ON THE EXPONENT OF COHOMOLOGY OF DISCRETE GROUPS

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Introduction

Let Γ be a discrete group of finite virtual cohomological dimension (v.c.d.). This means that it contains a (normal) subgroup Γ' of finite index and of finite cohomological dimension. The Euler Characteristic of Γ is the well-defined rational number

$$\chi(\Gamma) = \chi(\Gamma')/[\Gamma:\Gamma'].$$

In [5], K. Brown proved that if m is the least common multiple of the orders of finite subgroups of Γ, then $m\cdot\chi(\Gamma)\in\mathbf{Z}$. Motivated by this and the fact that the Farrell cohomology of Γ, $\hat{H}^*(\Gamma,\mathbf{Z})$, depends on the finite subgroups of Γ, Brown [6] raised the following question.

QUESTION. Does the integer m annihilate $\hat{H}^*(\Gamma,\mathbf{Z})$?

In this note we will construct a counterexample to this question, by observing that if $G = \Gamma/\Gamma'$ and X is a contractible, finite dimensional, proper Γ-complex, then $\hat{H}^*(\Gamma,\mathbf{Z})$ is the G-equivariant Tate cohomology of X/Γ'. We use this more familiar description to prove:

THEOREM 2.1. *There is a group Γ of v.c.d. 3 with no subgroups of order p^2 and with exponent p^2 occurring infinitely often in $\hat{H}^*(\Gamma,\mathbf{Z})$.*

In sufficiently high dimensions, Farrell cohomology coincides with ordinary cohomology. The torsion occurring asymptotically often arises from the finite subgroups in a simple way, but our example shows that a different type of cohomological twisting can appear, linked to the geometry of the singular set of the G-complex X/Γ'.

We obtain a general bound on the exponent of $\hat{H}^*(\Gamma,\mathbf{Z})$ by combining the description given previously with a recent result on torsion in equivariant cohomology:

THEOREM 1.5. *Under the above conditions,*

$$\exp \hat{H}^*(\Gamma,\mathbf{Z}) \mid \prod_{i=1}^{r(\Gamma)} \exp y_i$$

where $r(\Gamma) = \max\{p\text{-rank}\,\Gamma_x \mid \Gamma_x$ is a finite subgroup of $\Gamma\}$, p ranges over all prime divisors of $[\Gamma:\Gamma']$ and $y_1,\ldots,y_{r(\Gamma)}$ are cohomology classes in $H^*(G,\mathbf{Z})$.

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This result shows that the exponent will depend on how the finite subgroups of Γ relate *cohomologically* to G. Pursuing this approach further we obtain a sufficient condition for the question to have a positive response, namely that G is elementary abelian.

We also provide a homotopy-theoretic definition of Farrell homology. This involves using a transfer map, which is well defined because $[\Gamma:\Gamma']$ is *finite*. We measure the difference between ordinary homology and Farrell homology in terms of the homotopy groups of certain 'homotopy fixed-points'. For virtual duality groups this can be expressed homologically by using the dualizing module.

1. *Farrell (co)homology*

We recall the definition of a complete resolution and of Farrell cohomology.

DEFINITION 1.1. (1) A *complete resolution* for Γ is an acyclic chain complex F of projective $\mathbf{Z}\Gamma$-modules together with an ordinary projective resolution $\varepsilon: P \to \mathbf{Z}$ such that F and P coincide in sufficiently high dimensions.

(2) The *Farrell cohomology* of Γ is defined as $\hat{H}^*(\Gamma, \mathbf{Z}) \cong H^*(\mathrm{Hom}_\Gamma(F, \mathbf{Z}))$.

Given Γ of finite virtual cohomological dimension, we can choose a torsion-free normal subgroup Γ' in Γ; let $G = \Gamma/\Gamma'$. Now let X be a finite dimensional, contractible, proper Γ-complex. Proper means that the isotropy subgroups are all *finite*. The existence of X follows from work of Serre [**10**]. In addition, X can be constructed so that X^H is contractible if and only if H is a finite subgroup of Γ. The projection $\Gamma \to G$ maps the finite subgroups Γ_x of Γ isomorphically onto isotropy subgroups of G, with respect to the action on X/Γ'. Now let Γ act on the universal free G-space EG through this projection. It is easy to verify that the induced diagonal action on $X \times EG$ is free, and hence that we have a homotopy equivalence $B\Gamma \simeq X/\Gamma' \times_G EG$. The fact that X/Γ' is a finite dimensional space allows us to extract meaningful information about this Borel construction by using methods from equivariant cohomology; the geometry of the singular set plays a crucial role in this. Note that the G-Tate cohomology of X/Γ' is also well-defined; the following lemma identifies it.

LEMMA 1.2. $\hat{H}(\Gamma, \mathbf{Z}) \cong \hat{H}^*_G(X/\Gamma')$.

Proof. Let F_* be a complete resolution for G, and let Γ act on it through the quotient map. Then, if $C_*(X)$ is the integral cellular chain complex of X, $C_*(X) \otimes F_*$ is a complete Γ-resolution. The result now follows by using this to compute $\hat{H}^*(\Gamma, \mathbf{Z})$.

Next we will describe a homotopy-theoretic definition of Farrell homology, using a transfer map. As an application of it we will describe the difference between $\hat{H}_*(\Gamma, \mathbf{Z})$ and $H_*(\Gamma, \mathbf{Z})$ in terms of the homotopy groups of a certain homotopy fixed-point set. In case Γ is a virtual duality group, this is a homological invariant of the dualizing module.

Take a simplicial model of X and of EG. Given a simplicial complex Y, denote by $\mathbf{Z}(Y)$ the free abelian functor applied to Y; its n-simplices are the free abelian group on the n-simplices in Y. The important fact is that $\pi_*(|\mathbf{Z}(Y)|) \cong H_*(|Y|, \mathbf{Z})$, where the vertical bars denote realization. We have a well-defined transfer map

$$|\mathbf{Z}(X/\Gamma' \times_G EG)| \longrightarrow |\mathbf{Z}(X/\Gamma')|^G \longrightarrow \mathrm{Map}_G(|EG|, |\mathbf{Z}(X/\Gamma')|).$$

We are using the fact that the transfer falls in the fixed-point set, and then pushing into the 'homotopy fixed-points' by using the map $|EG| \to *$. The term on the right will be abbreviated as $|\mathbf{Z}(X/\Gamma')|^{hG}$. In [2] we proved that the homotopy groups of the fibre of this composition are $\hat{H}^G_*(X/\Gamma')$ for $* \geqslant 0$. Combining this with Lemma 1.2 yields the following.

THEOREM 1.3. *Let $\mathfrak{F}(\Gamma)$ denote the fibre of the transfer map defined above. Then for all $* \geqslant 0$,*

$$\pi_*(\mathfrak{F}(\Gamma)) \cong \hat{H}_*(\Gamma, \mathbf{Z}).$$

Let $n = \text{v.c.d.}\,\Gamma$, then by 1.3 we have a long exact sequence of homotopy groups, which become

$$0 \longrightarrow \hat{H}_n(\Gamma, \mathbf{Z}) \longrightarrow H_n(\Gamma, \mathbf{Z}) \longrightarrow \pi_n(|\mathbf{Z}(X/\Gamma')|^{hG}) \longrightarrow \hat{H}_{n-1}(\Gamma, \mathbf{Z}) \longrightarrow \cdots$$
$$\cdots \longrightarrow H_0(\Gamma, \mathbf{Z}) \longrightarrow \pi_0(|\mathbf{Z}(X/\Gamma')|^{hG} \longrightarrow \hat{H}_{-1}(\Gamma, \mathbf{Z}) \longrightarrow 0.$$

(Compare this with [7, p. 280].)

Assume now that Γ is a virtual duality group, that is, that we can take Γ' to be a duality group in the sense of Bieri and Eckmann [3]. The dualizing module is $D = H^n(\Gamma, \mathbf{Z}\Gamma)$. If D is $\mathbf{Z}$-free, Farrell [8] showed that there is a long exact sequence involving $\hat{H}_*(\Gamma, \mathbf{Z}), H_*(\Gamma, \mathbf{Z})$ and H^{n-*} $(\Gamma, \mathrm{Hom}\,(D, \mathbf{Z}))$. Comparing it to the one above and doing the necessary homological identifications leads to the following theorem.

THEOREM 1.4. *Let Γ be a virtual duality group of virtual cohomological dimension n with a $\mathbf{Z}$-free dualizing module D. Then for all $* \geqslant 0$*

$$\pi_*(|\mathbf{Z}(X/\Gamma')|^{hG}) \cong H^{n-*}(\Gamma, \mathrm{Hom}\,(D, \mathbf{Z})).$$

We can use our description of Farrell cohomology to show it is independent of the choice of Γ'. Indeed, if Γ'' is another torsion-free normal subgroup in Γ, then $\Gamma' \cap \Gamma''$ is normal of finite index in both Γ and Γ', and the quotient group $\Gamma'/\Gamma' \cap \Gamma''$ acts *freely* on $X/\Gamma' \cap \Gamma''$. From this it follows that

$$\hat{H}^*_{\Gamma/\Gamma'}(X/\Gamma') \cong \hat{H}^*_{\Gamma/\Gamma' \cap \Gamma''}(X/\Gamma' \cap \Gamma'').$$

Other properties of Farrell cohomology can be proved quite directly using this approach. In particular the facts that periodicity of the group and cohomological triviality of a module are determined on finite subgroups are easy consequences of 1.2.

Now factorizing the map of a G-orbit to a point through X/Γ' induces a commutative diagram:

$$\begin{array}{ccc} \hat{H}^*(G, \mathbf{Z}) & \longrightarrow & \hat{H}^*(\Gamma, \mathbf{Z}) \\ & \searrow & \downarrow \\ & & \hat{H}^*(\Gamma_x, \mathbf{Z}) \end{array}$$

from which we conclude that the integer $m = \mathrm{lcm}\,\{|\Gamma_x|\}$ must divide the exponent of $\hat{H}^*(\Gamma, \mathbf{Z})$. However, equality does not hold in general, as our counterexample will show. First we establish a general bound on the exponent of $\hat{H}^*(\Gamma, \mathbf{Z})$. If y is a single cohomology class, we denote its order by $\exp y$. Similarly if A is a finitely generated abelian group, $\exp A$ is the smallest positive integer which annihilates all its elements.

THEOREM 1.5. *Let Γ be a group of finite virtual cohomological dimension containing a normal, torsion-free subgroup Γ'. Then, if $G = \Gamma/\Gamma'$,*

$$\exp \hat{H}^*(\Gamma, \mathbf{Z}) \,\Big|\, \prod_{i=1}^{r(\Gamma)} \exp y_i$$

where $r(\Gamma) = \max\{p\text{-rank } of\ \Gamma_x \mid \Gamma_x \text{ is a finite subgroup of } \Gamma\}$, p ranges over all prime divisors of $[\Gamma:\Gamma']$, and $y_1, \ldots, y_{r(\Gamma)} \in H^*(G, \mathbf{Z})$.

Proof. We use the same notation as before; then $\hat{H}^*(\Gamma, \mathbf{Z})$ can be identified with $\hat{H}^*_G(X/\Gamma')$. Note that the set of distinct finite subgroups in Γ coincides with the set of distinct isotropy subgroups in G. Let $y \in H^k(G, \mathbf{Z})$ and denote by $\Omega^k(\mathbf{Z})$ the k-fold dimension-shift of $\mathbf{Z}$, the trivial $\mathbf{Z}G$-module. It is not hard to see that y can be represented by the map on the right of a short exact sequence

$$0 \longrightarrow M \longrightarrow \Omega^k(\mathbf{Z}) \xrightarrow{\ \rho\ } \mathbf{Z} \longrightarrow 0.$$

Now the fundamental result in [1] is that if Y is a finite dimensional G-CW complex, there exist modules $M_1, \ldots, M_{r(Y)}$ chosen as above, such that

$$C^*(Y) \otimes M_1 \otimes \ldots \otimes M_{r(Y)}$$

is a projective G-cochain complex. Here $r(Y) = \max\{p\text{-rank of } G_x\}$, as p ranges over all prime divisors of $|G|$, and G_x over all isotropy subgroups. Applying this to $Y = X/\Gamma'$ and noticing that $r(Y) = r(\Gamma)$, we can choose $M_1, \ldots, M_{r(\Gamma)}$ such that $C^*(X/\Gamma') \otimes M_1 \otimes \ldots \otimes M_{r(\Gamma)}$ is projective. Letting Γ act through G, this means that $M_1 \otimes \ldots \otimes M_{r(\Gamma)}$ is a cohomologically trivial Γ-module.

Let ρ_i denote the map associated to M_i and $y_i \in H^{s_i}(G, \mathbf{Z})$ the class it represents. Successively tensoring the short exact sequences with $C^*(X/\Gamma')$ yields (at the ith stage):

$$0 \longrightarrow C^*(X/\Gamma') \otimes M_1 \otimes \ldots \otimes M_i \longrightarrow C^*(X/\Gamma') \otimes M_1 \otimes \ldots \otimes M_{i-1} \otimes \Omega^{s_i}(\mathbf{Z}) \longrightarrow C^*(X/\Gamma') \otimes M_1 \otimes \ldots \otimes M_{i-1} \longrightarrow 0.$$

Applying $\hat{H}^*(G, -\!-)$ to this, using 1.2 and letting Γ act through G, we obtain long exact sequences in cohomology:

$$\ldots \longrightarrow \hat{H}^{j-1}(\Gamma, M_1 \otimes \ldots \otimes M_{i-1}) \longrightarrow \hat{H}^j(\Gamma, M_1 \otimes \ldots \otimes M_i) \longrightarrow \hat{H}^{j-s_i}(\Gamma, M_1 \otimes \ldots \otimes M_{i-1}) \xrightarrow{\ \cup y_i\ } \hat{H}^j(\Gamma, M_1 \otimes \ldots \otimes M_{i-1}) \longrightarrow \ldots$$

The key ingredient is that the maps ρ_i induce cup product by the class y_i in $H^{s_i}(G, \mathbf{Z})$. Doing this successively we obtain divisibility relations:

$$\exp \hat{H}^*(\Gamma, M_1 \otimes \ldots \otimes M_{i-1}) / \exp \hat{H}^*(\Gamma, M_1 \otimes \ldots \otimes M_i) \mid \exp y_i.$$

Multiplying these out for all values of i between 1 and $r(\Gamma)$ and recalling that the Γ-module $M_1 \otimes \ldots \otimes M_{r(\Gamma)}$ is cohomologically trivial completes the proof. We refer to [1] for complete details about this technique, which is based on methods from complexity theory.

As we shall see from the counterexample, this divisibility relation cannot in general be sharpened to an equality. However, in the special case when G is p-elementary abelian it does hold (see [**1**]), from which we derive a sufficient condition for the original question to have a positive response.

COROLLARY 1.6. *Let Γ be a group of finite virtual cohomological dimension. If Γ' is a torsion-free normal subgroup such that $G = \Gamma/\Gamma'$ is elementary abelian, then the exponent of $\hat{H}^*(\Gamma, \mathbf{Z})$ is the least common multiple of the orders of its finite subgroups.*

2. *The counterexample*

The following counterexample was motivated by a construction due to W. Browder in [**4**].

THEOREM 2.1. *There is a group Γ of v.c.d. 3 with no subgroups of order p^2 and with exponent p^2 occurring infinitely often in $\hat{H}^*(\Gamma, \mathbf{Z})$.*

Proof. Let $M = M(r, s, t)$ be the closed 3-manifold obtained by intersecting the complex algebraic surface $z_1^r + z_2^s + z_3^t = 0$ with the unit sphere $|z_1|^2 + |z_2|^2 + |z_3|^2 = 1$. These are known as Brieskorn manifolds, and they are known to be integral homology spheres if and only if r, s, t are pairwise relatively prime. Milnor [**9**] proved that if $r^{-1} + s^{-1} + t^{-1} < 1$ (the hyperbolic case) then M is diffeomorphic to a coset space $\Gamma\backslash\widetilde{SL_2(\mathbf{R})}$, where $\widetilde{SL_2(\mathbf{R})}$ is the universal covering group of $SL_2(\mathbf{R})$ and Γ is a discrete subgroup.

For any prime p the map

$$\rho(z_1, z_2, z_3) = (e^{2\pi sti/p} z_1, e^{2\pi rti/p} z_2, e^{2\pi rsi/p} z_3)$$

on M defines a $\mathbf{Z}/p$ action which is free if r, s, t are pairwise relatively prime to p. Hence if we take r, s, t to be sufficiently large numbers which are pairwise relatively prime and not divisible by p, then $M = M(r, s, t)$ is an aspherical 3-manifold which is an integral homology sphere and admits a free $\mathbf{Z}/p$ action.

The fundamental group $\pi_1(M)$ is the commutator subgroup of the centrally extended triangle group T, which is generated by three elements $\gamma_1, \gamma_2, \gamma_3$ subject to the relations

$$\gamma_1^r = \gamma_2^s = \gamma_3^t = \gamma_1\gamma_2\gamma_3.$$

Denote $G = \mathbf{Z}/p^2$ and let $H \subset G$ be a subgroup of order p with quotient $Q = G/H$. Then the infinite lens space $L_p = (EG)/H$ has a free Q action. If we make Q act freely on M as indicated above, we obtain a free diagonal Q action on $M \times L_p$.

We have a fibration of aspherical spaces

$$\begin{array}{ccc} L_p & \longrightarrow & M \times_Q L_p \\ & & \downarrow \\ & & M/Q \end{array}$$

from which we deduce the existence of an extension

$$1 \longrightarrow H \longrightarrow \pi_1(M \times_Q L_p) \longrightarrow \pi_1(M/Q) \longrightarrow 1.$$

As $\pi_1(M/Q)$ is torsion-free and $H \cong \mathbf{Z}/p$, it follows that $\Gamma = \pi_1(M \times_Q L_p)$ is a group of virtual cohomological dimension 3, with no subgroups of order p^2.

To compute the cohomology of Γ, we let G act on M through the projection $G \to Q$ so that

$$M \times_Q L_p = M \times_Q (EG)/H = (M \times EG/H)/Q = M \times_G EG.$$

We compare this Borel construction with the one associated to the Q action on M. The spectral sequences to compute their integral cohomology are of the form

$$E_2^{p,q}(G) = H^p(G, H^q(M, \mathbf{Z})), \quad E_2^{p,q}(Q) = H^p(Q, H^q(M, \mathbf{Z})).$$

We analyse the first differential in both:

$$\begin{array}{ccc} H^3(M,\mathbf{Z}) & \xrightarrow{d_4} & H^4(Q,\mathbf{Z}) \\ & \searrow^{d_4'} & \downarrow \\ & & H^4(G,\mathbf{Z}). \end{array}$$

As the vertical map is trivial in dimension 4, $d_4' = 0$, that is, the spectral sequence for $M \times_G EG$ degenerates, and in particular we have an injection

$$H^*(BG, \mathbf{Z}) \hookrightarrow H^*(M \times_G EG, \mathbf{Z}).$$

As an $H^*(G)$-module, $H^*(\Gamma, \mathbf{Z}) \cong H^*(G) \otimes H^*(S^3)$.

In sufficiently high dimensions,

$$H^*(M \times_G EG, \mathbf{Z}) \cong \hat{H}_G^*(M) \cong \hat{H}^*(\Gamma, \mathbf{Z}).$$

Hence $\hat{H}^*(\Gamma, \mathbf{Z})$ has exponent p^2 infinitely often, but as we have seen, no subgroups of order p^2, giving us the desired counterexample.

Using the Borel Construction for G, we obtain an extension

$$1 \longrightarrow \pi_1(M) \longrightarrow \Gamma \longrightarrow G \longrightarrow 1.$$

Applying Theorem 1.5 to this example yields $r(\Gamma) = 1, \Gamma_x \cong \mathbf{Z}/p$, and we must choose $y_1 \in H^*(G, \mathbf{Z})$ to be a class of exponent p^2, as cup product with it is necessarily an isomorphism.

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