2. *Let* $T \in \mathrm{SL}(k;\mathbf{Z})$ *with* $T^p = 1$ *and with* $X^{p-1} + X^{p+2} + \cdots + X + 1$ *as an elementary divisor over* $\mathbf{F}_2$. *Then* T *is not realizable on* $H^n((S^n)^k;\mathbf{Z})$ *by any map* $(S^n)^k \to (S^n)^k$ *for* $n \neq 1,3,7$. $\square$

The characteristic polynomial of a linear map is the product of its elementary divisors, so that Corollary 2.2 implies that T is not realizable on $H^n((S^n)^k;\mathbf{Z})$, $n \neq 1,3,7$, if its characteristic polynomial over $\mathbf{F}_2$ has the form

$$(X^p - 1)^s (X^{p-1} + \cdots + X + 1)^t, \qquad t \geq 1.$$

Applying this to the decomposition theorem for $\mathbf{Z}[\mathbf{Z}/p]$-modules, we obtain

PROPOSITION 2.3. *Let* $n \neq 1,3,7$. *There is no* $\mathbf{Z}/p$-*action on* $X = (S^n)^k$ *such that, as a G-module,* $H^n(X;\mathbf{Z})$ *is of type* (r,s,t) *with* $r > t$.

PROOF. If $H^n(X;\mathbf{Z})$ is of type (r,s,t), $r > t$, then the characteristic polynomial of the action mod 2 would be

$$(X^p - 1)^{s+t} (X^{p-1} + \cdots + X + 1)^{r-t}, \qquad r - t > 0. \quad \square$$

The case $X = (S^n)^k$, n even, has fewer cohomology automorphisms than when n is odd:

PROPOSITION 2.4. *Let* $R^* = H^*((S^n)^k;\mathbf{Z})$, n *even, and* $T\colon R^* \to R^*$ *an automorphism of finite period. Then* $T^{(n)}\colon R^{(n)} \to R^{(n)}$ *can be represented by a signed permutation matrix.*

PROOF. $R^* = \bigwedge^*(e_1,\ldots,e_k)$, e_i n-dimensional, i.e. R^* is an exterior algebra on k n-dimensional generators, and as n is even, they commute.

Let

$$Te_i = \sum_{j=1}^{k} \alpha_{ji} e_j.$$

Then, as $e_i^2 = 0$ and T is a ring automorphism

$$0 = T(e_i^2) = T(e_i)^2 = \left(\sum \alpha_{ji} e_j\right)^2$$
$$\Rightarrow \sum_{\substack{j=1,\dots,k \\ l=1,\dots,k \\ j\le l}} 2\alpha_{ji}\alpha_{li} e_j e_l = 0, \qquad \alpha_{ji}\alpha_{li} = 0,\ j \neq l,$$

as the $e_j e_l$, $j \neq l$, are a basis for $R^{(2n)}$.

T is an automorphism, so some $\alpha_{ji} \neq 0$ and therefore $\alpha_{li} = 0$, $\forall l \neq j$.

$\det T^{(n)} = \pm 1$ as $T^{(n)}$ is invertible; thus $\alpha_{ji} = \pm 1$ and $T^{(n)}$ is a signed permutation matrix. □

For an automorphism of order $p = 2$ this is not restrictive, but for p odd it is. There must necessarily be an even number of signs in the matrix, and hence by a change of basis it becomes a permutation matrix. We have

PROPOSITION 2.5. *Let $\mathbf{Z}/p$, p odd, act on $X \sim (S^n)^k$ with n even. Then*

$$H^n(X;\mathbf{Z}) \cong \left(\bigoplus_1^s \mathbf{Z}[\mathbf{Z}/p]\right) \oplus \left(\bigoplus_1^t \mathbf{Z}\right). \qquad \square$$

In particular, $H^n(X;\mathbf{Z})$ is of type $(0,s,t)$.

3. Applications of the Lefschetz Fixed-Point Theorem. Given a map $(S^n)^k \xrightarrow{f} (S^n)^k$, the most elementary tool available to decide the existence of fixed points is the Lefschetz Fixed-Point Theorem.

If f_m is the induced map $f_m\colon H_m((S^n)^k;Q) \to H_m((S^n)^k;Q)$ the Lefschetz number of f is

$$L(f) = \sum_{i=0}^{nk} (-1)^i \operatorname{tr} f_i.$$

The theorem says that if $L(f) \neq 0$, then f has a fixed point.

For $(S^n)^k$ we can use the trace in integral cohomology instead, and so

$$L(f) = \sum_{i=0}^{nk} (-1)^i \operatorname{tr} f_i = \sum_{i=0}^{k} (-1)^{in} \operatorname{tr} \bigwedge^i (f^n).$$

Now let f be an automorphism of prime period p on $(S^n)^k$. We deal with the case n even first.

If n is even, then

$$L(f) = \sum_{i=0}^{k} \operatorname{tr} \bigwedge^i (f^n).$$

To help analyze $L(f)$, we can study the algebraic properties of the following function on $k \times k$ matrices:

DEFINITION 3.1. $L\colon M_{k\times k}(\mathbf{C}) \to \mathbf{C}$,

$$L(A) = \sum_{i=0}^{k} \operatorname{tr} \bigwedge^{i}(A).$$

LEMMA 3.2.

$$(1) \qquad L(A \oplus B) = L(A) \cdot L(B),$$

$$(2) \qquad L(QAQ^{-1}) = L(A), \qquad Q \in M_{k\times k}(\mathbf{C}).$$

PROOF. (1) follows from the identity for modules

$$\bigwedge^{i}(M \oplus N) \cong \sum_{k+j=i} \bigwedge^{j}(M) \otimes \bigwedge^{k}(N)$$

and (2) from $\bigwedge^i(QAQ^{-1}) = \bigwedge^i(Q)\bigwedge^i(A)\bigwedge^i(Q)^{-1}$. □

By Proposition 2.4, $f^{(n)}$ is a signed permutation module in a certain basis. Changing signs and reordering, we may suppose that

$$f^n = (P_1 \oplus \cdots \oplus P_r) \oplus (-I_{s\times s}) \oplus (I_{t\times t}),$$

I the identity matrix, P_j permutation matrices of length p. For p odd, $s = 0$.

It is easy to show that $L(1) = 2$, $L(-1) = 0$, $L(P_i) = 2$, and so on.

PROPOSITION 3.3. *Let f be an automorphism of period p on $(S^n)^k$, n even.*

(1) If p is an odd prime then $L(f) \neq 0$ and hence f has a fixed point.

(2) If $p = 2$ and $f^{(n)} = (\bigoplus_1^r P_i) \oplus I_{t\times t}$, then $L(f) \neq 0$ and hence f has a fixed point. □

Note that for both cases $L(f) = 2^{r+t}$. (1) also follows from the result due to Conner and Floyd for $\mathbf{Z}/p$-actions:

$$\chi(X^{\mathbf{Z}/p}) \equiv \chi(X) \mod p.$$

For n even $\chi((S^n)^k) = 2^k$ and $2^k \not\equiv 0 \mod p$ if $p \neq 2$.

We now consider the case n odd, and so

$$L(f) = \sum_{i=0}^{k} (-1)^i \operatorname{tr} \bigwedge^{i}(f^{(n)})$$

where $\bigwedge^i$ is the usual exterior power on odd dimensional generators.

As before we use the function L on $k \times k$ matrices. The trace of a matrix is the sum of its eigenvalues. This can be generalized to exterior powers to prove that $\operatorname{tr}(\bigwedge^i(A)) = i$th symmetric function of eigenvalues of A.

Clearly $L(1) = 0$. Now let T represent the action of $\mathbf{Z}/p$ on an indecomposable projective module M. The eigenvalues of T are $1, \lambda, \lambda^2, \ldots, \lambda^{p-1}$, where λ is a primitive pth root of unity

$$\operatorname{tr} T = 1 + \lambda + \cdots + \lambda^{p-1} = 0.$$

We have shown that if M is projective, $\bigwedge^i(M)$ is projective for $0 < i < p$. From this it follows that

$$\operatorname{tr} \bigwedge^{i}(T) = 0, \qquad 0 < i < p.$$

So

$$L(T) = \operatorname{tr} \bigwedge^{0}(T) + (-1)^p \operatorname{tr} \bigwedge^{p}(T) = 0$$
$$\left(\bigwedge^{p}(T) = 1,\ p \text{ odd and } -1 \text{ for } p = 2\right).$$

If we take M to be an indecomposable of rank $p-1$, then T has eigenvalues $\lambda, \lambda^2, \ldots, \lambda^{p-1}$

$$\operatorname{Symm}_i(\lambda, \lambda^2, \ldots, \lambda^{p-1}) = (-1)^i.$$

Therefore in this case

$$L(T) = \sum_{i=0}^{p-1} (-1)^i \operatorname{tr} \bigwedge^{i}(T) = \sum_{i=0}^{p-1} (-1)^{2i} = P.$$

This is summarized as follows.

PROPOSITION 3.4 (n ODD). *Let $G \cong \mathbf{Z}/p$ with generator g act on $(S^n)^k$, with $H^n((S^n)^k; \mathbf{Z})$ of type (r, s, t) as a $\mathbf{Z}/p$-module. Then*
(1) *If s or t are nonzero, $L(g) = 0$.*
(2) *If $s = t = 0$, $L(g) = p^r$ and the action has a fixed point.* □

4. Results from cohomological methods. In this section we will apply the cohomological methods due to Borel. The spaces will be taken to be finitistic (e.g. compact or finite dimensional) and an appropriate (co)homology theory is used. Otherwise we can restrict ourselves to G-CW complexes to simplify technical details. The standard reference for this is [**Bre1**, Chapter VII].

Let G be a compact Lie group and EG a free contractible G-space. Then G acts diagonally on $X \times EG$, and the Borel construction of X is defined as

$$X \times_G EG = (X \times EG)/G.$$

This is the associated bundle over $EG/G = BG$ with fiber X. In other words the projection $\pi \colon EG \to BG$ induces a fibration

$$\begin{array}{ccc} X & \to & X \times_G EG \\ & & \downarrow \\ & & BG \end{array}$$

with fiber X.

Therefore there is an associated spectral sequence

$$E_2^{p,q} = H^p(BG; H^q(X; R)) \Rightarrow H^{p+q}(X \times_G EG; R)$$

where $H^*(X; R)$ is twisted by the action of $\pi_1(BG) = G$, i.e.

$$H^p(BG; H^q(X; R)) = H^p(G; H^q(X; R)), \quad \text{the group cohomology,}$$

provided G is finite.

An important property of this gadget is the following theorem.

THEOREM 4.1. (SEE [**Bre 1**] FOR PROOF.) *Let $G \cong \mathbf{Z}/p$ act on X with fixed-point set F. Denote by $j\colon F \to X$ the natural inclusion. Suppose that $H^k(X; R) = 0$ for $k > N$. Then $j_G^*\colon H^k(X \times_G EG; R) \to H^k(F \times BG; R)$ is an isomorphism for $k > N$, j_G the map induced on the Borel construction by j.* □

This fundamental theorem allows us to relate F with the type of action on $H^*(X; R)$.

The following lemma is a direct consequence of Theorem 4.1.

LEMMA 4.2. *$G \cong \mathbf{Z}/p$, X a G-space with*

$$H^i(X; \mathbf{F}_p) \cong \begin{cases} \mathbf{F}_p, & i = N, \\ 0, & i > N. \end{cases}$$

If the generator $\mu \in H^N(X; \mathbf{F}_p)$ is in the image of i^*, where $i\colon X \to X \times_G EG$ is the fiber inclusion, then the action has fixed points.

PROOF. Recall that for $G \cong \mathbf{Z}/p$

$$H^*(BG; \mathbf{F}_p) = \begin{cases} P(x), & \dim x = 1,\ p = 2, \\ E(x) \otimes P(y), & \dim x = 1,\ \dim y = 2,\ p \,\text{odd}. \end{cases}$$

In both cases there is a nonzero element t in $H^1(BG; \mathbf{F}_p)$.

Now the fact that $\mu \in \operatorname{im} i^*$ is equivalent to $\mu \in E_\infty^{0,N}$, i.e. it is a permanent cocycle. Then $0 \neq t\mu$ is also a permanent cocycle and cannot be killed in the spectral sequence by our hypothesis on the fiber.

Therefore $E_\infty^{1,N} \neq 0$, and so $H^{N+1}(X \times_G EG; \mathbf{F}_p) \neq 0$. By Theorem 4.1, this implies $F \neq \varnothing$. □

This lemma will be used several times later on. If X has the cohomology of an orientable closed manifold, it can be stated as saying that if i^* is nonzero on the orientation class mod p, then $X^G \neq \varnothing$.

Let X be a paracompact G-space, $G \cong \mathbf{Z}/p$ (e.g. a G-CW complex). Then G acts on $\prod^p X$ via the permutation action. We recall a construction due to Steenrod [**S-E**] for cell-complexes and extended by Bredon [**Bre1**] to paracompact spaces.

PROPOSITION 4.3. *There exists a natural map (not a homomorphism)*

$$P\colon H^n(X; \mathbf{F}_p) \to H^{pn}\left(\left(\prod^p X\right) \times_G EG; \mathbf{F}_p\right)$$

for each n, satisfying the following properties:

(1) *If $k\colon \prod^p X \to (\prod^p X) \times_G EG$ then*

$$k^* p(a) = a \times \cdots \times a, \qquad a \in H^n(X; \mathbf{F}_p).$$

(2) *Let $\Delta\colon X \to \prod^p X$ be the inclusion of the fixed point set (the diagonal). Then*

$$\Delta_G^* P(a) = \sum_{i=0}^{n} P^i(a) \otimes \mu_{n-i} \in H^*(X; \mathbf{F}_p) \otimes H^*(BG; \mathbf{F}_p)$$

where $\mu_{n-i} \in H^{n-i}(BG; \mathbf{F}_p)$ is a basis element and P^i is used in the definition of the Steenrod operations. □

There is a G-equivariant map

$$h\colon X \to \prod^p X, \qquad x \mapsto (x, Tx, \ldots, T^{p-1}x),$$

T the action on X. The following lemma using h is also due to Bredon [**Bre2**].

LEMMA 4.4. *If $i\colon X \to X \times_G EG$, then*
(1) $i^* h_G^* P(a) = a \cup T^* a \cup \cdots \cup (T^{p-1})^* a$,
(2) $j_G^* h_G^* P(a) = \sum_{i=0}^n P^i(a|F) \otimes \mu_{n-i}$,
where $F = FIX\ SET$ of X, $j\colon F \to X$.

PROOF. (1) The diagram

$$\begin{array}{ccc} X & \xrightarrow{i} & X \times_G EG \\ h \downarrow & & \downarrow h_G \\ \prod^p X & \xrightarrow{k} & (\prod^p X) \times_G EG \end{array}$$

commutes. Therefore

$$i^* h_G^* P(a) = h^* k^* P(a) = h^*(a \times \cdots \times a) = a \cup T^* a \cup \cdots \cup (T^{p-1})^* a.$$

(2) In this case

$$\begin{array}{ccc} X \times_G EG & \xrightarrow{h_G} & (\prod^p X) \times_G EG \\ j_G \uparrow & & \uparrow \Delta_G \\ F \times BG & \xrightarrow{j \times 1} & X \times BG \end{array}$$

commutes. Therefore

$$j_G^* h_G^* p(a) = (j \times 1)^* \Delta_G^* p(a) = (j \times 1)^* \left(\sum_{i=0}^n P^i(a) \otimes \mu_{n-i} \right)$$
$$= \sum_{i=0}^n P^i(a|F) \otimes \mu_{n-i}.$$

The preceding machinery will now be applied to the case $X \sim (S^n)^k$, $G \cong \mathbf{Z}/p$ acting on X. Given a certain G-module structure on $H^n(X; \mathbf{Z})$, we will derive some cohomological results on F.

THEOREM 4.5. *Let $G \cong \mathbf{Z}/p$ act on the finitistic space $X \sim (S^n)^k$ such that $H^n(X; \mathbf{Z})$ is projective (of type $(0, s, 0)$) as a $\mathbf{Z}G$-module. Then $F \neq \varnothing$ and $F \sim_p (S^n)^s$.*

PROOF. As $H^n(X; \mathbf{Z})$ is of type $(0, s, 0)$, then $H^n(X; \mathbf{F}_p) \cong \bigoplus^s \mathbf{F}_p G$, free $\mathbf{F}_p G$-module.

For each summand, choose x_i so that $x_i, T^* x_i, \ldots, (T^{p-1})^* x_i$ are linearly independent, where $T\colon X \to X$ represents the action. This is possible because $1 \in \mathbf{F}_p G$ satisfies this (the identity element of G).

Consider the elements

$$\eta_i = x_i \cup T^* x_i \cup \cdots \cup T^{p-1^*} x_i \in H^{pn}(X; \mathbf{F}_p).$$

By Bredon's lemma (4.4(1)), each of them lies in $\operatorname{im} i^*$, $i\colon X \to X \times_G EG$.

We use the mod p spectral sequence associated to $X \times_G EG \to BG$, $E_2^{p,q} = H^p(G; H^q(X; \mathbf{F}_p))$.

The elements η_i originate from different summands, so they are linearly independent and in fact generate an exterior algebra $\wedge \subset \operatorname{im} i^* = E_\infty^{0,*}$. The product $\nu = \eta_1 \cdots \eta_s$ is a nonzero multiple of the orientation class, so by Lemma 4.2, $F \neq \varnothing$.

Choose $t \in H^2(BG; \mathbf{F}_p)$ such that $t^k \neq 0\ \forall k$. Then, as ν has highest fiber degree $0 \neq t^k\nu \in E_\infty^{2k,psn}$, i.e. it cannot be killed in the spectral sequence. The same must be true of $t^k\eta_i$ for each i. Otherwise suppose $t^k\eta_i = d(\xi)$ for some i.

Then

$$d\left((\xi)\left(\prod_{j\neq i}\eta_j\right)\right) = d(\xi)\left(\prod \eta_j\right) = \pm t^k\nu$$

(d is a derivation, the η_j are permanent cocycles). The elements $t^k\eta_i$ are represented by $t^k h_G^* P(x_i) \in H^{2k+pn}(X \times_G EG; \mathbf{F}_p)$.

Therefore, for sufficiently large k

$$j_G^*(t^k h_G^* P(x_i)) \neq 0,$$
$$t^k j_G^*(h_G^* P(x_i)) \neq 0 \quad \forall i, \text{ and by 4.4(2)},$$
$$x_i|_F \neq 0 \quad \forall i.$$

We will show that the $x_i|_F$ generate $H^*(F; \mathbf{F}_p)$ as an exterior algebra; first we bound its rank.

For k large

$$(*) \qquad H^k(X \times_G EG; \mathbf{F}_p) \cong H^k(F \times BG; \mathbf{F}_p) \cong \sum_{n=0}^{k} H^{k-n}(F; \mathbf{F}_p) \otimes H^n(BG; \mathbf{F}_p)$$

(Künneth formula).

By hypothesis, $H^n(X; \mathbf{F}_p) \cong \bigoplus^s \mathbf{F}_p G$,

$$\bigwedge^i (H^n(X; \mathbf{F}_p)) \cong \bigoplus_{j_1+\cdots+j_s=i} \bigwedge^{j_1}(\mathbf{F}_p) \otimes \cdots \otimes \bigwedge^{j_s}(\mathbf{F}_p G).$$

If for some j_r, $p \nmid j_r$, then the summand is G-cohomologically trivial. If $i = pl$, then up to cohomology the right side is

$$\bigoplus_{\substack{j_1+\cdots+j_s=pl \\ p|j_r,\ \forall r}} \mathbf{F}_p, \qquad \mathbf{F}_p \text{ trivial } G\text{-module.}$$

From this it follows that

$$H^k(G; H^{ni}(X; \mathbf{F}_p)) = \begin{cases} 0, & \text{if } p \nmid i, \\ H^k(G; \bigoplus^{\binom{s}{l}} \mathbf{F}_p), & i = pl. \end{cases}$$

From $(*)$

$$\mathrm{rk}_p H^*(F; \mathbf{F}_p) = \sum_{p+q=k} \mathrm{rk}_p E_\infty^{p,q} \leq \sum_{p+q=k} \mathrm{rk}_p E_2^{p,q}.$$

We conclude that

$$\mathrm{rk}_p H^*(F; \mathbf{F}_p) \leq \sum_{l=0}^{s} \binom{s}{l} = 2^s.$$

Take $t \in H^2(BG; \mathbf{F}_p)$ as before; the $t, \eta_1, \ldots, \eta_s$ generate a subalgebra in E_∞^{**} of the form $P(t) \otimes \bigwedge(\eta_1, \ldots, \eta_s)$. Hence for sufficiently large k they generate a subspace of rank 2^s in $\sum_{p+q=k} E_\infty^{p,q}$, and so all of it.

Suppose some element

$$Z|_F = \sum a_{i_1\cdots i_r}(x_{i_1}|_F)\cdots(x_{i_r}|_F) = 0.$$

Then

$$(1\otimes t^k)\left(\sum_{j=0}^{rn} P^j(Z|_F)\otimes \mu_{rn-j}\right) = 0 \quad \forall k,\ t^k \text{ as before}$$
$$\Rightarrow t^k \cdot j_G^* h_G^* P(Z) = 0.$$

For k large enough this implies $t^k h_G^* P(Z) = 0$. But this element represents

$$t^k\left(\sum a_{i_1\cdots i_r}\eta_{i_1}\cdots\eta_{i_r}\right) \Rightarrow \sum a_{i_1}\cdots a_{i_r}\eta_{i_1}\cdots\eta_{i_r} = 0.$$

Therefore $\{x_i|_F\}$ generate an exterior algebra in $H^*(F;\mathbf{F}_p)$ of rank 2^s.

$$\Rightarrow \langle\{x_i|_F\}\rangle = H^*(F;\mathbf{F}_p)$$

and so $F \sim_p (S^n)^s$. $\square$

THEOREM 4.6. *Let $G \cong \mathbf{Z}/p$ act on $X \sim (S^n)^{(p-1)r}$, n odd, and suppose that $H^n(X;\mathbf{Z})$ is of type $(r,0,0)$ as a G-module. Then $F \neq \emptyset$, $H^*(F;\mathbf{Z}_{(p)})$ is torsion free, zero in odd dimensions, and of rank p^r if reduced* $\operatorname{mod} p$.

PROOF. By Proposition 1.10

$$(*)\quad H^k\left(G;\bigwedge^i(M)\right) \cong \sum_{j_1+\cdots+j_n=i} H^{k+i}(G;\mathbf{Z}), \qquad k>0,\ 0\le j_q\le p-1.$$

Let $E_r^{p,q}$ be the integral spectral sequence associated to the Borel construction. The possible nonzero differentials are of the form $E_{ln+1}^{k,jn} \xrightarrow{d} E_{ln+1}^{k+ln+1,(j-l)n}$.

Then, by $(*)$, $E_2^{k,tn} = 0$ if $k+t$ is odd, and as n is odd this is equivalent to $k+tn$ odd. Therefore the above differentials are all 0 if $k>0$.

Choose k sufficiently large so that $E^{k,jn}$ cannot be killed by any element on the vertical edge of the spectral sequence. Then by the preceding observation, $E_2^{k,jn} = E_\infty^{k,jn}$. In other words the spectral sequence degenerates in large total degree.

Fix N large: the E_∞^N term is an $\mathbf{F}_p$-vector space and

$$\sum_{p+q=N} E_\infty^{p,q} = \sum_{0\le t\le r(p-1)} E_2^{N-tn,tn}$$
$$\cong \sum_{0\le t\le r(p-1)}\left(\sum_{\substack{j_1+\cdots+j_r=t\\ 0\le j_q\le p-1}} H^N(G;\mathbf{Z})\right).$$

(Recall n is odd.) Therefore

$$\operatorname{rk}_p E_\infty^N = \begin{cases} 0, & N \text{ odd},\\ \sum_{t=0}^{r(p-1)} P_r(t), & N \text{ even}.\end{cases}$$

$P_r(t)$ denotes the number of partitions of t by r integers between (and including) 0 and $p-1$. By induction, it can be shown that

$$\sum_{t=0}^{r(p-1)} p_r(t) = p^r.$$

Thus we have obtained

$$\mathrm{rk}_p E_\infty^N = \begin{cases} 0, & N \text{ odd}, \\ p^r, & N \text{ even}. \end{cases}$$

For N large, p annihilates $H^N(X \times_G EG; \mathbf{Z})$ as this is a finitely generated module over $H^*(BG; \mathbf{Z})$. Therefore, using Theorem 4.4(2)

$$\mathrm{rk}_p H^N(F \times BG; \mathbf{Z}) = \begin{cases} 0, & N \text{ odd}, \\ p^r, & N \text{ even}. \end{cases}$$

$H^*(BG; \mathbf{Z}) = P(x)$, a polynomial algebra over $\mathbf{F}_p$ on one generator in dimension 2. By the Künneth formula, $H^*(F; \mathbf{Z}_{(p)}) = 0$ for k odd and $H^*(F; \mathbf{Z}_{(p)})$ has no p-torsion (otherwise two consecutive nonzero terms would appear in $H^*(F \times BG; \mathbf{Z})$).

Finally if we reduce mod p

$$\mathrm{rk}_p H^N(F \times BG; \mathbf{F}_p) = p^r \quad \text{for any } N \text{ sufficiently large}$$

and therefore $\mathrm{rk}_p H^*(F; \mathbf{F}_p) = p^r$. □

Note. If p is odd, n must be odd by Proposition 2.5. For $p = 2$, $\mathbf{Z}/2$ acts freely on S^{2n} changing orientation, i.e. $H^{2n}(X) \cong A$ where A is the twisted action on $\mathbf{Z}$, $1 \to -1$ (Example 3, §2).

The case $(r, s, 0)$ is not as complete as the preceding ones: the structure of $H^*(F; \mathbf{F}_p)$ is not easy to determine.

THEOREM 4.7. *Let $G \cong \mathbf{Z}/p$, p odd, act on $X \sim (S^n)^{(p-1)r+ps}$ and suppose that $H^n(X; \mathbf{Z})$ is of type $(r, s, 0)$ as a G-module. Then $F \neq 0$, $\mathrm{rk}_p H^*(F; \mathbf{F}_p) \geq 2^{r+s}$ and $H^*(F; \mathbf{F}_p)$ contains an exterior algebra on s n-dimensional generators.*

PROOF. Let

$$H^n(X; \mathbf{Z}) \cong_G \left(\bigoplus_1^r A_i \right) \oplus \left(\bigoplus_1^s P_j \right).$$

Then

$$\bigwedge^{(p-1)r+ps} (H^n(X; \mathbf{Z})) \cong_G \left(\bigwedge^{p-1}(A_1) \otimes \cdots \otimes \bigwedge^{p-1}(A_r) \right) \otimes \left(\bigwedge^{p}(P_1) \otimes \cdots \otimes \bigwedge^{p}(P_s) \right).$$

As p is odd, $\bigwedge^{p-1}(A_i) \cong \mathbf{Z} \cong \bigwedge^p(P_j)$, trivial G-modules. Let ν_i generate $\bigwedge^{p-1}(A_i)$.

Reducing mod p, the analogous isomorphism holds. Choose $\eta_i \in H^{pn}(X; \mathbf{F}_p)$ as in Theorem 4.5: i.e. they generate $\bigwedge^p(P_i) \otimes \mathbf{F}_p$ and are in the image of i^*, $i \colon X \hookrightarrow X \times_G EG$. Then $(\overline{\nu}_1 \cdots \overline{\nu}_r)(\mu_1 \cdots \mu_s)$ is a nonzero multiple of the orientation class mod p, and $\mu_1 \cdots \mu_s \in \operatorname{im} i^*$, $\overline{\nu}_i$ the reduction of ν_i. Therefore if we can show $\nu_i \in \operatorname{im} i^*$ integrally, then $\overline{\nu}_i \in \operatorname{im} i^*$ (mod p) and so by Lemma 4.2 $F \neq \varnothing$.

Consider the spectral sequence associated to $X \times_G EG$ with integral coefficients: $E_2^{p,q} = H^p(G; H^q(S; \mathbf{Z}))$. Then $\nu_i \in E_2^{0,(p-1)n}$.

Claim. ν_i is a permanent cocycle.

Look at the differentials $E_{jn+1}^{0,(p-1)n} \to E_{jn+1}^{jn+1,(p-j-1)n}$. These are the only differentials which can possibly involve the ν_i.

$$H^{jn+1}(G; H^{(p-j-1)n}(X;\mathbf{Z})) \cong H^{jn+1}\left(G; \bigwedge^{p-j-1}\left(\left(\bigoplus A_i\right) \oplus \left(\bigoplus P_j\right)\right)\right)$$

$$\cong H^{jn+1}\left(G; \bigwedge^{p-j-1}\left(\bigoplus A_i\right)\right), \quad P_j \text{ are projective and } p-j-1 < p,$$

$$\cong \sum H^{jn+1+p-j-1}(G;\mathbf{Z}) \quad \text{(Proposition 1.10)}.$$

However if p is odd, n must be odd (Proposition 2.5). So $j(n-1)+p$ is odd, hence

$$H^{j(n-1)+p}(G;\mathbf{Z}) = 0$$
$$\Rightarrow E_{jn+1}^{jn+1,(p-j-1)n} = 0 \quad \forall j$$
$$\Rightarrow \nu_i \text{ are permanent cocycles, hence } \nu_i \in E_\infty^{0,(p-1)n} = \operatorname{im} i^*.$$

We have proved $F \neq \emptyset$.

The $\overline{\nu}_i, \eta_j$ generate a free $H^*(BG;\mathbf{F}_p)$ module of rank 2^{r+s}. This is verified as in the proof of 4.5. Also as in 4.5, if $\eta_j = a_j \cup T^* a_j \cup \cdots \cup T^{p-1*} a_j$, then the $\{a_j|_F\}$ generate an exterior algebra in $H^*(F;\mathbf{F}_p)$. □

Note. The case $p=2$ can be approached differently. *Assume* that $F \neq \emptyset$; then the $\overline{\nu}_i$ are transgressive, and must transgress to 0, i.e. they are permanent cocycles. Then we proceed as above; but by rank considerations $\operatorname{rk}_p H^*(F;\mathbf{F}_p) = 2^{r+s}$ and in fact the spectral sequence degenerates. This result is due to Bredon [**Bre2**, p. 273].

Given the classification Theorem 1.4, Theorem 4.7 has the following corollary.

COROLLARY 4.8. *Let $G \cong \mathbf{Z}/p$ act freely on $X \sim (S^n)^k$, p odd. Then $H^n(X;\mathbf{Z})$ has a trivial direct summand as a G-module.* □

The cup product on $H^*(X;\mathbf{Z})$ restricts its G-module structure. If we only require that X have the homology of $(S^n)^k$, then 4.8 is false. Counterexamples can easily be constructed without cohomology products.

Many of the cohomological techniques available for $\mathbf{Z}/p$ can be extended to $(\mathbf{Z}/p)^l$, and one may expect a version of 4.8. The following example is helpful in formulating a plausible extension of it.

EXAMPLE 4.9. $G = \mathbf{Z}/p \times \mathbf{Z}/p$. Let S, T be generators of G. Denote by $\rho\colon S^{2n-1} \to S^{2n-1}$ the free $\mathbf{Z}/p$-action on S^{2n-1}, $\overline{x} \to e^{2\pi i/p}\overline{x}$, and by $P\colon (S^{2n-1})^p \to (S^{2n-1})^p$ the permutation action $P(x_1, \ldots, x_p) = (x_2, x_3, \ldots, x_p, x_1)$.

We define a G-action on $(S^{2n-1})^{p+1}$ as follows.

$$S(x_1, \ldots, x_{p+1}) = (\rho x_1, \ldots, \rho x_p, x_{p+1}),$$
$$T(x_1, \ldots, x_{p+1}) = (P(x_1, \ldots, x_p), \rho x_{p+1}).$$

S, T commute and define an effective G-action on $(S^{2n-1})^{p+1}$. Suppose $\bar{z}$ is fixed under an element in G, i.e.

$$\begin{aligned} &S^iT^j(z_1, \ldots, z_{p+1}) = (z_1, \ldots, z_{p+1}) \\ &\quad \Rightarrow (P^j(\rho^i z_1, \ldots, \rho^i z_p), \rho^j z_{p+1}) = (z_1, \ldots, z_{p+1}), \\ &\rho^j x_{p+1} = x_{p+1} \Rightarrow j \equiv 0 \mod p. \end{aligned}$$

Hence $P^j = 1$ and so $\rho^i z_k = z_k$ and hence $i \equiv 0 \bmod p \Rightarrow S^iT^j = 1$ and the action is free.

As a $\mathbf{Z}[\langle S \rangle]$-module, $H^n((S^{2n-1})^{p+1}; \mathbf{Z}) \cong (\mathbf{Z})^{p+1}$, a trivial module of rank $p+1$. As a module over $\mathbf{Z}[\langle T \rangle]$, $H^n((S^{2n-1})^{p+1}; \mathbf{Z}) \cong \mathbf{Z}[\langle T \rangle] \oplus \mathbf{Z}$.

Clearly then, the n-dimensional cohomology module does not split off a trivial $\mathbf{Z}G$-module of rank 2, but nevertheless

$$\mathrm{rk}_{\mathbf{Z}} H^n((S^{2n-1})^{p+1}; \mathbf{Z})^G = 2.$$

This leads us to the following conjecture for p odd (if $p = 2$, use $\mathbf{F}_2$):

CONJECTURE. Let $G = (\mathbf{Z}/p)^l$ act freely on $X \sim (S^n)^k$; then

$$l \leq \mathrm{rk}(H^n(X; \mathbf{Z}))^G. \quad \square$$

This conjecture is true when the action is trivial in homology (see [**Ca**, **Bro**]), but the result can be proved in that case without the cup product structure.

Another approach is to restrict a $(\mathbf{Z}/p)^l$-action to its cyclic subgroups. In this way we can show that many modules can only be realized on $H^n((S^n)^k; \mathbf{Z})$ under certain conditions. For example:

PROPOSITION 4.10. *Let $G = (\mathbf{Z}/p)^l$ act on $X \sim (S^n)^k$ such that $H^n(X; \mathbf{Z})$ is* (1) *projective or* (2) *cohomologous to $\Omega^{2m+1}(\mathbf{Z})$. Then every cyclic subgroup has a fixed point.*

PROOF. If $H^n(X; \mathbf{Z})$ is $\mathbf{Z}G$-projective, then $H^n(X; \mathbf{F}_p)$ is $\mathbf{F}_p[G]$-free. Therefore it is free restricted to each cyclic subgroup, and so they must have fixed points (4.5).

By the definition of $\Omega^{2m+1}(\mathbf{Z})$, in cohomology we have

$$\hat{H}^*(H; \Omega^{2m+1}(\mathbf{Z})) \cong \hat{H}^{*-2m-1}(H; \mathbf{Z})$$

for every subgroup $H \subseteq G$. In particular for $H \cong \mathbf{Z}/p$

$$\hat{H}^i(H; \Omega^{2m+1}(\mathbf{Z})) \cong \hat{H}^{i-2m-1}(\mathbf{Z}/p; \mathbf{Z}) = \begin{cases} \mathbf{Z}/p, & i \text{ odd}, \\ 0, & i \text{ even}, \end{cases}$$

because $2m+1$ is odd. Hence $\Omega^{2m+1}(\mathbf{Z})$ is of type $(1, s, 0)$ and H has fixed points (4.7). $\square$

To conclude this paper, we give a topological application of 4.8.

THEOREM 4.11. *Let X be a CW-complex dominated by a finite m-dimensional complex such that*

(1) $\Pi_1(X) \cong \mathbf{Z}/p$, *$p$ odd,*

(2) $\tilde{X} \sim (S^n)^k$, *$n > 1$, $m = kn$.*

$\tilde{X}$ universal cover of X.

Then X is equivalent to a finite complex of dimension $\max(3, m)$.

PROOF. The proof requires computing the finiteness obstruction

$$\mathcal{O}(X) \in \tilde{K}_0(\mathbf{Z}[\mathbf{Z}/p]).$$

The reference for this is Wall's paper [**W**].

We recall how it is defined. Let $C_i = C_i(\tilde{X})$; then these are free $\mathbf{Z}[\Pi_1(X)] = \mathbf{Z}[\mathbf{Z}/p]$-modules. The short exact sequence

$$0 \to \partial C_{m+1} \to C_m \to C_m/\partial C_{m+1} \to 0$$

splits over $\mathbf{Z}[\mathbf{Z}/p]$ by our hypothesis, and by definition

$$\mathcal{O}(X) = [C_m/\partial C_{m+1}] \in \tilde{K}_0(\mathbf{Z}[\mathbf{Z}/p]).$$

From the chain complex $\mathcal{O}(X) \to C_{m-1} \to \cdots \to C_0$ we can compute $\mathcal{O}(X)$ as an element in $G_0(\mathbf{Z}[\mathbf{Z}/p])$ (see [**SW**] for the definition).

The Euler characteristic formula implies

$$(*) \qquad \sum_{i=0}^{m}(-1)^i[H_i(\tilde{X})] = (-1)^m[\mathcal{O}(X)] + \sum_{i=0}^{m-1}(-1)^i[C_i]$$

in $G_0(\mathbf{Z}[\mathbf{Z}/p])$.

By Poincaré duality

$$H^{m-i}(\tilde{X};\mathbf{Z}) \cong H_i(\tilde{X};\mathbf{Z}) \quad \text{as } \mathbf{Z}/p\text{-modules.}$$

Therefore

$$\begin{aligned}\sum_{i=0}^{m}(-1)^i[H_i(\tilde{X})] &= \sum_{i=0}^{m}(-1)^i[H^{m-i}(\tilde{X})] \\ &= \sum_{i=0}^{m}(-1)^{2i-m}(-1)^{m-i}[H^{m-i}(\tilde{X})] \\ &= (-1)^m\left(\sum_{i=0}^{k}(-1)^i\left[\bigwedge^i(H^n(\tilde{X}))\right]\right) \qquad (n \text{ is odd}).\end{aligned}$$

We define the following element in $G_0(\mathbf{Z}[\mathbf{Z}/p]))$.

$$L([M]) = \sum_{i=0}^{\mathrm{rk}\, M}(-1)^i\left[\bigwedge_{\mathbf{Z}}^i(M)\right] \qquad (M \text{ torsion-free});$$

using the isomorphism $\bigwedge^*(M \oplus N) \cong \bigwedge^*(M) \otimes \bigwedge^*(N)$ it is not hard to show

$$L([M \oplus N]) = L([M]) \cdot L([N]) \quad \text{in } G_0(\mathbf{Z}[\mathbf{Z}/p]).$$

[In fact L is well defined as a function on $G_0^{\mathbf{Z}}(\mathbf{Z}[\mathbf{Z}/p])$.]

Now $\Pi_1(X) \cong \mathbf{Z}/p$ acts freely on $\tilde{X}$; hence by Corollary 4.8

$$H^n(\tilde{X};\mathbf{Z}) \cong M \oplus \mathbf{Z} \quad \text{as } \mathbf{Z}[\mathbf{Z}/p]\text{-modules.}$$

Trivially $L([\mathbf{Z}]) = 0$, so $L([H^n(\tilde{X})]) = 0$.

From $(*)$ we conclude

$$\mathcal{O}(X) = (-1)^{m+1}\left(\sum_{i=0}^{m-1}(-1)^i[C_i]\right)$$

as elements in $G_0(\mathbf{Z}[\mathbf{Z}/p])$. However, for $G = \mathbf{Z}/p$, the Cartan map $K_0(\mathbf{Z}[\mathbf{Z}/p]) \to G_0(\mathbf{Z}[\mathbf{Z}/p])$ is a monomorphism (see [**Sw**])

$$\Rightarrow \mathcal{O}(X) = (-1)^{m+1}\left(\sum_{i=0}^{m-1}(-1)^i[C_i]\right)$$

in $K_0(\mathbf{Z}[\mathbf{Z}/p])$; but the C_i are free, hence in the reduced group $\tilde{K}_0(\mathbf{Z}[\mathbf{Z}/p])$, $\mathcal{O}(X) = 0$.

Wall's theorem implies the result. □

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