# ON THE STRUCTURE OF SPACES OF COMMUTING ELEMENTS IN COMPACT LIE GROUPS

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Abstract. In this note we study topological invariants of the spaces of homomorphisms $\mathrm{Hom}(\pi, G)$, where $\pi$ is a finitely generated abelian group and $G$ is a compact Lie group arising as an arbitrary finite product of the classical groups $SU(r)$, $U(q)$ and $Sp(k)$.

## 1. Introduction

Let $\mathcal{P}$ denote the class of compact Lie groups arising as arbitrary finite products of the classical groups $SU(r)$, $U(q)$ and $Sp(k)$. In this article we use methods from algebraic topology to study the spaces of homomorphisms $\mathrm{Hom}(\pi, G)$ where $\pi$ denotes a finitely generated abelian group and $G \in \mathcal{P}$. Our main interest is the computation of invariants associated to these spaces such as their cohomology and stable homotopy type, as well as their equivariant $K$-theory with respect to the natural conjugation action. The natural quotient space under this action is the space of representations $\mathrm{Rep}(\pi, G)$, which can be identified with the moduli space of isomorphism classes of flat connections on principal $G$–bundles over $M$, where $M$ is a compact connected manifold with $\pi_1(M) = \pi$. Thus our results provide insight into these geometric invariants in the important case when $\pi_1(M)$ is a finitely generated abelian group.

Our starting point is the observation (see [3]) that when $G \in \mathcal{P}$ and $\pi$ is a finitely generated abelian group, the conjugation action of $G$ on the space of homomorphisms $\mathrm{Hom}(\pi, G)$ satisfies the following property: for every element $x \in \mathrm{Hom}(\pi, G)$ the isotropy subgroup $G_x$ is connected and of maximal rank. This property plays a central part in our analysis. Indeed, let $T \subset G$ be a maximal torus; in general if a compact Lie group $G$ acts on a compact space $X$ with connected maximal rank isotropy subgroups then there is an associated action of $W$ on the fixed–point set $X^T$ and many properties of the space $X$ are determined by the action of $W$ on $X^T$ (see [3], [8]). For our examples this means that a detailed understanding of the $W$-action on the subspace $\mathrm{Hom}(\pi, G)^T = \mathrm{Hom}(\pi, T)$ can be used to describe key homotopy–theoretic invariants for the original space of homomorphisms.

This approach can be used for example to obtain an explicit description of the number of path–connected components in $\mathrm{Hom}(\pi, G)$. Indeed we show that if $\pi = \mathbb{Z}^n \oplus A$, where

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$A$ is a finite abelian group, then the number of path–connected components in $\mathrm{Hom}(\pi, G)$ equals the number of distinct orbits for the action of $W$ on $\mathrm{Hom}(A, T)$

In [1] a stable splitting for the spaces of commuting $n$-tuples in $G$, $\mathrm{Hom}(\mathbb{Z}^n, G)$, was derived for any Lie group $G$ that is a closed subgroup of $GL_n(\mathbb{C})$. Here we show that this splitting can be generalized to the spaces of homomorphisms $\mathrm{Hom}(\pi, G)$ when $G \in \mathcal{P}$ and $\pi$ is any finitely generated abelian group. This is done by constructing a stable splitting on $\mathrm{Hom}(\pi, G)^T = \mathrm{Hom}(\pi, T)$ and proving that this splitting lifts to the space $\mathrm{Hom}(\pi, G)$. Suppose that $\pi = \mathbb{Z}/(q_1) \oplus \cdots \oplus \mathbb{Z}/(q_n)$, where $n \geq 0$ and $q_1, \ldots, q_n$ are integers. Here we allow some of the $q_i$'s to be 0 and in that case $\mathbb{Z}/(0) = \mathbb{Z}$. Thus $\mathrm{Hom}(\pi, G)$ can be seen as the subspace of $G^n$ consisting of those commuting $n$-tuples $(x_1, \ldots, x_n)$ such that $x_i^{q_i} = 1_G$ for all $1 \leq i \leq n$. For $1 \leq r \leq n$ let $J_{n,r}$ denote the set of all sequences of the form $\mathfrak{m} := \{1 \leq m_1 < \cdots < m_r \leq n\}$. Given such a sequence $\mathfrak{m}$ let $P_{\mathfrak{m}}(\pi) := \mathbb{Z}/(q_{m_1}) \oplus \cdots \oplus \mathbb{Z}/(q_{m_r})$ be a quotient of $\pi$. Let $S_1(P_{\mathfrak{m}}(\pi), G)$ be the subspace of $\mathrm{Hom}(P_{\mathfrak{m}}(\pi), G)$ consisting of those $r$-tuples $(x_{m_1}, \ldots, x_{m_r})$ in $\mathrm{Hom}(P_{\mathfrak{m}}(\pi), G)$ for which at least one of the $x_{m_i}$'s is equal to $1_G$.

**Theorem 1.1.** *Suppose that $G \in \mathcal{P}$ and that $\pi$ is a finitely generated abelian group. Then there is a $G$-equivariant homotopy equivalence*

$$\Theta : \Sigma \, \mathrm{Hom}(\pi, G) \to \bigvee_{1 \leq r \leq n} \Sigma \left( \bigvee_{\mathfrak{m} \in J_{n,r}} \mathrm{Hom}(P_{\mathfrak{m}}(\pi), G)/S_1(P_{\mathfrak{m}}(\pi), G) \right).$$

In Section 4 we determine the homotopy type of the stable factors appearing in the previous theorem for certain particular cases. In particular we determine the stable homotopy type of $\mathrm{Hom}(\pi, SU(2))$ for any finitely generated abelian group.

Suppose now that $G$ is any compact Lie group. The fundamental group of the spaces of homomorphisms of the form $\mathrm{Hom}(\mathbb{Z}^n, G)$ was computed in [7]. Let $\mathbb{1} \in \mathrm{Hom}(\mathbb{Z}^n, G)$ be the trivial representation. If $\mathbb{1}$ is chosen as the base point, then by [7, Theorem1.1] there is a natural isomorphism $\pi_1(\mathrm{Hom}(\mathbb{Z}^n, G)) \cong (\pi_1(G))^n$. Here we show that the methods applied in [7] can be used to compute $\pi_1(\mathrm{Hom}(\pi, G))$ for any choice of base point if we further require that $G \in \mathcal{P}$ and that $\pi$ is a finitely generated abelian group. Write $\pi$ in the form $\pi = \mathbb{Z}^n \oplus A$, with $A$ a finite abelian group. Then the space of homomorphisms $\mathrm{Hom}(\pi, G)$ can naturally be identified as a subspace of the product $\mathrm{Hom}(\mathbb{Z}^n, G) \times \mathrm{Hom}(A, G)$. Given $f \in \mathrm{Hom}(A, T)$ let

$$\mathbb{1}_f := \mathbb{1} \times f \in \mathrm{Hom}(\pi, G) \subset \mathrm{Hom}(\mathbb{Z}^n, G) \times \mathrm{Hom}(A, G).$$

Every path–connected component in $\mathrm{Hom}(\pi, G)$ contains some $\mathbb{1}_f$ and thus it suffices to consider the elements of the form $\mathbb{1}_f$ as base points in $\mathrm{Hom}(\pi, G)$. With this in mind we have the following.

**Theorem 1.2.** *Let $\pi = \mathbb{Z}^n \oplus A$, with $A$ a finite abelian group and let $G \in \mathcal{P}$. Suppose $f \in \mathrm{Hom}(A, T)$ and take $\mathbb{1}_f$ as the base point of $\mathrm{Hom}(\pi, G)$. Then there is a natural* *isomorphism* $\pi_1(\mathrm{Hom}(\pi, G)) \cong (\pi_1(G_f))^n$ *where* $G_f = Z_G(f)$ *is the subgroup of elements in* $G$ *commuting with* $f(x)$ *for all* $x \in A$.

In Section 6 we study the equivariant $K$-theory of the spaces of homomorphisms $\mathrm{Hom}(\pi, G)$ with respect to the conjugation action by $G$. When $\pi$ is a finite group, then $\mathrm{Hom}(\pi, G)$ is the disjoint union of homogeneous spaces of the form $G/H$ where $H$ is a maximal rank subgroup. Using this it is easy to see that $K^*_G(\mathrm{Hom}(\pi, G))$ is a free module over the representation ring of rank $|\mathrm{Hom}(\pi, T)|$. This result can be generalized for finitely generated abelian groups of rank 1 in the following way.

**Theorem 1.3.** *Suppose that* $G \in \mathcal{P}$ *is simply connected and of rank* $r$. *Let* $\pi = \mathbb{Z} \oplus A$ *where* $A$ *is a finite abelian group. Then* $K^*_G(\mathrm{Hom}(\pi, G))$ *is a free* $R(G)$*-module of rank* $2^r \cdot |\mathrm{Hom}(A, T)|$.

It turns out that $K^*_G(\mathrm{Hom}(\pi, G))$ is not always free as a module over $R(G)$. In fact, as was pointed out in [3], the $R(SU(2))$-module $K^*_{SU(2)}(\mathrm{Hom}(\mathbb{Z}^2, SU(2)))$ is not free. However, $K^*_{SU(2)}(\mathrm{Hom}(\mathbb{Z}^2, SU(2)))\otimes\mathbb{Q}$ turns out to be free as a module over $R(SU(2))\otimes\mathbb{Q}$. The next theorem shows that a similar result holds for all the spaces of homomorphisms that we consider here.

**Theorem 1.4.** *Suppose that* $G \in \mathcal{P}$ *is of rank* $r$ *and that* $\pi$ *is a finitely generated abelian group written in the form* $\pi = \mathbb{Z}^n \oplus A$, *where* $A$ *is a finite abelian group. Then* $K^*_G(\mathrm{Hom}(\pi, G)) \otimes \mathbb{Q}$ *is a free module over* $R(G) \otimes \mathbb{Q}$ *of rank* $2^{nr} \cdot |\mathrm{Hom}(A, T)|$.

The layout of this article is as follows. In Section 2 some general properties of the spaces of homomorphisms $\mathrm{Hom}(\pi, G)$ are determined. In Section 3 we study the cohomology groups with rational coefficients of these spaces. In Section 4 Theorem 1.1 is proved and some explicit examples are computed. In Section 5 the fundamental group of the spaces $\mathrm{Hom}(\pi, G)$ are computed for any choice of base point. Finally, in Section 6 we study the problem of computing $K^*_G(\mathrm{Hom}(\pi, G))$, where $G$ acts by conjugation on $\mathrm{Hom}(\pi, G)$.

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## 2. Preliminaries on spaces of commuting elements

Let $\pi$ be a finitely generated discrete group and $G$ a Lie group. Consider the set of homomorphisms from $\pi$ to $G$, $\mathrm{Hom}(\pi, G)$. This set can be given a topology as a subspace of a finite product of copies of $G$ in the following way. Fix a set of generators $e_1, \dots, e_n$ of $\pi$ and let $F_n$ be the free group on $n$-letters. By mapping the generators of $F_n$ onto the different $e_i$'s we obtain a surjective homomorphism $F_n \to \pi$. This surjection induces an inclusion of sets $\mathrm{Hom}(\pi, G) \hookrightarrow \mathrm{Hom}(F_n, G) \cong G^n$. This way $\mathrm{Hom}(\pi, G)$ can be given the subspace topology. It is easy to see that this topology is independent of the generators chosen for $\pi$. In case $\pi$ happens to be abelian, then any map $F_n \to \pi$ factors through $F_n \to \mathbb{Z}^n \to \pi$ yielding an inclusion of spaces $\mathrm{Hom}(\pi, G) \hookrightarrow \mathrm{Hom}(\mathbb{Z}^n, G) \hookrightarrow G^n$. Thus the space of homomorphisms $\mathrm{Hom}(\pi, G)$ can be seen as a subspace of the space of commuting $n$-tuples in $G$, $\mathrm{Hom}(\mathbb{Z}^n, G)$.

In this note we collect some facts about these spaces of homomorphisms in the particular case that $\pi$ is a finitely generated abelian group and $G$ belongs to a suitable family of Lie groups. We are mainly interested in the following family of Lie groups.

**Definition 2.1.** Let $\mathcal{P}$ denote the collection of all compact Lie groups arising as finite cartesian products of the groups $SU(r)$, $U(q)$ and $Sp(k)$.

Whenever $G$ belongs to the family $\mathcal{P}$ the space of homomorphisms $\mathrm{Hom}(\pi, G)$ satisfies the following crucial condition as we prove below in Proposition 2.3.

**Definition 2.2.** Let $X$ be a $G$-space. The action of $G$ on $X$ is said to have connected maximal rank isotropy subgroups if for every $x \in X$, the isotropy group $G_x$ is a connected subgroup of maximal rank; that is, for every $x \in X$ we can find a maximal torus $T_x$ in $G$ such that $T_x \subset G_x$.

**Proposition 2.3.** *Suppose that $\pi$ is a finitely generated abelian group and $G \in \mathcal{P}$. Then the conjugation action of $G$ on* $\mathrm{Hom}(\pi, G)$ *has connected maximal rank isotropy subgroups.*

**Proof:** Choose generators $e_1, \dots, e_n$ of $\pi$. As pointed out above we can use these generators to obtain an inclusion of $G$-spaces $\mathrm{Hom}(\pi, G) \hookrightarrow \mathrm{Hom}(\mathbb{Z}^n, G)$. Given this inclusion it suffices to show that the conjugation action of $G$ on $\mathrm{Hom}(\mathbb{Z}^n, G)$ has connected maximal rank isotropy groups. In [3, Example 2.4] it was proven that the action of $G$ on $\mathrm{Hom}(\mathbb{Z}^n, G)$ has connected maximal rank isotropy subgroups if and only if $\mathrm{Hom}(\mathbb{Z}^{n+1}, G)$ is path–connected. The proposition follows by noting that $\mathrm{Hom}(\mathbb{Z}^k, G)$ is path–connected for all $k \geq 0$ whenever $G \in \mathcal{P}$. □

Suppose that a compact Lie group $G$ acts on a space $X$ with connected maximal rank isotropy subgroups. Choose a maximal torus $T$ in $G$ and let $W$ be the Weyl group. By passing to the level of $T$-fixed points, the action of $G$ on $X$ induces an action of the Weyl group $W$ on $X^T$. Many properties of the action of $G$ on $X$ are determined by the action of $W$ on $X^T$ as explained in [8] and in some situations the former is completely determined by the latter up to isomorphism. For example, we can use this approach to produce $G$-CW complex structures on the spaces of homomorphisms as is proved next.

**Corollary 2.4.** *Suppose that $\pi$ is a finitely generated abelian group and $G \in \mathcal{P}$. Then* $\mathrm{Hom}(\pi, G)$ *with the conjugation action has the structure of a $G$-CW complex.*

**Proof:** Since $\pi$ is a finitely generated abelian group it can be written in the form $\pi = \mathbb{Z}^n \oplus A$, where $A$ is a finite abelian group. Let $X := \mathrm{Hom}(\pi, G)$ with the conjugation action of $G$. Note that $X^T = \text{Hom}(\pi, G)^T = T^n \times \text{Hom}(A, T)$. Since $\text{Hom}(A, T)$ is a discrete set, it follows that $X^T$ has the structure of a smooth manifold on which $W$ acts smoothly. In particular, by [9, Theorem 1] it follows that $X^T$ has the structure of a $W$-CW complex. Since the conjugation action of $G$ on $X$ has connected maximal rank isotropy subgroups then by [3, Theorem 2.2] it follows that this $W$-CW complex structure on $X^T$ induces a $G$-CW complex on $X$. □

This approach can also be used to determine explicitly the structure of these spaces of homomorphisms whenever $\pi$ is a finite abelian group.

**Proposition 2.5.** *Suppose that $\pi$ is a finite abelian group and $G \in \mathcal{P}$. Then there is a $G$-equivariant homeomorphism*

$$\Phi : \text{Hom}(\pi, G) \to \bigsqcup_{[f] \in \text{Hom}(\pi, T)/W} G/G_f.$$

*Here $[f]$ runs through a system of representatives of the $W$-orbits in $\text{Hom}(\pi, T)$ and each $G_f$ is a maximal rank subgroup with $W(G_f) = W_f$.*

**Proof:** Consider the $G$-space $X := \text{Hom}(\pi, G)$. Note that $X^T = \text{Hom}(\pi, T)$ is a discrete set endowed with an action of $W$. By decomposing $X^T$ into the different $W$-orbits we obtain a $W$-equivariant homeomorphism

$$X^T \cong \bigsqcup_{[f] \in \text{Hom}(\pi, T)/W} W/W_f.$$

Here $[f]$ runs through a set of representatives for the action of $W$ on $\text{Hom}(\pi, T)$. For each $f \in \text{Hom}(\pi, T)$ let $G_f$ denote the subgroup of elements in $G$ commuting with $f(x)$ for all $x \in \pi$. This group is a maximal rank subgroup in $G$ as $T \subset G_f$. Moreover, by [8, Theorem 1.1] it follows that $W(G_f) = W_f$. Also note that if we let $G$ act on the left on the homogeneous space $G/G_f$ then $(G/G_f)^T = W/W_f$. Let

$$Y = \bigsqcup_{[f] \in \text{Hom}(\pi, T)/W} G/G_f.$$

The left action of $G$ on $Y$ has maximal rank isotropy and there is a $W$-equivariant homeomorphism $\phi : X^T \to Y^T$. By [8, Theorem 2.1] there is a unique $G$-equivariant extension $\Phi : X \to Y$ of $\phi$ and this map is in fact a homeomorphism. □

## 3. Rational cohomology and path–connected components

In this section we explore the set of path connected components and the rational cohomology groups of the spaces of homomorphisms $\text{Hom}(\pi, G)$.

Suppose that $G$ is a compact connected Lie group and let $T$ be a maximal torus in $G$. Assume that $G$ acts on a space $X$ of the homotopy type of a $G$-CW complex with maximal rank isotropy subgroups. Consider the continuous map

$$\phi : G \times X^T \to X$$
$$(g, x) \mapsto gx.$$

Since $G$ acts on $X$ with maximal rank isotropy subgroups for every $x \in X$ we can find a maximal torus $T_x$ in $G$ such that $T_x \subset G_x$. As every pair of maximal tori in $G$ are conjugate it follows that for every $x \in X$ we can find some $g \in G$ such that $gx \in X^T$. This shows that $\phi$ is a surjective map. The normalizer of $T$ in $G$, $N_G(T)$ acts on the right on $G \times X^T$ by $(g, x) \cdot n = (gn, n^{-1}x)$ and the map $\phi$ is invariant under this action. Thus $\phi$ descends to a surjective map

$$\varphi : G \times_{N_G(T)} X^T = G/T \times_W X^T \to X$$
$$[g, x] \mapsto gx$$

The map $\varphi$ is not injective in general. Indeed, as was proven in [4], given $x \in X$ there is a homeomorphism $\varphi^{-1}(x) \cong G_x^0/N_{G_x^0}(T)$, where $G_x^0$ denotes the path–connected component of $G_x$ containing the identity element. Let $\mathbb{F}$ be a field with characteristic relatively prime to $|W|$. Then as observed in [4] the space $G_x^0/N_{G_x^0}(T)$ has $\mathbb{F}$–acyclic cohomology. The Vietoris-Begle theorem shows that $\varphi$ induces an isomorphism in cohomology with $\mathbb{F}$-coefficients. As a consequence we obtain the following proposition (first proved in [4]).

**Proposition 3.1.** *Suppose that $G$ is a compact connected Lie group acting on a spaces $X$ with maximal rank isotropy subgroups. If $\mathbb{F}$ is a field with characteristic relatively prime to $|W|$ then $H^*(X; \mathbb{F}) \cong H^*(G/T \times_W X^T; \mathbb{F}) \cong H^*(G/T \times X^T; \mathbb{F})^W$.*

**Remark 3.2.** Suppose that $G$ acts on $X$ with *connected* maximal rank isotropy groups. As pointed out above the map $\varphi$ is not injective in general since $\varphi^{-1}(x) \cong G_x^0/N_{G_x^0}(T)$ for $x \in X$. Under the given hypothesis we have $G_x^0 = G_x$. By [8, Theorem 1.1] the assignment $(H) \mapsto (WH)$ defines a one to one correspondence between the set of conjugacy classes of isotropy subgroups of the action of $G$ on $X$ and the set of conjugacy classes of isotropy subgroups of the action of $W$ on $X^T$. Thus the different isotropy subgroups of the action of $W$ on $X^T$ determine how far the map $\varphi$ is from being injective. In particular, if $W$ acts freely on $X^T$ then $\varphi$ is a continuous bijection and thus a homeomorphism if for example $X^T$ is compact.

Suppose now that $G \in \mathcal{P}$ and let $\pi$ be a finitely generated abelian group. By Proposition 2.3 the conjugation action of $G$ on $\mathrm{Hom}(\pi, G)$ has connected maximal rank isotropy subgroups. In this case $\mathrm{Hom}(\pi, G)^T = \mathrm{Hom}(\pi, T)$. As a consequence of the previous result the following is obtained.

**Corollary 3.3.** *Suppose that $G \in \mathcal{P}$ and let $\pi$ be a finitely generated abelian group. Then there is an isomorphism $H^*(\mathrm{Hom}(\pi, G); \mathbb{Q}) \cong H^*(G/T \times \mathrm{Hom}(\pi, T); \mathbb{Q})^W$.*

As an application of Corollary 3.3 the following can be derived.

**Corollary 3.4.** *Suppose that $G \in \mathcal{P}$ and let $\pi$ be a finitely generated abelian group written in the form $\pi = \mathbb{Z}^n \oplus A$. Then the number of path–connected components in* $\mathrm{Hom}(\pi, G)$ *equals the number of different orbits of the action of $W$ on* $\mathrm{Hom}(A, T)$

## 4. Stable splittings

In this section we show that the fat wedge filtration on a finite product of copies of $G$ induces a natural filtration on the spaces of homomorphisms $\mathrm{Hom}(\pi, G)$. It turns out that this filtration splits stably after one suspension whenever $\pi$ is a finitely generated abelian group and $G \in \mathcal{P}$.

Suppose that $\pi$ is a finitely generated abelian group. Using the fundamental theorem of finitely generated abelian groups $\pi$ can be written in the form

$$\pi = \mathbb{Z}/(q_1) \oplus \cdots \oplus \mathbb{Z}/(q_n),$$

where $n \geq 0$ and $q_1, \ldots, q_n$ are integers. Here we allow some of the $q_i$'s to be 0 and in that case $\mathbb{Z}/(0) = \mathbb{Z}$. This way we can see $\mathrm{Hom}(\pi, G)$ as the subspace of $G^n$ consisting of those commuting $n$-tuples $(x_1, \ldots, x_n)$ such that $x_i^{q_i} = 1_G$ for all $1 \leq i \leq n$. The fat wedge filtration on $G^n$ induces a natural filtration on the space of homomorphisms $\mathrm{Hom}(\pi, G)$. To be more precise, for each $1 \leq j \leq n$ let

$$S_j(\pi, G) = \{(x_1, \ldots, x_n) \in \mathrm{Hom}(\pi, G) \subset G^n \mid x_i = 1_G \text{ for at least } j \text{ of the } x_i\text{'s}\}.$$

This way we obtain a filtration of $\mathrm{Hom}(\pi, G)$

$$\{(1_G, ..., 1_G)\} = S_n(\pi, G) \subset S_{n-1}(\pi, G) \subset \cdots \subset S_0(\pi, G) = \mathrm{Hom}(\pi, G). \tag{1}$$

Note that each $S_j(\pi, G)$ is invariant under the conjugation action of $G$. In particular each $S_j(\pi, G)$ can be seen as a $G$-space that has connected maximal rank isotropy subgroups. On the level of the $T$-fixed points the filtration (1) induces a filtration of $\mathrm{Hom}(\pi, G)^T$

$$\{(1_G, ..., 1_G)\} = S_n(\pi, G)^T \subset S_{n-1}(\pi, G)^T \subset \cdots \subset S_0(\pi, G)^T = \mathrm{Hom}(\pi, G)^T. \tag{2}$$

For each $1 \leq i \leq n$ consider $\mathrm{Hom}(\mathbb{Z}/q_i, T) = \{t \in T \mid t^{q_i} = 1\}$. Note that each $\mathrm{Hom}(\mathbb{Z}/q_i, T)$ is a space endowed with the action of $W$. Whenever $q_i = 0$ we have $\mathrm{Hom}(\mathbb{Z}/q_i, T) = T$ and if $q_i \neq 0$ then $\mathrm{Hom}(\mathbb{Z}/q_i, T)$ is a discrete set. Since $T$ is abelian it follows that

$$\mathrm{Hom}(\pi, G)^T = \mathrm{Hom}(\pi, T) = \mathrm{Hom}(\mathbb{Z}/q_1, T) \times \cdots \times \mathrm{Hom}(\mathbb{Z}/q_n, T).$$

Moreover, the filtration (2) is precisely the fat wedge filtration of $\mathrm{Hom}(\pi, G)^T$ where we identify $\mathrm{Hom}(\pi, G)^T$ with the above product. It is well known that the fat wedge filtration on a product of spaces splits stably after one suspension. More precisely, for each $0 \leq j \leq n-1$ we can find a continuous map

$$r_j : \Sigma S_j(\pi, G)^T \to \Sigma S_{j+1}(\pi, G)^T$$

in such a way that there is a homotopy $h_j$ between $r_j \circ \Sigma(i_j)$ and $1_{\Sigma(S_{j+1}(\pi,G)^T)}$. Here

$$i_j : S_{j+1}(\pi, G)^T \to S_j(\pi, G)^T$$

denotes the inclusion map. Moreover, both the map $r_j$ and the homotopy $h_j$ can be arranged in such a way that they are $W$-equivariant. The $W$-action that we have in sight is the diagonal action of $W$ on the product $\mathrm{Hom}(\mathbb{Z}/q_1, T)\times\cdots\times\mathrm{Hom}(\mathbb{Z}/q_n, T)$. Consider the action of $G$ on $\Sigma\,\mathrm{Hom}(\pi, G)$ with $G$ acting trivially on the suspension component. This action has connected maximal rank isotropy subgroups and $(\Sigma\,\mathrm{Hom}(\pi, G))^T = \Sigma\,\mathrm{Hom}(\pi, T)$. By [8, Theorem 2.1] we can find a unique $G$-equivariant extension

$$R_j : \Sigma S_j(\pi, G) \to \Sigma S_{j+1}(\pi, G)$$

of $r_j$ and a unique $G$-equivariant homotopy $H_j$ between $R_j \circ \Sigma(I_j)$ and $1_{\Sigma(S_{j+1}(\pi,G))}$ extending $h_j$. Here $I_j : S_{j+1}(\pi, G) \to S_j(\pi, G)$ as before denotes the inclusion map.

Let $J_{n,r}$ denote the set of all sequences of the form $\mathfrak{m} := \{1 \le m_1 < \cdots < m_r \le n\}$. Note that $J_{n,r}$ contains precisely $\binom{n}{r}$ elements. Given such a sequence $\mathfrak{m}$, there is an associated abelian group $P_{\mathfrak{m}}(\pi) := \mathbb{Z}/(q_{m_1}) \oplus \cdots \oplus \mathbb{Z}/(q_{m_r})$ obtained as a quotient of $\pi$ and also a $G$-equivariant projection map

$$\begin{aligned} P_{\mathfrak{m}} : \mathrm{Hom}(\pi, G) &\to \mathrm{Hom}(P_{\mathfrak{m}}(\pi), G) \\ (x_1, ..., x_n) &\mapsto (x_{m_1}, ..., x_{m_r}). \end{aligned}$$

The above can be used to prove the following theorem.

**Theorem 4.1.** *Suppose that $G \in \mathcal{P}$ and that $\pi$ is a finitely generated abelian group. Then there is a $G$-equivariant homotopy equivalence*

$$\Theta : \Sigma\,\mathrm{Hom}(\pi, G) \to \bigvee_{1\le r\le n} \Sigma \left( \bigvee_{\mathfrak{m}\in J_{n,r}} \mathrm{Hom}(P_{\mathfrak{m}}(\pi), G)/S_1(P_{\mathfrak{m}}(\pi), G) \right).$$

**Proof:** Note that each $S_j(\pi, G)^T$ has the homotopy type of a $W$-CW complex and this implies that each $S_j(\pi, G)$ has the homotopy type of a $G$-CW complex by [3, Theorem 2.2]. The different maps $R_j$ and the homotopies $H_j$ induce a $G$-equivariant homotopy equivalence

$$\Sigma\,\mathrm{Hom}(\pi, G) \simeq \bigvee_{0\le r\le n-1} \Sigma S_r(\pi, G)/S_{r+1}(\pi, G) = \bigvee_{1\le r\le n} \Sigma S_{n-r}(\pi, G)/S_{n-r+1}(\pi, G).$$

To finish the theorem we will show that for each $1 \le r \le n$ there is a $G$-equivariant homotopy equivalence

$$S_{n-r}(\pi, G)/S_{n-r+1}(\pi, G) \simeq \bigvee_{\mathfrak{m}\in J_{n,r}} \mathrm{Hom}(P_{\mathfrak{m}}(\pi), G)/S_1(P_{\mathfrak{m}}(\pi), G).$$

To see this note that the different projection maps $\{P_{\mathfrak{m}}\}_{\mathfrak{m}\in J_{n,r}}$ can be assembled to obtain a $G$-map

$$\eta : \mathrm{Hom}(\pi, G) \to \prod_{\mathfrak{m}\in J_{n,r}} \mathrm{Hom}(P_{\mathfrak{m}}(\pi), G)/S_1(P_{\mathfrak{m}}(\pi), G)$$
$$(x_1, ..., x_n) \mapsto \{\bar{P}_{\mathfrak{m}}(x_1, ..., x_n)\}_{\mathfrak{m}\in J_{n,r}}.$$

The map $\eta$ sends $S_{n-r}(\pi, G)$ onto $\bigvee_{\mathfrak{m}\in J_{n,r}} \mathrm{Hom}(P_{\mathfrak{m}}(\pi), G)/S_1(P_{\mathfrak{m}}(\pi), G)$ and $S_{n-r+1}(\pi, G)$ is mapped onto the base point. It is easy to see that $\eta$ induces a $G$-equivariant homeomorphism

$$S_{n-r}(\pi, G)/S_{n-r+1}(\pi, G) \cong \bigvee_{\mathfrak{m}\in J_{n,r}} \mathrm{Hom}(P_{\mathfrak{m}}(\pi), G)/S_1(P_{\mathfrak{m}}(\pi), G)$$

and the theorem follows. □

**Remark:** A case of particular importance in the previous theorem is $\pi = \mathbb{Z}^n$. In this case $\mathrm{Hom}(\mathbb{Z}^n, G)$ is precisely the space of commuting ordered $n$-tuples in $G$. The previous theorem provides a simple proof for the stable equivalence provided in [1] for the spaces $\mathrm{Hom}(\mathbb{Z}^n, G)$ whenever $G \in \mathcal{P}$.

**Example 4.2.** Suppose that $\pi = \mathbb{Z}^n$. Let $1 \le r \le n$. For any $\mathfrak{m} \in J_{n,r}$ we have

$$\mathrm{Hom}(P_{\mathfrak{m}}(\pi), G)/S_1(P_{\mathfrak{m}}(\pi), G) \cong \mathrm{Hom}(\mathbb{Z}^r, G)/S_r^1(G),$$

where $S_r^1(G) \subset \mathrm{Hom}(\mathbb{Z}^n, G)$ is the subspace of those commuting $n$-tuples $(x_1, \dots, x_n)$ with at least one of the $x_i$ equal to $1_G$. These stable factors were identified independently in [2], [5] and [6] in the particular case where $G = SU(2)$. Let $n\lambda_2$ denote the the Whitney sum of $n$-copies of the canonical vector bundle over $\mathbb{R}P^2$ and let $s_n$ denote its zero section. Then

$$\mathrm{Hom}(\mathbb{Z}^n, SU(2))/S_n^1(SU(2)) \cong \begin{cases} \mathbb{S}^3 & \text{if } n = 1, \\ (\mathbb{R}P^2)^{n\lambda_2}/s_n(\mathbb{R}P^2) & \text{if } n \ge 2. \end{cases}$$

**Example 4.3.** Suppose now that $\pi = \mathbb{Z}/(q_1) \oplus \cdots \oplus \mathbb{Z}/(q_n)$ is any finitely generated abelian group and $G = SU(2)$. Let $T$ be the maximal torus consisting of $2 \times 2$ diagonal matrices with entries in $\mathbb{S}^1$ and determinant 1. In this case $W = \mathbb{Z}/2$ acts by permuting the diagonal entries of elements in $T$. Next we determine the stable factors of the form

$$\mathrm{Hom}(P_{\mathfrak{m}}(\pi), SU(2))/S_1(P_{\mathfrak{m}}(\pi), SU(2)),$$

where $\mathfrak{m} = \{1 \le m_1 < \cdots < m_r \le n\}$ is fixed. We consider the following cases.

- Suppose $P_{\mathfrak{m}}(\pi)$ is a finite group so that $q_{m_i} \ne 0$ for all $1 \le i \le r$. Assume further that at least one of the $q_{m_i}$'s is odd. By Proposition 2.5 there is a homeomorphism

$$\mathrm{Hom}(P_{\mathfrak{m}}(\pi), SU(2)) \cong \bigsqcup_{[f]\in \mathrm{Hom}(P_{\mathfrak{m}}(\pi), T)/W} G/G_f.$$

Here $[f]$ runs through all the $W$-orbits in $\mathrm{Hom}(P_{\mathfrak{m}}(\pi), T)$. In this case

$$G/G_f = G/T \cong \mathbb{S}^2$$

for all orbits corresponding to elements $f$ for which $W_f$ is trivial. On the other hand, when $f$ is fixed by $W$ the corresponding orbit is $G/G_f = G/G = *$. Since we are assuming that one of the $q_{m_i}$'s is odd, then every $f \in \mathrm{Hom}(P_{\mathfrak{m}}(\pi), T)$ corresponding to $r$-tuples $(x_{m_1}, \dots, x_{m_r})$ in $\mathrm{Hom}(P_{\mathfrak{m}}(\pi), SU(2))$ with $x_{m_i} \neq 1_G$ for all $i$ satisfies $W_f = 1$. This shows that

$$\mathrm{Hom}(P_{\mathfrak{m}}(\pi), SU(2))/S_1(P_{\mathfrak{m}}(\pi), SU(2)) \cong (\bigsqcup_{A(\mathfrak{m},\pi)} \mathbb{S}^2)_+.$$

Here $A(\mathfrak{m}, \pi)$ is the number of $W$-orbits in $\mathrm{Hom}(P_{\mathfrak{m}}(\pi), T)$ corresponding to $r$-tuples that don't contain the element 1. This number is precisely

$$A(\mathfrak{m}, \pi) = \frac{1}{2}(q_{m_1} - 1) \cdots (q_{m_r} - 1).$$

- Suppose now that $q_{m_i} \neq 0$ is even for all $1 \leq i \leq r$. In this case we have two possibilities for the $W$-orbits in $\mathrm{Hom}(P_{\mathfrak{m}}(\pi), T)$. If $[f]$ represents the orbit $[(-1, \dots, -1)]$ then $W_f = W$ and the corresponding orbit is $G/G_f = *$. For all other orbits $[f] \in \mathrm{Hom}(P_{\mathfrak{m}}(\pi), T)/W$ corresponding to $r$-tuples $(x_{m_1}, \dots, x_{m_r})$ in $\mathrm{Hom}(P_{\mathfrak{m}}(\pi), SU(2))$ with $x_{m_i} \neq 1_G$ for all $i$ we have $W_f = 1$ and as before $G/G_f \cong \mathbb{S}^2$. This shows that

$$\mathrm{Hom}(P_{\mathfrak{m}}(\pi), SU(2))/S_1(P_{\mathfrak{m}}(\pi), SU(2)) \cong (\bigsqcup_{A(\mathfrak{m},\pi)} \mathbb{S}^2) \sqcup \mathbb{S}^0,$$

where now $A(\mathfrak{m}, \pi)$ is the number of $W$-orbits in $\mathrm{Hom}(P_{\mathfrak{m}}(\pi), T)$ corresponding to $r$-tuples in $\mathrm{Hom}(P_{\mathfrak{m}}(\pi), T)$ that don't contain the element 1 and that are different from $(-1, \dots, -1)$. This number is precisely

$$A(\mathfrak{m}, \pi) = \frac{1}{2}\left((q_{m_1} - 1) \cdots (q_{m_r} - 1) - 1\right).$$

- We now consider the case where $q_{m_i} = 0$ for some $1 \leq i \leq r$. If $q_{m_i} = 0$ for all $1 \leq i \leq r$ then

$$\mathrm{Hom}(P_{\mathfrak{m}}(\pi), SU(2))/S_1(P_{\mathfrak{m}}(\pi), SU(2)) = \mathrm{Hom}(\mathbb{Z}^r, SU(2))/S_r^1(SU(2))$$

and these stable factors are as in Example 4.2. Suppose then that $q_{m_i} \neq 0$ for some $i$. For simplicity and without loss of generality we may assume that

$$P_{\mathfrak{m}}(\pi) = \mathbb{Z}^k \oplus \mathbb{Z}/(q_{m_{k+1}}) \oplus \cdots \oplus \mathbb{Z}/(q_{m_r})$$

for some $1 \leq k < r$ and $q_{m_i} \neq 0$ for $k+1 \leq i \leq r$. Since the inclusion map $S_1(P_{\mathfrak{m}}(\pi), SU(2)) \hookrightarrow \mathrm{Hom}(P_{\mathfrak{m}}(\pi), G)$ is a cofibration, we have

$$\mathrm{Hom}(P_{\mathfrak{m}}(\pi), SU(2))/S_1(P_{\mathfrak{m}}(\pi), SU(2)) \cong (\mathrm{Hom}(P_{\mathfrak{m}}(\pi), SU(2)) \setminus S_1(P_{\mathfrak{m}}(\pi), SU(2)))^+.$$

Here if $X$ is a locally compact space then $X^+$ denotes its one point compactification. Consider the map

$$\varphi_{\mathfrak{m}} : G/T \times_W \mathrm{Hom}(P_{\mathfrak{m}}(\pi), T) \to \mathrm{Hom}(P_{\mathfrak{m}}(\pi), G)$$
$$(g, (t_{m_1}, \ldots, t_{m_r})) \mapsto (gt_{m_1}g^{-1}, \ldots, gt_{m_r}g^{-1}).$$

This map is surjective as the action of $G$ on $\mathrm{Hom}(P_{\mathfrak{m}}(\pi), G)$ has connected maximal rank isotropy. Moreover

$$\varphi_{\mathfrak{m}}(g, (t_{m_1}, \ldots, t_{m_r})) \in S_1(P_{\mathfrak{m}}(\pi), SU(2))$$

if and only if $t_{m_i} = 1$ for some $1 \leq i \leq r$. Let $Q(P_{\mathfrak{m}}(\pi), T)$ denote the subset of

$$\mathrm{Hom}(\mathbb{Z}/(q_{m_{k+1}}) \oplus \cdots \oplus \mathbb{Z}/(q_{m_r}), T)$$

consisting of those $(r-k)$-tuples $(t_{m_{k+1}}, \ldots t_{m_r})$ in $T$ such that $t_{m_i} \neq 1$ for $k+1 \leq i \leq r$. Using the Cayley map as in [2, Section 7] we can find a $W$-equivariant homeomorphism

$$T \setminus \{1\} \cong \mathfrak{t}.$$

Here $\mathfrak{t}$ denotes the Lie algebra of $T$. Using this identification and the restriction of the map $\varphi_{\mathfrak{m}}$, we obtain a surjective map

$$\psi_{\mathfrak{m}} : G/T \times_W (\mathfrak{t}^k \times Q(P_{\mathfrak{m}}(\pi), T)) \to \mathrm{Hom}(P_{\mathfrak{m}}(\pi), G) \setminus S_1(P_{\mathfrak{m}}(\pi), SU(2))$$

Moreover, $\psi_{\mathfrak{m}}$ is injective except where the action of $W$ on $\mathfrak{t}^k \times Q(P_{\mathfrak{m}}(\pi), T)$ is not free. We need to consider two cases.

- Suppose first that that $q_{m_i}$ is odd for some $k+1 \leq i \leq r$. In that case $W$ acts freely on $\mathfrak{t}^k \times Q(P_{\mathfrak{m}}(\pi), T)$ and we have a $W$-equivariant homeomorphism

  $$\mathfrak{t}^k \times Q(P_{\mathfrak{m}}(\pi), T) \cong \bigsqcup_{A(\mathfrak{m},\pi)} \mathfrak{t}^k \times W$$

  Here $A(\mathfrak{m}, \pi)$ is the number of $W$-orbits in $Q(P_{\mathfrak{m}}(\pi), T)$. This number is precisely

  $$A(\mathfrak{m}, \pi) = \frac{1}{2}(q_{m_{k+1}} - 1) \cdots (q_{m_r} - 1).$$

  Therefore

  $$(G/T \times_W (\mathfrak{t}^k \times Q(P_{\mathfrak{m}}(\pi), T)))^+ \cong (\bigsqcup_{A(\mathfrak{m},\pi)} G/T \times_W (\mathfrak{t}^k \times W))^+ \cong \bigvee_{A(\mathfrak{m},\pi)} (G/T \times \mathfrak{t}^k)^+.$$

  Note that $G/T = \mathbb{S}^2$ and thus

  $$(G/T \times \mathfrak{t}^k)^+ \cong \Sigma^k(\mathbb{S}^2_+) \cong \mathbb{S}^{k+2} \vee \mathbb{S}^k.$$

  In this case the map $\psi_{\mathfrak{m}}$ is a homeomorphism as the action of $W$ on $\mathfrak{t}^k \times Q(P_{\mathfrak{m}}(\pi), T)$ is free. This shows that if $q_{m_i}$ is odd for some $k+1 \leq i \leq r$ then

  $$\mathrm{Hom}(P_{\mathfrak{m}}(\pi), SU(2))/S_1(P_{\mathfrak{m}}(\pi), SU(2)) \cong (\bigvee_{A(\mathfrak{m},\pi)} \mathbb{S}^{k+2}) \vee (\bigvee_{A(\mathfrak{m},\pi)} \mathbb{S}^k).$$

- Suppose now that $P_{\mathfrak{m}}(\pi)=\mathbb{Z}^k\oplus\mathbb{Z}/(q_{m_{k+1}})\cdots\oplus\mathbb{Z}/(q_{m_r})$ and that $q_{m_i}$ is even for every $k+1\leq i\leq r$. In this case we have two kinds of elements in $Q(P_{\mathfrak{m}}(\pi),T)$. On the one hand we have the $(r-k)$-tuple $(-1,\dots,-1)$ on which $W$ acts trivially. For all other elements in $Q(P_{\mathfrak{m}}(\pi),T)$ the action of $W$ is free. This shows that there is a $W$-equivariant homeomorphism
$$\mathfrak{t}^k\times Q(P_{\mathfrak{m}}(\pi),T)\cong\mathfrak{t}^k\sqcup(\bigsqcup_{A(\mathfrak{m},\pi)}\mathfrak{t}^k\times W).$$
Here $A(\mathfrak{m},\pi)$ denotes the number of $W$-orbits in $Q(P_{\mathfrak{m}}(\pi),T)$ different from the trivial orbit $[(-1,\dots,-1)]$. This number is precisely
$$A(\mathfrak{m},\pi)=\frac{1}{2}\left((q_{m_1}-1)\cdots(q_{m_r}-1)-1\right).$$
The map $\psi_{\mathfrak{m}}$ is no longer injective. Note that $G/T\times_W\mathfrak{t}^k$ is the Whitney sum of $k$-copies of the canonical vector bundle over $\mathbb{R}P^2$ and $\psi$ maps the zero section onto the $n$-tuple $(-1,\dots,-1)$. On the other hand, the restriction of $\psi_{\mathfrak{m}}$ onto the factor
$$G/T\times_W\left(\bigsqcup_{A(\mathfrak{m},\pi)}\mathfrak{t}^k\times W\right)\cong\bigsqcup_{A(\mathfrak{m},\pi)}G/T\times\mathfrak{t}^k$$
is injective. This shows that if $q_{m_i}$ is even for every $k+1\leq i\leq r$ then
$$\operatorname{Hom}(P_{\mathfrak{m}}(\pi),SU(2))/S_1(P_{\mathfrak{m}}(\pi),SU(2))\cong(\mathbb{R}P^2)^{k\lambda_2}/s_k(\mathbb{R}P^2)\vee(\bigvee_{A(\mathfrak{m},\pi)}\mathbb{S}^{k+2})\vee(\bigvee_{A(\mathfrak{m},\pi)}\mathbb{S}^k).$$

We can use the above for example to establish the stable homotopy type of the space of homomorphisms $\operatorname{Hom}(\mathbb{Z}^2\oplus\mathbb{Z}/(2)\oplus\mathbb{Z}/(3),SU(2))$. In this case we have that after one suspension $\operatorname{Hom}(\mathbb{Z}^2\oplus\mathbb{Z}/(2)\oplus\mathbb{Z}/(3),SU(2))$ is homotopy equivalent to
$$(\bigvee^2(\mathbb{R}P^2)^{2\lambda_2}/s_2(\mathbb{R}P^2))\vee(\bigvee^2\mathbb{S}^4)\vee(\bigvee^8\mathbb{S}^3)\vee(\bigvee^2\mathbb{S}^2)\vee(\bigvee^2\mathbb{S}^2_+)\vee(\bigvee^4\mathbb{S}^1)\vee\mathbb{S}^0.$$

**Example 4.4.** Suppose that $\pi=\mathbb{Z}\oplus A$, where $A$ is a finite abelian group. Choose $G\in\mathcal{P}$ and assume that $A$ is such that the action of $W$ on $\operatorname{Hom}(A,T)\setminus\{1\}$ is free. Since $W$ fixes the trivial homomorphism $1\in\operatorname{Hom}(A,T)$, then the decomposition of $\operatorname{Hom}(A,T)$ into $W$-orbits shows that in particular $|W|$ divides $(|\operatorname{Hom}(A,T)|-1)$ under this assumption.

We will show that in this case
$$\Sigma\operatorname{Hom}(\pi,G)\simeq\Sigma(\bigvee_k T)\vee\Sigma(\bigvee_k G/T\wedge T)\vee\Sigma G\vee(\bigsqcup_k G/T)_+$$
Here $k:=(|\operatorname{Hom}(A,T)|-1)/|W|$ is the number of distinct $W$-orbits on the set $\operatorname{Hom}(A,T)$ that are different from the one corresponding to the trivial homomorphism.

Indeed, using Theorem 4.1 we obtain a homotopy equivalence
$$\Sigma\operatorname{Hom}(\pi,G)\simeq\Sigma\operatorname{Hom}(\pi,G)/S_1(\pi,G)\vee\Sigma\operatorname{Hom}(\mathbb{Z},G)/S_1(\mathbb{Z},G)\vee\Sigma\operatorname{Hom}(A,G)/S_1(A,G).$$

Trivially $\mathrm{Hom}(\mathbb{Z},G)/S_1(\mathbb{Z},G) = G$. Also, since $A$ is a finite abelian group then by Proposition 2.5 we have

$$\mathrm{Hom}(A,G) \cong \bigsqcup_{[f]\in \mathrm{Hom}(A,T)/W} G/G_f.$$

Here $[f]$ runs through all the $W$-orbits in the finite set $\mathrm{Hom}(A,T)$ and $G_f$ is a maximal rank subgroup such that $W(G_f) = W_f$. In $\mathrm{Hom}(A,T)$ we have two different kinds of orbits. On the one hand, we have the orbit corresponding to the trivial homomorphism in $\mathrm{Hom}(A,T)$. For this orbit we have $W_f = W$ and $G_f = G$. The assumptions on $A$ imply that for all other orbits in $\mathrm{Hom}(A,T)/W$ we have $W_f = 1$ and thus $G_f = T$. This shows that

$$\mathrm{Hom}(A,G)/S_1(A,G) = (\bigsqcup_k G/T)_+.$$

We now determine the stable factor $\mathrm{Hom}(\pi,G)/S_1(\pi,G)$. For this consider the map

$$\varphi : G/T \times_W \mathrm{Hom}(\pi,T) \to \mathrm{Hom}(\pi,G).$$

Since the action of $G$ on $\mathrm{Hom}(\pi,G)$ has maximal rank isotropy subgroups $\varphi$ is surjective. Moreover, the restriction of $\varphi$ induces a surjective map

$$\varphi_| : G/T \times_W ((T\setminus\{1\}) \times (\mathrm{Hom}(A,T)\setminus\{1\})) \to \mathrm{Hom}(\pi,G)\setminus S_1(\pi,G).$$

Since the action of $W$ on $\mathrm{Hom}(A,T)\setminus\{1\}$ is free we have that this restriction map is a homeomorphism. Also

$$G/T \times_W ((T\setminus\{1\}) \times (\mathrm{Hom}(A,T)\setminus\{1\})) \cong \bigsqcup_k G/T \times (T\setminus\{1\}).$$

This shows that

$$\mathrm{Hom}(\pi,G)/S_1(\pi,G) \cong \bigvee_k (G/T \times (T\setminus\{1\}))^+$$

Note that

$$(G/T \times (T\setminus\{1\}))^+ \cong (G/T\times T)/(G/T\times\{1\}).$$

and it is easy to see that there is a homotopy equivalence

$$\Sigma((G/T\times T)/(G/T\times\{1\})) \simeq \Sigma T \vee \Sigma G/T \wedge T.$$

This shows that

$$\Sigma\,\mathrm{Hom}(\pi,G)/S_1(\pi,G) \cong \Sigma(\bigvee_k T) \vee \Sigma(\bigvee_k G/T\wedge T)$$

proving the claim.

## 5. Fundamental group

In this section we study the fundamental group of the spaces of homomorphisms $\mathrm{Hom}(\pi, G)$ under different choices of base point.

Suppose first that $\pi = \mathbb{Z}^n$ and that $G$ is a compact Lie group. Let $\mathbb{1} \in \mathrm{Hom}(\mathbb{Z}^n, G)$ be the trivial representation. If we give $\mathrm{Hom}(\mathbb{Z}^n, G)$ the base point $\mathbb{1}$ then by [7, Theorem 1.1] there is a natural isomorphism $\pi_1(\mathrm{Hom}(\mathbb{Z}^n, G)) \cong (\pi_1(G))^n$. The tools applied in [7] can be used to generalize this result to the class of spaces of homomorphisms $\mathrm{Hom}(\pi, G)$. Here we need to assume that $\pi$ is a finitely generated abelian group and that $G$ is a Lie group in the class $\mathcal{P}$. Under these assumptions [7, Theorem 1.1] can be generalized for any choice of base point in $\mathrm{Hom}(\pi, G)$. Write $\pi$ in the form $\pi = \mathbb{Z}^n \oplus A$, where $A$ is a finite abelian group. Suppose first that $n = 0$ and thus $\pi$ is a finite group. In this case by Proposition 2.5 there is a homeomorphism

$$\Phi : \mathrm{Hom}(\pi, G) \to \bigsqcup_{[f] \in \mathrm{Hom}(\pi, T)/W} G/G_f,$$

where each $G_f$ is a maximal rank isotropy subgroup with $W(G_f) = W_f$. For each maximal rank subgroup $H \subset G$ we have $\pi_1(G/H) = 1$. It follows that $\pi_1(\mathrm{Hom}(\pi, G)) = 1$ for any choice of base point in this case. This handles the case of finite groups. Suppose then that $n \geq 1$. Let $T \subset G$ be a maximal torus. Note that $\mathrm{Hom}(\pi, G)^T = \mathrm{Hom}(\pi, T)$ and since $T$ is abelian we have $\mathrm{Hom}(\pi, T) = \mathrm{Hom}(\mathbb{Z}^n, T) \times \mathrm{Hom}(A, T) = T^n \times \mathrm{Hom}(A, T)$. Choose $f \in \mathrm{Hom}(A, T)$ and let $\mathbb{1} \in \mathrm{Hom}(\mathbb{Z}^n, T)$ denote the trivial representation. Let

$$\mathbb{1}_f := \mathbb{1} \times f \in \mathrm{Hom}(\mathbb{Z}^n, T) \times \mathrm{Hom}(A, T) = \mathrm{Hom}(\pi, T) \hookrightarrow \mathrm{Hom}(\mathbb{Z}^n, G)$$

and denote by $\mathrm{Hom}(\pi, G)_{\mathbb{1}_f}$ the path–connected component of $\mathrm{Hom}(\pi, G)$ containing $\mathbb{1}_f$. It is easy to see that

$$\mathrm{Hom}(\pi, G) = \bigsqcup_{[f] \in \mathrm{Hom}(A, T)/W} \mathrm{Hom}(\pi, G)_{\mathbb{1}_f}.$$

Since the fundamental group does not depend, up to isomorphism, on the base point chosen on a path–connected space, it suffices to compute $\pi_1(\mathrm{Hom}(\pi, G)_{\mathbb{1}_f})$ for any $f \in \mathrm{Hom}(A, T)$, where $\mathrm{Hom}(\pi, G)_{\mathbb{1}_f}$ is given the base point $\mathbb{1}_f$. Fix $f \in \mathrm{Hom}(A, T)$. Note that $\mathrm{Hom}(\pi, G)_{\mathbb{1}_f}$ is invariant under the conjugation action of $G$ and this action has connected maximal isotropy subgroups. In this case $\mathrm{Hom}(\pi, G)^T_{\mathbb{1}_f} \cong T^n \times W/W_f$. As pointed out in Section 3 we have a surjective map

$$\varphi : G/T \times_W \mathrm{Hom}(\pi, G)^T_{\mathbb{1}_f} \cong G/T \times_{W_f} T^n \to \mathrm{Hom}(\pi, G)_{\mathbb{1}_f}$$

that has connected fibers. As observed before this map is not injective in general; however, there is a large set on which this has this desirable property. Let $\mathcal{F}(\pi, f)$ be the subset of $\mathrm{Hom}(\pi, G)^T_{\mathbb{1}_f}$ on which $W$ acts freely. Then the restriction of $\varphi$

$$\varphi_| : G/T \times_W \mathcal{F}(\pi, f) \to \varphi(G/T \times_W \mathcal{F}(\pi, f)) \subset \mathrm{Hom}(\pi, G)_{\mathbb{1}_f}$$

is a homeomorphism onto its image. Assume further that $G$ is not a torus. Then under this additional assumption the complement of $G/T \times_W \mathcal{F}(\pi, f)$ is an analytic subspace of $G/T \times_W (T^n \times W/W_f)$ of co-dimension at least 2. Indeed, if $G$ is not a torus then $G/T$ is a smooth manifold of dimension at least 2 and $\mathrm{Hom}(\pi, G)^T_{\mathbb{1}_f} \setminus \mathcal{F}(\pi, f)$ submanifold of $\mathrm{Hom}(\pi, G)^T_{\mathbb{1}_f}$ of co-dimension at least 1. This proves the claim. Note that when $G$ is a torus then $W$ is trivial, $\mathcal{F}(\pi, f) = \mathrm{Hom}(\pi, G)^T_{\mathbb{1}_f}$ and the map $\varphi$ is a homeomorphism.

Following [7] we have the following definition.

**Definition 5.1.** Define $\mathcal{H}^r_f$ to be the image of $G/T \times_W \mathcal{F}(\pi, f)$ under the map $\varphi$. We refer to $\mathcal{H}^r_f$ as the regular part of $\mathrm{Hom}(\pi, G)_{\mathbb{1}_f}$. Also define $\mathcal{H}^s_f := \mathrm{Hom}(\pi, G)_{\mathbb{1}_f} \setminus \mathcal{H}^r_f$. We refer to $\mathcal{H}^s_f$ as the singular part of $\mathrm{Hom}(\pi, G)_{\mathbb{1}_f}$.

Note that $\mathrm{Hom}(\pi, G)_{\mathbb{1}_f}$ is a compact real analytic space and since $\mathcal{H}^s_f$ is the image of the compact analytic space $(G/T \times_W T^n \times W/W_f) \setminus (G/T \times_W \mathcal{F}(\pi, f))$ under the analytic map $\varphi$, it follows that $\mathcal{H}^s_f$ is a compact analytic subspace of $\mathrm{Hom}(\pi, G)_{\mathbb{1}_f}$. As a consequence of the Whitney stratification theorem it follows that $\mathrm{Hom}(\pi, G)_{\mathbb{1}_f}$ can be given the structure of a simplicial complex in such a way that $\mathcal{H}^s_f$ is a subcomplex. Note that when $G$ is a torus $\mathcal{H}^s_f$ is empty. On the other hand, if $G$ is not a torus then using the fact that the complement of $G/T \times_W \mathcal{F}(\pi, f)$ is an analytic subspace of $G/T \times_W (T^n \times W/W_f)$ of co-dimension at least 2 and an argument similar to the one provided in [7, Lemma 2.4] the following lemma is obtained for any $G \in \mathcal{P}$.

**Lemma 5.2.** *The space* $\mathrm{Hom}(\pi, G)_{\mathbb{1}_f}$ *is a compact simplicial complex and the singular part* $\mathcal{H}^s_f$ *is a subcomplex. Also,* $\mathcal{H}^s_f$ *is nowhere dense and does not disconnect connected open subsets of* $\mathrm{Hom}(\pi, G)_{\mathbb{1}_f}$.

The previous lemma can be used as a first step for the computation of $\pi_1(\mathrm{Hom}(\pi, G)_{\mathbb{1}_f})$. Indeed, suppose that $X$ is a compact simplicial complex and that $Y \subset X$ is a subcomplex. Assume that $X \setminus Y$ is dense and that $Y$ does not separate any connected open set in $X$. If $x_0 \in X \setminus Y$ is the base point then by [7, Lemma 2.5] the inclusion map $i : X \setminus Y \to X$ induces a surjective homomorphism $i_* : \pi_1(X \setminus Y, x_0) \to \pi_1(X, x_0)$. This can be applied in our situation. Choose $x_0 \in \mathcal{H}^r_f$ as the base point. Using Lemma 5.2 we obtain that the inclusion $i : \mathcal{H}^r_f \hookrightarrow \mathrm{Hom}(\pi, G)_{\mathbb{1}_f}$ induces a surjective homomorphism $i_* : \pi_1(\mathcal{H}^r_f) \to \pi_1(\mathrm{Hom}(\pi, G)_{\mathbb{1}_f})$. The same argument shows that the inclusion map

$$i : G/T \times_W \mathcal{F}(\pi, f) \hookrightarrow G/T \times_W \mathrm{Hom}(\pi, G)^T_{\mathbb{1}_f}$$

induces a surjective homomorphism

$$i_* : \pi_1(G/T \times_W \mathcal{F}(\pi, f)) \to \pi_1(G/T \times_W \mathrm{Hom}(\pi, G)^T_{\mathbb{1}_f}).$$

Since $\varphi_| : G/T \times_W \mathcal{F}(\pi, f) \to \mathcal{H}^r$ is a homeomorphism and the fundamental group of a connected space does not depend on the choice of base point, up homeomorphism, we obtain the following proposition (compare [7, Corollary 2.6]).

**Proposition 5.3.** *Suppose that $G \in \mathcal{P}$ and that $\pi$ is a finitely generated abelian group. Then the map*

$$\varphi : G/T \times_W \mathrm{Hom}(\pi, G)^T_{\mathbb{1}_f} \to \mathrm{Hom}(\pi, G)_{\mathbb{1}_f}$$

*is $\pi_1$-surjective.*

Note that

$$G/T \times_W \mathrm{Hom}(\pi, G)^T_{\mathbb{1}_f} \cong G/T \times_{W_f} T^n.$$

Since $W_f$ acts freely on $G/T$ the projection map $p$ induces a fibration sequence

$$T^n \to G/T \times_{W_f} T^n \xrightarrow{p} (G/T)/W_f \cong G/N_{G_f}(T).$$

The tail of the homotopy long exact sequence associated to this fibration is the exact sequence

$$\pi_1(T^n) \to \pi_1(G/T \times_{W_f} T^n) \to \pi_1(G/N_{G_f}(T)) \to 1. \tag{3}$$

Note that $\mathbb{1} \in T^n$ is a fixed point of $W_f$. Therefore the map

$$\begin{aligned} s : G/N_{G_f}(T) &\to G/T \times_{W_f} T^n \\ [g] &\mapsto [g \times \mathbb{1}] \end{aligned}$$

is a section of $p$ and in particular the sequence (3) splits. This proves that $\pi_1(G/T \times_{W_f} T^n)$ is generated by $\pi_1(T^n)$ and $s_*(\pi_1(G/N_{G_f}(T)))$. Next we prove the following lemma.

**Lemma 5.4.** *If $\alpha : [0,1] \to G/N_{G_f}(T)$ is a loop then $\varphi \circ s \circ \alpha$ is homotopic to the the trivial loop in $\mathrm{Hom}(\pi, G)_{\mathbb{1}_f}$. Therefore $s_*(\pi_1(G/N_{G_f}(T))) \subset Ker(\varphi_*)$.*

**Proof:** Let $\alpha : [0,1] \to G/N_{G_f}(T)$ be a loop. Note that $\mathrm{Hom}(\pi, G)_{\mathbb{1}_f}$ can be seen as a subspace of $\mathrm{Hom}(\mathbb{Z}^n, G) \times \mathrm{Hom}(A, G)$. Under this identification $\beta := \varphi \circ s \circ \alpha$ is the loop in $\mathrm{Hom}(\pi, G)_{\mathbb{1}_f}$ given by

$$\begin{aligned} \beta := \varphi \circ s \circ \alpha : [0,1] &\to \mathrm{Hom}(\pi, G)_{\mathbb{1}_f} \subset \mathrm{Hom}(\mathbb{Z}^n, G) \times \mathrm{Hom}(A, G) \\ t &\mapsto (\mathbb{1}, \alpha(t) f \alpha(t)^{-1}). \end{aligned}$$

Let $G_f = Z_G(f)$ be the subspace of elements in $G$ commuting with $f(x)$ for all $x \in A$ and $G \cdot f$ the space of elements in $\mathrm{Hom}(A, G)$ conjugated to $f$. Then

$$\beta : [0,1] \to \{\mathbb{1}\} \times G \cdot f \subset \mathrm{Hom}(\pi, G)_{\mathbb{1}_f}.$$

There is a homeomorphism $G \cdot f \cong G/G_f$ and $G_f$ is a maximal rank subgroup in $G$ as $T \subset G_f$. In particular the homogeneous space $G/G_f$ is simply connected. The simply connectedness of $G \cdot f$ shows that up to homotopy $\beta$ is the trivial loop in $\mathrm{Hom}(\pi, G)_{\mathbb{1}_f}$ proving the lemma. □

The previous lemma together with Proposition 5.3 and the fact that $\pi_1(G/T \times_{W_f} T^n)$ is generated by $\pi_1(T^n)$ and $s_*(\pi_1(G/N_{G_f}(T)))$ show that the map

$$\begin{aligned} \sigma_f : T^n &\to \mathrm{Hom}(\pi, G)_f \subset \mathrm{Hom}(\mathbb{Z}^n, G) \times \mathrm{Hom}(A, G) \\ (t_1, \dots, t_n) &\mapsto (t_1, \dots, t_n) \times \{f\} \end{aligned}$$

is $\pi_1$-surjective. On the other hand, the inclusion $T \subset G_f$ shows that $T^n \subset \mathrm{Hom}(\mathbb{Z}^n, G_f)$ and there is a commutative diagram

$$\begin{array}{ccc} T^n & \xrightarrow{\sigma_f} & \mathrm{Hom}(\pi, G)_{\mathbb{1}_f}. \\ \downarrow & \nearrow_{i_f} & \\ \mathrm{Hom}(\mathbb{Z}^n, G_f) & & \end{array} \tag{4}$$

In the previous commutative diagram $i_f$ denotes the map

$$\begin{aligned} i_f : \mathrm{Hom}(\mathbb{Z}^n, G_f) &\to \mathrm{Hom}(\pi, G)_f \subset \mathrm{Hom}(\mathbb{Z}^n, G) \times \mathrm{Hom}(A, G) \\ (x_1, \dots x_n) &\mapsto (x_1, \dots x_n) \times \{f\}. \end{aligned}$$

The inclusion map $T \subset G_f$ is $\pi_1$-surjective and by [7, Theorem 1.1] the map induced by the inclusion $\pi_1(\mathrm{Hom}(\mathbb{Z}^n, G_f)) \to \pi_1(G_f^n) \cong (\pi_1(G_f))^n$ is an isomorphism. This proves that the map $\pi_1(T^n) \to \pi_1(\mathrm{Hom}(\mathbb{Z}^n, G_f))$ is surjective. Using the commutativity of diagram (4) and the fact that $\sigma_f$ is $\pi_1$-surjective we obtain the following corollary.

**Corollary 5.5.** *Suppose that $\pi$ is a finitely generated abelian . Then the map*

$$i_f : \mathrm{Hom}(\mathbb{Z}^n, G_f) \to \mathrm{Hom}(\pi, G)_{\mathbb{1}_f}$$

*described above is $\pi_1$-surjective.*

We are now ready to prove the following theorem which is the main theorem of this section.

**Theorem 5.6.** *Let $\pi = \mathbb{Z}^n \oplus A$, with $A$ a finite abelian group and $G \in \mathcal{P}$. Let $f \in \mathrm{Hom}(A, T)$ and let $\mathbb{1} := \mathbb{1} \times f \in \mathrm{Hom}(\pi, G)$ be the base point of $\mathrm{Hom}(\pi, G)$. Then there is a natural isomorphism $\pi_1(\mathrm{Hom}(\pi, G)) \cong (\pi_1(G_f))^n$, where $G_f = Z_G(f)$ is the subgroup of elements in $G$ commuting with $f(x)$ for all $x \in A$.*

**Proof:** Suppose first that $\pi$ is a finite group and thus $n = 0$. Then as proved above $\pi_1(\mathrm{Hom}(\pi, G)) = 1$ for any choice of base point and the theorem is true in this case. Suppose then that $n \geq 1$. By Corollary 5.5 the map $i_f$ is $\pi_1$-surjective. We now show that in fact $(i_f)_* : \pi_1(\mathrm{Hom}(\mathbb{Z}^n, G_f)) \to \pi_1(\mathrm{Hom}(\pi, G)_{\mathbb{1}_f})$ is an isomorphism. This together with the isomorphism $\pi_1(\mathrm{Hom}(\mathbb{Z}^n, G_f)) \cong (\pi_1(G_f))^n$ provided by [7, Theorem 1.1] proves the theorem.

To start note that $G_f$ is such that $\pi_1(G_f)$ is torsion free. Therefore we can write $\pi_1(G_f) = \mathbb{Z}^a$ for some integer $a$ and the map

$$(i_f)_* : \pi_1(\mathrm{Hom}(\mathbb{Z}^n, G_f)) \cong \mathbb{Z}^{na} \to \pi_1(\mathrm{Hom}(\pi, G)_{\mathbb{1}_f})$$

is a surjection. This shows in particular that $\pi_1(\mathrm{Hom}(\pi, G)_{\mathbb{1}_f})$ is abelian and of rank at most $na$. We are going to show that in fact

$$r := \mathrm{rank}_{\mathbb{Z}}(\pi_1(\mathrm{Hom}(\pi, G)_{\mathbb{1}_f})) = na.$$

The only way this is possible is that $(i_f)_*$ is an isomorphism, proving the theorem. We now verify this. The universal coefficient theorem together with the Hurewicz theorem provide an isomorphism of $\mathbb{Q}$-vector spaces

$$H^1(\mathrm{Hom}(\pi, G)_{\mathbb{1}_f}; \mathbb{Q}) \cong \mathbb{Q}^r, \tag{5}$$

$$H^1(\mathrm{Hom}(\mathbb{Z}^n, G_f); \mathbb{Q}) \cong \mathbb{Q}^{na}. \tag{6}$$

On the other hand, since the conjugation action of $G$ on $\mathrm{Hom}(\pi, G)_{\mathbb{1}_f}$ has connected maximal rank isotropy subgroups, then by Theorem 3.1 there is an isomorphism

$$H^*(\mathrm{Hom}(\pi, G)_{\mathbb{1}_f}; \mathbb{Q}) \cong H^*(G/T \times_W (\mathrm{Hom}(\pi, G)_{\mathbb{1}_f})^T; \mathbb{Q}).$$

In this case

$$G/T \times_W (\mathrm{Hom}(\pi, G)_{\mathbb{1}_f})^T \cong G/T \times_{W_f} T^n.$$

In particular

$$H^1(\mathrm{Hom}(\pi, G)_{\mathbb{1}_f}; \mathbb{Q}) \cong H^1(G/T \times T^n; \mathbb{Q})^{W_f} \cong H^1(T^n; \mathbb{Q})^{W_f}. \tag{7}$$

The second isomorphism follows from the fact that $H^1(G/T; \mathbb{Q}) = 0$ as $G/T$ is simply connected. On the other hand, by [8, Theorem 1.1] it follows $W(G_f) = W_f$. The conjugation action of $G_f$ on $\mathrm{Hom}(\mathbb{Z}^n, G_f)$ also has maximal rank isotropy subgroups. The same argument as above yields the following isomorphisms

$$H^1(\mathrm{Hom}(\mathbb{Z}^n, G_f); \mathbb{Q}) \cong H^1(G_f/T \times T^n; \mathbb{Q})^{W_f} \cong H^1(T^n; \mathbb{Q})^{W_f}. \tag{8}$$

Equations (7) and (8) show that there is an isomorphism of $\mathbb{Q}$-vector spaces

$$H^1(\mathrm{Hom}(\mathbb{Z}^n, G_f); \mathbb{Q}) \cong H^1(\mathrm{Hom}(\pi, G)_{\mathbb{1}_f}; \mathbb{Q})$$

and thus $n = ra$ by (5). □

## 6. Equivariant $K$-theory

In this section we study the $G$-equivariant $K$-theory of the spaces of homomorphisms $\mathrm{Hom}(\pi, G)$. We divide our study according to the nature of the group $\pi$. From now on we fix $T$ a maximal torus in $G$ and let $W$ be the associated Weyl group.

6.1. **Finite abelian groups.** We first consider the case where $\pi$ is a finite abelian group.

Fix a finite abelian group $\pi$ and $G \in \mathcal{P}$. By Proposition 2.5 there is a $G$-equivariant homeomorphism

$$\Phi : \mathrm{Hom}(\pi, G) \to \bigsqcup_{[f]\in \mathrm{Hom}(\pi,T)/W} G/G_f.$$

Given a subgroup $H \subset G$ we have

$$K^q_G(G/H) \cong \begin{cases} R(H) & \text{if} \quad q \text{ is even,} \\ 0 & \text{if} \quad q \text{ is odd.} \end{cases}$$

By [10, Theorem 1] it follows that if $H \subset G$ is a subgroup of maximal rank then $R(H)$ is a free module over $R(G)$ of rank $|W|/|WH|$. As a corollary of this the following is obtained.

**Corollary 6.1.** *Let $G \in \mathcal{P}$ and $\pi$ be a finite abelian group. Then $K^0_G(\mathrm{Hom}(\pi, G))$ is a free $R(G)$-module of rank $|\mathrm{Hom}(\pi, T)|$ and $K^1_G(\mathrm{Hom}(\pi, G)) = 0$.*

**Proof:** Using Proposition 2.5 and the above we have $K^1_G(\mathrm{Hom}(\pi, G)) = 0$ and

$$K^0_G(\mathrm{Hom}(\pi, G)) \cong \bigoplus_{[f]\in \mathrm{Hom}(\pi,T)/W} R(G_f).$$

Each $R(G_f)$ is a free module over $R(G)$ of rank $|W|/|W_f|$. Note that $W(G_f) = W_f$, where $W_f$ denotes the isotropy subgroup at $f$, under the action of $W$ on the finite set $\mathrm{Hom}(\pi, T)$. The partition of $\mathrm{Hom}(\pi, T)$ into the different $W$-orbits provides the identity

$$|\mathrm{Hom}(\pi, T)| = \sum_{[f]\in \mathrm{Hom}(\pi,T)/W} |W|/|W_f|.$$

This proves that $K^0_G(\mathrm{Hom}(\pi, G))$ is free as a module over $R(G)$ of rank $|\mathrm{Hom}(\pi, T)|$. □

**Remark:** The previous corollary is not true in general if $G \notin \mathcal{P}$. For example, it can be seen that $K^*_{PU(3)}(\mathrm{Hom}((\mathbb{Z}/(3))^2, PU(3)))$ is not free as a module over $R(PU(3))$.

6.2. **Abelian groups of rank one.** We now consider the case where $\pi$ is a finitely generated abelian group of rank one. Thus we can write $\pi$ in the form $\pi = \mathbb{Z} \oplus A$ where $A$ is a finite abelian group.

Suppose that $X$ is a $G$-CW complex. The skeleton filtration of $X$ induces a multiplicative spectral sequence (see [11]) with

$$E_2^{p,q} = H^p_G(X; \mathcal{K}^q_G) \Longrightarrow K^{p+q}_G(X). \tag{9}$$

The $E_2$-term of this spectral sequence is the Bredon cohomology of $X$ with respect to the coefficient system $\mathcal{K}^q_G$ defined by $G/H \mapsto K^q_G(G/H)$.

Suppose that $G \in \mathcal{P}$ and that $\pi = \mathbb{Z} \oplus A$, where $A$ is a finite abelian group. Then Corollary 2.4 gives $X := \mathrm{Hom}(\pi, G)$ the structure of a $G$-CW complex and we can use the previous spectral sequence to compute $K_G^*(\mathrm{Hom}(\pi, G))$. In [3, Theorem 1.6] a criterion for the collapse of the spectral sequence (9) without extension problems was provided. This criterion can be used in this case to compute the structure of $K_G^*(\mathrm{Hom}(\pi, G))$ as a module over $R(G)$. Let $\Phi$ be the root system associated to $(G, T)$. Fix a subset $\Phi^+$ of positive roots of $\Phi$ and let $\Delta = \{\alpha_1, \dots, \alpha_r\}$ be an ordering of the corresponding set of simple roots. Suppose that $W_i \subset W$ is a reflection subgroup. Let $\Phi_i$ be the corresponding root system and $\Phi_i^+$ the corresponding positive roots. Define

$$W_i^\ell := \{w \in W \mid w(\Phi_i^+) \subset \Phi^+\}.$$

The set $W_i^\ell$ forms a system of representatives of the left cosets in $W/W_i$ by [12, Lemma 2.5]. In a precise way, this means that any element $w \in W$ can be factored in a unique way in the form $w = ux$ with $u \in W_i^\ell$ and $x \in W_i$. In order to apply the criterion provided in [3, Theorem 1.6] we must verify the hypothesis required there. In particular we need to show that $X^T$ has the structure of a $W$-CW complex in such a way that there is a CW-subcomplex $K$ of $X^T$ such that for every element $x \in X^T$ there is a unique $w \in W$ such that $wx \in K$. We construct such a subcomplex next. To start note that $X^T = \mathrm{Hom}(\pi, G)^T = T \times \mathrm{Hom}(A, T)$ and $\mathrm{Hom}(A, T)$ is a discrete set endowed with an action of $W$. If we assume that $G$ is simply connected then as pointed out in [3, Section 6.1] the (closed) alcoves in $T$ provide a structure of a $W$-CW complex in $T$ in such a way that $K(\Delta)$, the alcove determined by $\Delta$, is a sub CW-complex of $T$ such that any element in $T$ has a unique representative in $K(\Delta)$ under the $W$-action. Moreover, for each cell $\sigma$ in $K(\Delta)$, the isotropy subgroup $W_\sigma$ is a reflection subgroup of the form $W_I$ for some $I \subset \Delta$. Here $W_I$ denotes the reflection subgroup generated by the reflections of the form $s_\alpha$ for $\alpha \in I$. This can be used to produce a sub CW-complex in $\mathrm{Hom}(\pi, G)^T$ satisfying similar properties in the following way. Let $f_1, \dots, f_m$ be a set of representatives for the $W$-orbits in the discrete set $\mathrm{Hom}(A, T)$. We can choose each $f_i$ in such a way that the isotropy subgroup $W_{f_i}$ is a reflection subgroup of $W$ of the form $W_{I_i}$ for some $I_i \subset \Delta$. For each $1 \leq i \leq m$ let $W_{f_i}^\ell$ be a system of minimal length representatives of $W/W_{f_i}$ as defined above. Define

$$L(\Delta) := \bigcup_{i=1}^{m} \bigcup_{u \in W_{f_i}^\ell} \left(u^{-1}K(\Delta) \times \{f_i\}\right) \subset T \times \mathrm{Hom}(A, T) = X^T.$$

Defined in this way $L(\Delta) \subset X^T$ is a sub CW-complex. We now show that $L(\Delta)$ is such that for every element $x \in X^T$ there is a unique $w \in W$ such that $wx \in L(\Delta)$. To see this, since $K(\Delta) \subset T$ satisfies this property, it suffices to see that for any $i$ and any $v_1, v_2 \in W$ there are unique $v \in W$ and $u \in W_{f_i}^\ell$ such that $v_1 K(\Delta) \times v_2 f_i = v(u^{-1}K(\Delta) \times f_i)$. Indeed, suppose that $v_1, v_2 \in W$. Using the defining property of $W_{f_i}^\ell$ we can find unique $u \in W_{f_i}^\ell$ and $x \in W_{f_i}$ such that $v_1^{-1}v_2 = ux$. Let $v = v_1 u$. Then $v_1 = vu^{-1}$ and in particular $v_1 K(\Delta) = vu^{-1}K(\Delta)$. Also, $x = v^{-1}v_2 \in W_{f_i}$ and thus $v_2 f_i = v f_i$. This shows that $v \in W$ and $u \in W_{f_i}^\ell$ are the unique elements such that $v_1 K(\Delta) \times v_2 f_i = v(u^{-1}K(\Delta) \times f_i)$.

On the other hand, note that $H^*(X^T;\mathbb{Z})$ is torsion–free and of rank $2^r \cdot |\operatorname{Hom}(A,T)|$, where $r$ denotes the rank of the Lie group $G$. Also note that the isotropy subgroups of the action of $W$ on $\operatorname{Hom}(\pi,G)^T = \operatorname{Hom}(\pi,T)$ are of the form $W_I$, with $I \subset \Delta$.

The above work shows that the conditions of [3, Theorem 1.6] are satisfied yielding the next theorem.

**Theorem 6.2.** *Suppose that $G \in \mathcal{P}$ is simply connected and of rank $r$. Let $\pi = \mathbb{Z} \oplus A$ where $A$ is a finite abelian group. Then $K_G^*(\operatorname{Hom}(\pi,G))$ is a free $R(G)$-module of rank $2^r \cdot |\operatorname{Hom}(A,T)|$.*

**Remark:** Combining Corollary 6.1 and Theorem 6.2 it follows that $K_G^*(\operatorname{Hom}(\pi,G))$ is free as a module over $R(G)$ whenever $\pi$ is a finitely generated abelian group of rank less or equal to 1 and $G \in \mathcal{P}$ is simply connected. As already pointed out in [3] this result does not extend to all finitely generated abelian groups $\pi$ as $K_{SU(2)}^*(\operatorname{Hom}(\mathbb{Z}^2, SU(2)))$ contains torsion as a $R(SU(2))$-module.

However, if we tensor with the rational numbers the previous result does extend to the family of finitely generated abelian groups and all Lie groups $G \in \mathcal{P}$. This is done next.

6.3. **Finitely generated abelian groups.** We show that $K_G^*(\operatorname{Hom}(\pi,G)) \otimes \mathbb{Q}$ is a free $R(G) \otimes \mathbb{Q}$-module for all finitely generated abelian groups $\pi$ and all Lie groups $G \in \mathcal{P}$.

Let $G$ be a compact Lie group with $\pi_1(G)$ torsion–free act on a compact space $X$ with connected maximal rank isotropy. If we further assume that $X^T$ has the homotopy type of a $W$-CW complex then by [3, Theorem 1.1] $K_G^*(X) \otimes \mathbb{Q}$ is a free module over $R(G) \otimes \mathbb{Q}$ of rank equal to $\sum_{i \geq 0} \operatorname{rank}_{\mathbb{Q}} H^i(X^T;\mathbb{Q})$. This theorem can be applied in our situation. Let $\pi$ be a finitely generated abelian group and $G \in \mathcal{P}$. Let $X := \operatorname{Hom}(\pi,G)$. Then the conjugation action of $G$ on $X$ has connected maximal rank isotropy subgroups and $X$ has the homotopy type of a $G$-CW complex by Proposition 2.3 and Corollary 2.4. Note that $H^*(\operatorname{Hom}(\pi,G)^T;\mathbb{Q})$ is a $\mathbb{Q}$-vector space of rank $2^{nr} \cdot |\operatorname{Hom}(A,T)|$, where $r$ is the rank of $G$. This proves that the hypotheses of [3, Theorem 1.1] are satisfied in this case yielding the following.

**Theorem 6.3.** *Suppose that $G \in \mathcal{P}$ is of rank $r$ and that $\pi$ is a finitely generated abelian group written in the form $\pi = \mathbb{Z}^n \oplus A$, where $A$ is a finite abelian group. Then $K_G^*(\operatorname{Hom}(\pi,G)) \otimes \mathbb{Q}$ is a free module over $R(G) \otimes \mathbb{Q}$ of rank $2^{nr} \cdot |\operatorname{Hom}(A,T)|$.*

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