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COHOMOLOGICAL EXPONENTS OF $\mathbb{Z}G$-LATTICES

Alejandro ADEM*
Mathematics Department, Stanford University, Stanford, CA 94305-2125, U.S.A.

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1. Introduction

Let G be a finite group and M a finitely generated $\mathbb{Z}G$-module which is $\mathbb{Z}$-torsion free (also known as a $\mathbb{Z}G$-lattice). It is well known that $|G|$ annihilates $\hat{H}^*(G,M)$; however, there are few results relating the occurrence of exponents with representation-theoretic properties of M.

In this paper we find restrictions on $\mathbb{Z}G$-lattices with highest exponent in Tate cohomology. The first one holds for any finite group:

Theorem 1.1. *Let G be a finite group and let M be a finitely generated $\mathbb{Z}G$-lattice. If $\hat{H}^0(G,M)$ has exponent $|G|$, then $M \cong \mathbb{Z} \oplus K$ for some $\mathbb{Z}G$-lattice K.*

In particular, this implies that the property of having a direct summand of a module isomorphic to $\mathbb{Z}$ is locally determined.

Using dimension-shifting over p-subgroups of G, we obtain a restriction on $M \otimes \mathbb{F}_p$ when $\exp \hat{H}^i(G,M) = |G|$:

Corollary 1.3. *If M is a finitely generated $\mathbb{Z}G$-lattice with $\exp \hat{H}^i(G,M) = |G|$ and P is a p-subgroup of G, then*

$$(M \otimes \mathbb{F}_p)\,|_{\mathbb{F}_pP} \cong \Omega^i(\mathbb{F}_p) \oplus N \quad \text{for some } \mathbb{F}_pP\text{-module } N.$$

Here $\mathbb{F}_p$is the trivial $\mathbb{F}_pP$ module, and $\Omega^i(\mathbb{F}_p)$ is the ith Heller operator applied to it (see Section 1 for its definition).

Calculating dimensions yields

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Corollary 1.4. *Under the conditions of Corollary* 1.3,

$$\mathrm{rk}_{\mathbb{Z}} M \geq \max_{\substack{P \subset G \\ p\text{-group}}} \left\{ \left(\sum_{j=0}^{|i|-1} (-1)^{|i|+j+1} \dim H^j(P, \mathbb{F}_p) |P| \right) + (-1)^{|i|} \right\}.$$

Our final result is a cohomological characterization of a class of groups in terms of exponents.

Theorem 2.1. *A finite group G has elementary abelian p-Sylow subgroups if and only if there exists a square-free integer n such that* $n \cdot \hat{H}^i(G, \mathbb{Z}) = 0$ *for all i sufficiently large.*

The proofs of these results are an application of methods in cohomology of groups. I am particularly grateful to the referee for his constructive remarks. After submitting this paper, P. Symonds informed me that he has independently obtained a (different) proof of Theorem 1.1.

1. A cohomological splitting theorem

Theorem 1.1. *Let G be any finite group and let M be a finitely generated* $\mathbb{Z}G$*-lattice. If* $\hat{H}^0(G, M)$ *has exponent* $|G|$, *then* $M \cong K \oplus \mathbb{Z}$ *for a* $\mathbb{Z}G$*-lattice K.*

Proof. Recall $\hat{H}^0(G, M) = M^G / N(M)$, where $N = \sum_{g \in G} g$ is the norm. Let $x \in M^G$ such that $[x] \in M^G/N(M)$ has exponent $|G|$. Then there is a short exact sequence of $\mathbb{Z}G$-modules

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\ i\ } M \xrightarrow{\ \pi\ } K \longrightarrow 0$$

where $i(1) = x$ and hence represents a class of maximal exponent. Furthermore it is easy to choose x so that K has no torsion prime to $|G|$.

Suppose now there exists $l \mid |G|$ such that $l\pi(m) = 0$ for some $m \in M$. Then $\pi(lm) = 0$, hence $lm \in \operatorname{im} i \subset M^G$. As M is torsion-free, this means that $m \in M^G$. However,

$$|G| m = (|G|/l) lm \in NM$$

and if we write $lm = kx$, then $|G| \mid (|G|/l)k \Rightarrow l \mid k$ and so $m = (k/l)x$, i.e. $\pi(m) = 0$.

Therefore we may assume K is torsion-free, and so we have a dual exact sequence

$$0 \to K^* \to M^* \to \mathbb{Z}^* \to 0.$$

Consider the associated long exact sequence in cohomology

$$0 \to (K^*)^G \to (M^*)^G \to \mathbb{Z}^* \to H^1(G, K^*) \to H^1(G, M^*) \to 0$$

(recall that $H^1(G,\mathbb{Z})=0$). The map on the extreme right is the dual of $\hat{H}^{-1}(G,M)\to\hat{H}^{-1}(G,K)$, and by construction, this is an isomorphism. Hence its dual is too, and we have a short exact sequence of invariants

$$0\to(K^*)^G\to(M^*)^G\to\mathbb{Z}^*\to 0.$$

It follows that there is a G-map $\varphi: M\to\mathbb{Z}$ such that

$$\varphi\circ i=\mathrm{id}_{\mathbb{Z}}.$$

Hence we obtain a splitting $M\cong K\oplus\mathbb{Z}$. □

An easy consequence of Theorem 1.1 is

Corollary 1.2. *Let M be a finitely generated $\mathbb{Z}G$-lattice. Then M has $\mathbb{Z}$ as a direct summand if and only if $M_{(p)}$ has $\mathbb{Z}_{(p)}$ as a direct summand for all $p\mid |G|$.* □

In other words, the property of having a direct summand of a module isomorphic to $\mathbb{Z}$ is locally determined.

For the next application of Theorem 1.1 we recall the definition of the Heller operator $\Omega^i(N)$ for a finitely generated $\mathbb{F}_pG$-module N, when G is a p-group. Take a free module F which maps onto N, $\phi: F\to N$. We may write its kernel as $\Omega(N)\oplus F'$, where F' is projective, and $\Omega(N)$ has no projective submodules. $\Omega(N)$ is unique up to isomorphism and independent of the choice of F and ϕ (see [2]). Inductively, define $\Omega^n(N)=\Omega(\Omega^{n-1}(N))$ for $n>1$. Since $\mathbb{F}_pG$ is a Frobenius algebra we may define $\Omega^{-1}(N)$ dually and $\Omega^{-n}(N)=(\Omega^{-n+1}(N))$ for $n>1$.

Corollary 1.3. *If M is a $\mathbb{Z}G$-lattice such that* $\exp\hat{H}^i(G,M)=|G|$, *then for any p-subgroup P of G*

$$(M\otimes\mathbb{F}_p)\mid_{\mathbb{F}_pP}\cong\Omega^i(\mathbb{F}_p)\oplus N$$

where $\mathbb{F}_p$ is the trivial $\mathbb{F}_pP$-module.

Proof. We can assume G is a p-group. Now let $\bar{M}$ be a dimension-shift of M such that $\hat{H}^*(G,\bar{M})\cong\hat{H}^{*+i}(G,M)$. Then by Theorem 1.1, $\bar{M}\cong\mathbb{Z}\oplus K$. As this shifting is done with torsion-free modules, we may reduce mod p to obtain

$$\Omega^{-i}(M\otimes\mathbb{F}_p)\oplus F\cong\bar{M}\otimes\mathbb{F}_p\cong\mathbb{F}_p\oplus(K\otimes\mathbb{F}_p)$$

where F is projective. Applying Ω^i to both sides completes the proof. □

An explicit computation for $\dim_{\mathbb{F}_p}\Omega^i(\mathbb{F}_p)$ over the p-subgroups of G [5] yields

Corollary 1.4. *Let M be a $\mathbb{Z}G$-lattice such that* $\exp\hat{H}^i(G,M)=|G|$ *for some $i\in\mathbb{Z}$. Then*

$$\mathrm{rk}_{\mathbb{Z}}M \geq \max_{\substack{P\subset G\\ p\text{-group}}} \left\{ \left(\sum_{j=0}^{|i|-1} (-1)^{|i|+j+1} \dim H^j(P,\mathbb{F}_p)|P| \right) + (-1)^{|i|} \right\}. \quad \square$$

With additional work a result like Corollary 1.3 can be proved over $\mathbb{Z}_{(p)}$. A simple example of this is provided by K in the short exact sequence

$$0 \longrightarrow K \longrightarrow \mathbb{Z}G \stackrel{\varepsilon}{\longrightarrow} \mathbb{Z}/|G| \longrightarrow 0, \quad \text{where } G \text{ is a } p\text{-group}.$$

Here ε is the augmentation followed by reduction mod $|G|$. Then $K \otimes \mathbb{Z}_{(p)} \cong (IG \otimes \mathbb{Z}_{(p)}) \oplus \mathbb{Z}_{(p)}$.

2. A cohomological characterization using exponents

We have seen that lattices with highest exponent are quite restricted. In this section we show how cohomological exponents also greatly restrict the structure of a group when they are small.

Theorem 2.1. *A finite group G has elementary abelian p-Sylow subgroups if and only if there exists a square-free integer n such that $n \cdot H^i(G,\mathbb{Z}) = 0$ for all i sufficiently large.*

Proof. First assume all the p-Sylow subgroups of G are elementary abelian. Notice that

$$\exp H^i(G,\mathbb{Z}) \,\Big|\, \prod_{\substack{p\text{-Sylow}\\ \text{subgroups } P}} \exp H^i(P,\mathbb{Z}).$$

Hence $\exp H^i(G,\mathbb{Z}) \mid \prod_{p \mid |G|,\, p \text{ prime}} p$ and the only if part follows.

Now assume $nH^i(G,\mathbb{Z}) = 0$, where n has no repeated prime divisors, and for i sufficiently large. From Even's Theorem [1] on finite generation in cohomology of groups, it is clear that for any $g \in G$, its order must divide n. ($H^*(\langle g\rangle,\mathbb{Z})$ is a finitely generated $H^*(G,\mathbb{Z})$-module).

Hence if P is a p-Sylow subgroup of G, it can only have elements of order p. To complete the proof, we need only show that P is abelian.

Assume it is not: then it contains a minimal non-abelian p-group, with all elements of order p. For $p = 2$ there are *none* (see [4]) and for p odd there is only one, of order p^3, described as follows:

generators: A, B, C,

relations: $C = [A,B],\ [C,A] = [C,B] = 1 = A^p = B^p = C^p$.

However a cohomology calculation by Lewis [3] shows that this group has exponent p^2 infinitely often in its cohomology. Using Even's result again, we would have $p^2 \mid n$. In any case we get a contradiction, thus completing the proof. $\square$

To conclude the paper we mention a conjecture generalizing Theorem 2.1 for p-groups.

Conjecture. Let G be a finite p-group. If $H^*(G, \mathbb{Z})$ contains an element of exponent p^r, then it contains infinitely many.

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