$\mathcal{A}_5$-INVARIANTS, THE COHOMOLOGY OF $L_3(4)$ AND RELATED EXTENSIONS

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ABSTRACT

Five of the sporadic simple groups have maximal subgroups involving $M_{21} = L_3(4)$. They are M_{22}, M_{23}, M^{c}L, HS and O'N. In this paper we study the cohomology rings of these maximal subgroups and several further groups closely associated with $L_3(4)$.

Contents

0. *Introduction*

Denote by $L_3(4)$ the simple finite group $\mathrm{PSL}_3(\mathbb{F}_4)$ of order 20, 160. It is also the Mathieu group M_{21} and is a maximal, index-22 subgroup of the Mathieu group M_{22}. Besides this, it is an important subgroup or subquotient in several other sporadics M^{c}L, HS, and O'N. Recall that $\mathrm{Out}(L_3(4)) \cong \mathbb{Z}/2 \times \mathcal{S}_3$. The two generators for the $\mathbb{Z}/2 \times \mathbb{Z}/2$ Sylow subgroup of $\mathrm{Out}(L_3(4))$ can be represented by 2_3: $a \to a^{-t}$, and 2_2: $a \to a^{\phi}$ where ϕ: $\mathbb{F}_4 \to \mathbb{F}_4$ is the Galois automorphism. The automorphism 2_1 is the composition of these automorphisms. (Note that since the centre of $L_3(4)$ is trivial, the only extensions are the split ones.) The group $L_3(4) : 2_2$ is a maximal odd-index subgroup in M_{23} and M^{c}L (the McLaughlin group), while $L_3(4) : 2_1$ is maximal in HS, the Higman–Sims group. In addition, an extension of the form $4 \cdot L_3(4) : 2_1$ is 2-local, maximal, and of odd index in the O'Nan group O'N. It thus follows that $L_3(4)$ and its related extensions play a pivotal role in the cohomological structure at the prime 2 of the sporadic groups. This discussion can be summarized in Table 1. There are also connections with the Held group He, the Conway group Co_3 and $G_2(4)$ ($G_2(4)$ contains a 3-local maximal subgroup $3 \cdot L_3(4) : 2_3$).

It is also important to note that the cohomology of a finite group of Lie type at its characteristic (especially 2) is quite inaccessible in general. An important calculation in this direction was $H^*(\mathrm{GL}_4(\mathbb{F}_2) ; \mathbb{F}_2)$ [**1**] (see also [**12, 10**]) using the

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TABLE 1

	Group	Maximal	Odd index	2-Local
M_{22}	$L_3(4)$	yes	no	no
M_{23}	$L_3(4):2_2$	yes	yes	no
$\mathrm{M^cL}$	$L_3(4):2_2$	yes	yes	no
HS	$L_3(4):2_1$	yes	no	no
O'N	$4\cdot L_3(4):2_1$	yes	yes	yes

isomorphism $\mathscr{A}_8 \cong \mathrm{GL}_4(\mathbb{F}_2)$. The next level of difficulty from this point of view is represented by $L_3(4)$ (note the rather special fact that $|\mathscr{A}_8| = |L_3(4)|$).

The cohomology of $L_3(4)$ seems to have been first studied by Yagita **[14]**; however, there seem to be some errors in the results there. Using techniques very close to those of **[14]**, David Benson gave a correct calculation of $H^*(\mathrm{Syl}_2(L_3(4)))$ which he was kind enough to show to one of us. We reproduce Benson's and Yagita's calculation in § 2. From that point on, however, our approach differs considerably from that of **[14]**.

To analyse the cohomology of $L_3(4)$, we first decompose it using the Tits building $T(L_3(4))$. In this case it is a graph with an edge-transitive action of $L_3(4)$ and orbit space

$$P_1 \bullet\!\!\xrightarrow{\quad B \quad}\!\!\bullet\, P_2$$

where the vertex stabilizers P_1 and P_2 are the usual parabolic subgroups (in this case $2^4:\mathscr{A}_5$ since $L_2(4) \cong \mathscr{A}_5$) and B is the Borel subgroup of upper triangular matrices ($2^4:\mathscr{A}_4$ in this case). From the well-known fact that $H_1(T(L_3(4)))$ is a projective $\mathbb{F}_2(L_3(4))$ module—indeed it is the Steinberg module $\mathrm{St}_3(4)$—we obtain the cohomological decomposition (see also **[13]**)

$$H^*(L_3(4)) \oplus H^*(B) \cong H^*(P_1) \oplus H^*(P_2).$$

This type of decomposition can be modified to deal with extensions of $L_3(4)$ as well.

The groups P_1, P_2 are isomorphic, and, as has been noted, are given as semidirect products

$$1 \to (\mathbb{Z}/2)^4 \to P \to \mathscr{A}_5 \to 1.$$

In § 1 we show how $H^*(P)$ decomposes via the formula

$$H^*(P) \oplus H^*((\mathbb{Z}/2)^4)^{\mathscr{A}_4} \cong H^*(\mathrm{Syl}_2(L_3(4)))^{\mathbb{Z}/3} \oplus H^*((\mathbb{Z}/2)^4)^{\mathscr{A}_5}.$$

Using this, the calculation is reduced to computing $H^*(B)$ (§ 3) and to calculating the rings of invariants (§ 5).

Next we turn to $L_3(4):2_2$ and its subgroups $2^4:\mathscr{S}_4$ and $2^4:\mathscr{S}_5$. We first determine $H^*(\mathrm{Syl}_2(L_3(4):2_2))$ and then, using double coset decompositions, we work our way from $H^*(2^4:\mathscr{S}_4)$ to $H^*(2^4:\mathscr{S}_5)$ and finally to $H^*(L_3(4):2_2)$. We point out that

$$\mathrm{Syl}_2(L_3(4):2_2) \cong \mathrm{Syl}_2(M_{22});$$

hence our explicit calculation of the cohomology of this group will be useful in the cohomological analysis of it (work in progress). Specifically, there are two odd-index maximal 2-locals in M_{22}, the group $2^4:\mathcal{S}_5$ above and a group $2^4:\mathcal{A}_6$ which is not a subgroup of $L_3(4):2_2$. Copies can be found in M_{22} so that their intersection is the group $2^4:\mathcal{S}_4$ and the discussion in **[2]** shows that this configuration and the resulting restriction maps in cohomology control $H^*(M_{22})$.

In the final two sections (§§ 6 and 7), we study the extensions $2^5:\mathcal{A}_5$ and $2\cdot L_3(4)$ involved in O'N. The group O'N has sectional rank 5 and this forces us to study an invariant ring of the form $H^*(2^5)^{\mathcal{A}_5}$ where the action is a non-split extension of the action above of $\mathcal{A}_5$ on $H^*(2^4)$ by the trivial action on $H^*(2)$. Otherwise the arguments are similar to those in § 4. In **[3]** we use these results, together with a determination of the poset space $\mathcal{A}_2(\text{O'N})$, to calculate $H^*(\text{O'N})$.

Perhaps the core of the work here is the determination of the $\mathcal{A}_5$-invariants in $H^*(2^4)$ for the actions above and perhaps some perspective on it would be helpful. In the calculation of $H^*(\mathcal{A}_8)$ **[1]**, essential use was made of the ring of Dickson invariants $H^*((\mathbb{Z}/2)^3)^{GL_3(2)}$. Indeed, the Dickson classes play a fundamental role in the cohomology of the finite alternating and symmetric groups **[9]**. The ring $H^*((\mathbb{Z}/2)^4)^{\mathcal{A}_5}$ plays the same crucial role in our calculations and represents an important new element in our knowledge of cohomologically significant rings of invariants. Indeed, it becomes apparent that the difficulty in carrying out cohomology calculations at this level (rank 4 and beyond) goes hand-in-hand with the intricate structure of these rings of invariants. For example, the calculation of $H^*(M_{22})$, as a consequence of the results here, reduces to understanding $H^*((\mathbb{Z}/2)^4)^{\mathcal{A}_6}$, where $\mathcal{A}_6 \subseteq \text{Sp}_4(2) \cong \mathcal{S}_6$.† Explicitly we have

THEOREM 3.2. *The ring of invariants $H^*((\mathbb{Z}/2)^4)^{\mathcal{A}_5}$ has Poincaré series*

$$L(x) = \frac{\begin{array}{c}1+2x^3+3x^6+x^8+6x^9+2x^{11}+9x^{12}+x^{14}+10x^{15}\\ +x^{16}+9x^{18}+2x^{19}+6x^{21}+x^{22}+3x^{24}+2x^{27}+x^{30}\end{array}}{(1-x^5)^2(1-x^{12})^2}.$$

It is a 60-dimensional module, free over a polynomial subalgebra having two 5-dimensional and two 12-dimensional generators. Moreover, as an algebra it has the polynomial generators above together with two 3-dimensional generators, an 8-dimensional generator and two 9-dimensional generators.

The essence of the calculation is a careful argument using known invariant subrings and some facts about Ext-groups over $\mathbb{F}_2[\mathcal{A}_5]$ to reduce the determination of these invariants to a finite calculation, and then using a computer to complete the determination.

Related invariant subrings are described in Proposition 2.3, the discussion before (3.1), (4.3), (4.10), the discussion after (4.12), and Lemmas 5.1, 6.4, and 7.3.

As regards cohomology we obtain, for example,

† In fact, the ring $H^*((\mathbb{Z}/2)^4)^{\mathcal{A}_6}$ has recently been determined by one of us. It has the form $\mathbb{F}_2[x_3, x_5, d_8, d_{12}](1, \gamma_9, b_{15}, \gamma_9 b_{15})$.

THEOREM 3.4. *The cohomology of $L_3(4)$ has Poincaré series $G(x)$ which is given as*

$$\frac{x^{21}+2x^{20}-2x^{19}+x^{18}+x^{17}-2x^{16}+4x^{15}+3x^{14}-x^{13}+2x^{12} + 3x^{11}-x^{10}+4x^9+3x^8-x^7+2x^6+2x^5-x^4+x^3+2x^2+1}{(1-x^3)(1-x^4)(1-x^5)(1-x^{12})}.$$

The actual cohomology ring is described in (3.6). We point out that this is probably the first known instance of a simple group whose mod 2 cohomology has been calculated and which is not Cohen–Macaulay, i.e., free and finitely generated over a polynomial subalgebra (see [**5**, § 9]). However, from the fact that M_{22} has maximal elementary abelian subgroups of different rank, it will also be true that $H^*(M_{22}, \mathbb{F}_2)$ is not Cohen–Macaulay, yet its cohomology is still undetermined. Note that the maximal 2-tori in $L_3(4)$ *do* have the same rank.

Throughout this paper the coefficients will be assumed to lie in a field of characteristic 2, either $\mathbb{F}_2$ or $\mathbb{F}_4$ depending on whether we need a primitive third root of unity or not. We use the notation from the ATLAS [**6**] for many of our extensions. The conventions needed are $n = \mathbb{Z}/n$, $\cdot$ denotes non-split extension, . or : denotes semidirect product.

COMMENT. This paper contains many core results aimed towards the determination of $H^*(\text{O'N}\,;\mathbb{F}_2)$ and $H^*(M_{22}\,;\mathbb{F}_2)$, $H^*(M_{23}\,;\mathbb{F}_2)$. Indeed, as has been indicated, all that is really needed to complete the determination of these last two rings is $H^*((\mathbb{Z}/2)^4)^{\mathscr{A}_6}$. For $H^*(\text{O'N}\,;\mathbb{F}_2)$ we only need $H^*(4\cdot L_3(4):2_1)$. (Here we analyse $H^*(2\cdot L_3(4))$ leaving the extension to [**3**].) Similarly, in the case of J_2, J_3, and $\tilde{A}_8$ (a group which seems central in understanding certain operations in homotopy theory) only a little needs to be added to our results here. For example, for J_2, J_3 we need to determine $H^*((\mathbb{Z}/2)^4)^{\mathscr{A}_5}$ where this action is given via the exotic embedding of $\mathscr{A}_5$ in $\text{GL}_4(2)$. We expect to complete these results in the sequels to this paper.

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1. *The Tits building for* $L_3(4)$

In this section we recall the cohomological decompositions induced by Tits buildings. For later use we begin with the group $\mathscr{A}_5 \cong L_2(4) = \text{SL}_2(\mathbb{F}_4)$. For this group the Tits building $T(\mathscr{A}_5)$ has as its vertices the one-dimensional subspaces in $(\mathbb{F}_4)^2$, and no higher-dimensional simplices. The group $\mathscr{A}_5$ acts transitively on this configuration of five points, with stabilizer the Borel subgroup of upper triangular matrices, $B \cong \mathscr{A}_4$. Hence we obtain a decomposition of $\mathbb{F}_2\mathscr{A}_5$-modules:

$$\mathbb{F}_2[\mathscr{A}_5/\mathscr{A}_4] \cong \mathbb{F}_2 \oplus \text{St}_2(4), \tag{1.1}$$

where $\text{St}_2(4)$ (the Steinberg module) is projective. Now let us consider the case where $G = L_3(4) = \text{PSL}_3(\mathbb{F}_4)$. The building in this case is a finite graph, with an edge transitive action of G, and two orbits of vertices. The stabilizers of the vertices are the usual parabolic subgroups P_1, P_2 and the edge stabilizer is the

Borel subgroup of upper triangular matrices. From the fact that the isotropy groups contain the 2-Sylow subgroup, and that the first homology group is the Steinberg module $\mathrm{St}_3(4)$, we again obtain a decomposition

$$\mathbb{F}_2 \oplus \mathbb{F}_2[G/B] \cong \mathbb{F}_2[G/P_1] \oplus \mathbb{F}_2[G/P_2] \oplus \mathrm{St}_3(4). \tag{1.2}$$

We can easily identify $P_1 \cong P_2 \cong 2^4 : \mathscr{A}_5$, $B \cong 2^4 : \mathscr{A}_4$, where as usual 2^4 denotes $(\mathbb{Z}/2)^4$ and : indicates a split semidirect product.

From (1.2) we derive the following very useful formula, which works over any field of characteristic 2:

$$H^*(L_3(4)) \oplus H^*(2^4 : \mathscr{A}_4) \cong H^*(2^4 : \mathscr{A}_5) \oplus H^*(2^4 : \mathscr{A}_5). \tag{1.3}$$

Let W be a 2-group fitting into an extension

$$1 \to W \to G \to \mathscr{A}_5 \to 1.$$

We now describe a decomposition of $H^*(G)$ coming from the Tits building associated to $\mathscr{A}_5 = \mathrm{SL}_2(\mathbb{F}_4)$. Let EG be a free contractible G-space. Then $\mathscr{A}_5 \cong G/W$ acts freely on $EG/W \simeq BW$, whence $C^*(EG/W)$ is a free $\mathscr{A}_5$ cochain complex. Tensoring (1.1) with the last chain complex and applying $H^*(\mathscr{A}_5, -)$ to it yields

$$H^*(G) \oplus [H^*(W) \otimes \mathrm{St}]^{\mathscr{A}_5} \cong H^*(\bar{G}),$$

where $\bar{G}$ is the extension restricted to $\mathscr{A}_4$. Now applying (1.1) to $H^*(W)$ as an $\mathscr{A}_5$-module yields

$$H^*(G) \oplus H^*(W)^{\mathscr{A}_4} \cong H^*(\bar{G}) \oplus H^*(W)^{\mathscr{A}_5},$$

which completes the proof of

PROPOSITION 1.4. *Let W be a finite 2-group fitting into an extension diagram*

$$1 \to W \to G \to \mathscr{A}_5 \to 1.$$

Then

$$H^*(G) \oplus H^*(W)^{\mathscr{A}_4} \cong H^*(\mathrm{Syl}_2(G))^{\mathbb{Z}/3} \oplus H^*(W)^{\mathscr{A}_5}.$$

The group $\mathscr{A}_5$ has a double coset decomposition corresponding to the (B, N)-pair $(\mathscr{A}_4, \mathscr{S}_3)$, $T = \mathscr{S}_3 \cap \mathscr{A}_4 = \mathbb{Z}/3$, into two double cosets,

$$\mathscr{A}_5 = \mathscr{A}_4 \sqcup \mathscr{A}_4 \tau \mathscr{A}_4,$$

and consequently G above has a double coset decomposition $G = \bar{G} \sqcup \bar{G}\tau\bar{G}$. Note that $\mathscr{A}_4 \cap \tau\mathscr{A}_4\tau^{-1} = T = \mathbb{Z}/3$ so that $\bar{G} \cap \tau\bar{G}\tau^{-1}$ is an extension of the form $H = W \times_T \mathbb{Z}/3$. Here τ acts on T as inversion so $\langle \tau T \rangle = \mathscr{S}_3$ and we have the sequence of inclusions

$$\begin{array}{ccccc} & & & W \times_T S_3 & & \\ & & \nearrow & & \searrow & \\ W & \hookrightarrow & H & & & G \\ & & \searrow & & \nearrow & \\ & & & \bar{G} & & \end{array}$$

With respect to this diagram we have

THEOREM 1.5. (1) *The function* res: $H^*(G;\mathbb{F}_2)\to H^*(\bar{G};\mathbb{F}_2)$ *is an injection.*
(2) *An element* $\theta\in H^*(\bar{G};\mathbb{F}_2)$ *is in the image from* $H^*(G;\mathbb{F}_2)$ *if and only if*

$$\mathrm{res}^W_{\bar{G}}(\theta)\in \mathrm{im}(H^*(W\times_T\mathcal{S}_3;\mathbb{F}_2)),$$

that is, if and only if $\mathrm{res}^W_{\bar{G}}(\theta)\in H^*(W;\mathbb{F}_2)^{\mathcal{S}_3}$.

(This is a direct application of the double coset formula.)

REMARK. If $\mathrm{res}^W_{\bar{G}}(\theta)\in H^*(W)^{\mathcal{S}_3}$ then $\mathrm{res}^W_{\bar{G}}(\theta)\in H^*(W)^{\mathcal{A}_5}$. Indeed, $\langle\tau,\mathcal{A}_4\rangle=\mathcal{A}_5$. In particular, $\mathrm{Syl}_2(G)\lhd\bar{G}$ with quotient $\mathbb{Z}/3$ and we have

COROLLARY 1.6. *Let* $\mu\in H^*(\mathrm{Syl}_2(G);\mathbb{F}_2)$. *Then* $\mu\in\mathrm{res}^{\mathrm{Syl}_2(G)}_G(H^*(G;\mathbb{F}_2))$ *if and only if*
(1) $\mu\in H^*(\mathrm{Syl}_2(G);\mathbb{F}_2)^{\mathbb{Z}/3}$ *and*
(2) $\mathrm{res}^W_{\mathrm{Syl}_2(G)}(\mu)\in H^*(W;\mathbb{F}_2)^{\mathcal{A}_5}$.

We can apply Proposition 1.4 to our decomposition (1.3), yielding

$$H^*(L_3(4))\oplus[H^*(2^4)^{\mathcal{A}_4}]^2\cong H^*(\mathrm{Syl}_2(L_3(4)))^{\mathbb{Z}/3}\oplus[H^*(2^4)^{\mathcal{A}_5}]^2. \tag{1.7}$$

The terms in this equation will be calculated in the following sections.

2. *The cohomology of* $\mathrm{Syl}_2(L_3(4))$

The calculation below is a recreation of a calculation, due to Yagita and first shown to us by David Benson, for this Sylow 2-subgroup. There is a central extension

$$1\to(\mathbb{Z}/2)^2\to\mathrm{Syl}_2(L_3(\mathbb{F}_4))\to(\mathbb{Z}/2)^4\to 1$$

with generators

$$A=\begin{pmatrix}1&0&0\\0&1&1\\0&0&1\end{pmatrix},\quad B=\begin{pmatrix}1&0&0\\0&1&\zeta_3\\0&0&1\end{pmatrix},\quad C=\begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix},$$

$$D=\begin{pmatrix}1&0&\zeta_3\\0&1&0\\0&0&1\end{pmatrix},\quad E=\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix},\quad F=\begin{pmatrix}1&\zeta_3&0\\0&1&0\\0&0&1\end{pmatrix}.$$

Here the central $(\mathbb{Z}/2)^2$ is generated by C, D, and the $(\mathbb{Z}/2)^4$ quotient is $\langle A,B,E,F\rangle$. Note that $\langle A,B\rangle$ and $\langle E,F\rangle$ are copies of $(\mathbb{Z}/2)^2$ in $\mathrm{Syl}_2(L_3(4))$ though the two subgroups do not commute with each other.

Additionally there is an action of $\mathbb{Z}/3$ with generator T on $\mathrm{Syl}_2(L_3(\mathbb{F}_4))$ defined by conjugation with the matrix $T=\begin{pmatrix}\zeta_3&0&0\\0&1&0\\0&0&\zeta_3^2\end{pmatrix}$. Let a be dual to A, b be dual to B, c be dual to C, d dual to D, and e,f dual to E,F respectively. Then the E_2-term of the spectral sequence for the central extension above has the form

$$\mathbb{F}_2[a,b,e,f]\otimes\mathbb{F}_2[c,d],$$

where a, b, e and f all have filtration $(1, 0)$ while c and d, lying on the fibre, have filtration $(0, 1)$. We have, taking account of the extension data, that $d_2(c) = ae + bf$, $d_2(d) = be + af + bf$, while the d_3 differential on the fibre terms is given by

$$d_3(c^2) = a^2e + ae^2 + b^2f + bf^2,$$
$$d_3(d^2) = b^2e + be^2 + a^2f + af^2 + b^2f + bf^2.$$

To understand better the structure of this spectral sequence and the action of T, we tensor with $\mathbb{F}_4$ and replace the basis above with the new basis (consisting of eigenvectors for the T-action):

Generator	Name	Eigenvalue
$\zeta e + \zeta^2 f$	$\mathscr{E}$	ζ
$\zeta^2 e + \zeta f$	$\mathscr{F}$	ζ^2
$\zeta a + \zeta^2 b$	$\mathscr{A}$	ζ
$\zeta^2 a + \zeta b$	$\mathscr{B}$	ζ^2
$\zeta^2 c + \zeta d$	$\mathscr{D}$	ζ
$\zeta c + \zeta^2 d$	$\mathscr{C}$	ζ^2

The differentials in the spectral sequence commute with the $\mathbb{Z}/3$-action, and the E_i-term of the tensored spectral sequence is the tensor product of the original E_i-term with $\mathbb{F}_4$. The same is true for the ring structure. However, since there is more room now to change bases, it is certainly possible that the tensored E_i-term has a simpler description. For example, this is already the case with the d_2-differentials. Since $\mathscr{C}$ and $\mathscr{D}$ are eigenvectors for T, the same is true for $d_2(\mathscr{C})$ and $d_2(\mathscr{D})$. It follows that $d_2(\mathscr{C}) = \varepsilon\mathscr{A}\mathscr{E}$ and $d_2(\mathscr{D}) = \varepsilon'\mathscr{B}\mathscr{F}$ for some non-zero coefficients ε, ε'. Checking for the coefficients we find that

$$d_2(\mathscr{C}) = \zeta^2\mathscr{A}\mathscr{E}, \quad d_2(\mathscr{D}) = \zeta\mathscr{B}\mathscr{F}, \quad d_3(\mathscr{D}^2) = \zeta^2(\mathscr{B}^2\mathscr{E} + \mathscr{A}\mathscr{F}^2),$$
$$d_3(\mathscr{C}^2) = \zeta(\mathscr{A}^2\mathscr{F} + \mathscr{B}\mathscr{E}^2).$$

Consequently, we can write the E_3-term of the spectral sequence as

$$\mathbb{F}_4[\mathscr{C}^4, \mathscr{D}^4](1, \mathscr{C}^2, \mathscr{D}^2, \mathscr{C}^2\mathscr{D}^2) \otimes (\mathbb{F}_4[\mathscr{A}] \oplus \mathbb{F}_4[\mathscr{B}]\mathscr{B} \oplus \mathbb{F}_4[\mathscr{E}]\mathscr{E} \oplus \mathbb{F}_4[\mathscr{F}]\mathscr{F}$$
$$\oplus\, \mathbb{F}_4[\mathscr{A}, \mathscr{B}]\mathscr{A}\mathscr{B} \oplus \mathbb{F}_4[\mathscr{E}, \mathscr{B}]\mathscr{E}\mathscr{B}$$
$$\oplus\, \mathbb{F}_4[\mathscr{A}, \mathscr{F}]\mathscr{A}\mathscr{F} \oplus \mathbb{F}_4[\mathscr{E}, \mathscr{F}]\mathscr{E}\mathscr{F}).$$

When we do this, the differential on the E_3-term decomposes quite nicely. However, to explain it better, we first evaluate the differential only on the terms above cupped with $\mathscr{C}^2$. Then, after these homology groups have been determined, we take the homology of the resulting complex with respect to $\mathscr{D}^2$. Thus we have

$$\mathbb{F}_4[\mathscr{A}]\mathscr{A}\mathscr{C}^2 \mapsto \mathbb{F}_4[\mathscr{A}]\mathscr{A}^3\mathscr{F},$$
$$\mathbb{F}_4[\mathscr{B}]\mathscr{B}\mathscr{C}^2 \mapsto \mathbb{F}_4[\mathscr{B}]\mathscr{B}^2\mathscr{E}^2,$$
$$\mathbb{F}_4[\mathscr{E}]\mathscr{E}\mathscr{C}^2 \mapsto \mathbb{F}_4[\mathscr{E}]\mathscr{B}\mathscr{E}^3,$$
$$\mathbb{F}_4[\mathscr{F}]\mathscr{F}\mathscr{C}^2 \mapsto \mathbb{F}_4[\mathscr{F}]\mathscr{A}^2\mathscr{F}^2,$$
$$\mathbb{F}_4[\mathscr{E}, \mathscr{B}]\mathscr{E}\mathscr{B}\mathscr{C}^2 \mapsto \mathbb{F}_4[\mathscr{E}, \mathscr{B}]\mathscr{B}^2\mathscr{E}^3,$$
$$\mathbb{F}_4[\mathscr{A}, \mathscr{F}]\mathscr{A}\mathscr{F}\mathscr{C}^2 \mapsto \mathbb{F}_4[\mathscr{A}, \mathscr{F}]\mathscr{A}^3\mathscr{F}^2,$$
$$\mathbb{F}_4[\mathscr{A}, \mathscr{B}]\mathscr{A}\mathscr{B}\mathscr{C}^2 \mapsto 0,$$
$$\mathbb{F}_4[\mathscr{E}, \mathscr{F}]\mathscr{E}\mathscr{F}\mathscr{C}^2 \mapsto 0.$$

Clearly this gives us the quotient

$$\mathbb{F}_4[\mathscr{C}^4, \mathscr{D}^4](1, \mathscr{D}^2) \otimes (\mathbb{F}_4[\mathscr{A}, \mathscr{B}](1, \mathscr{A}\mathscr{B}\mathscr{C}^2) \oplus \mathbb{F}_4[\mathscr{E}, \mathscr{F}](1, \mathscr{E}\mathscr{F}\mathscr{C}^2) \oplus (\mathscr{A}^2\mathscr{F} = \mathscr{B}\mathscr{E}^2, \mathbb{F}_4[\mathscr{F}]\mathscr{A}\mathscr{F} \oplus \mathbb{F}_4[\mathscr{B}]\mathscr{B}\mathscr{E})).$$

Note that $d_3(\mathscr{A}\mathscr{B}\mathscr{C}^2\mathscr{D}^2) = d_3(\mathscr{E}\mathscr{F}\mathscr{C}^2\mathscr{D}^2) = 0$, and, if we write

$$\mathbb{F}_4[\mathscr{A}, \mathscr{B}] = 1 \oplus \mathbb{F}_4[\mathscr{A}]\mathscr{A} \oplus \mathbb{F}_4[\mathscr{B}]\mathscr{B} \oplus \mathbb{F}_4[\mathscr{A}, \mathscr{B}]\mathscr{A}\mathscr{B},$$

we have

$$d_3(\mathbb{F}_4[\mathscr{A}]\mathscr{A}\mathscr{D}^2) = \mathbb{F}_4[\mathscr{A}]\mathscr{A}^2\mathscr{F}^2,$$
$$d_3(\mathbb{F}_4[\mathscr{B}]\mathscr{B}\mathscr{D}^2) = \mathbb{F}_4[\mathscr{B}]\mathscr{E}\mathscr{B}^3,$$
$$d_3(\mathbb{F}_4[\mathscr{A}, \mathscr{B}]\mathscr{A}\mathscr{B}\mathscr{C}^2) = 0.$$

Similarly for $\mathbb{F}_4[\mathscr{E}, \mathscr{F}]\mathscr{D}^2$. But $\mathscr{A}^2\mathscr{F}^2$ has already been killed using $\mathscr{C}^2$ and similarly for $\mathscr{B}^2\mathscr{E}^2$. Consequently, there are two cycles in dimension 3:

$$L = \mathscr{E}\mathscr{D}^2 + \mathscr{B}\mathscr{C}^2 \quad \text{and} \quad M = \mathscr{A}\mathscr{D}^2 + \mathscr{F}\mathscr{C}^2.$$

Additionally, the quotient by the boundary maps above consists of four elements,

$$\mathscr{B}\mathscr{E}, \ \mathscr{A}\mathscr{F}, \ \mathscr{F}\mathscr{A}^2 = \mathscr{B}\mathscr{E}^2, \ \mathscr{E}\mathscr{B}^2 = \mathscr{A}\mathscr{F}^2,$$

and we have the products $\mathscr{A}L = \mathscr{A}\mathscr{B}\mathscr{C}^2$, $\mathscr{F}L = \mathscr{E}\mathscr{F}\mathscr{D}^2$, $\mathscr{B}M = \mathscr{A}\mathscr{B}\mathscr{D}^2$, and $\mathscr{E}M = \mathscr{E}\mathscr{F}\mathscr{C}^2$. The terms

$$\mathscr{F}\mathscr{A}^2\mathscr{D}^2 = \mathscr{B}\mathscr{E}^2\mathscr{D}^2, \ \mathbb{F}_4[\mathscr{F}]\mathscr{A}\mathscr{F}\mathscr{D}^2 \oplus \mathbb{F}_4[\mathscr{B}]\mathscr{B}\mathscr{E}\mathscr{D}^2$$

are also infinite cycles, and from the above remarks are divisible by L or M. Finally, note that

$$LM = (\mathscr{E}\mathscr{F} + \mathscr{A}\mathscr{B})\mathscr{C}^2\mathscr{D}^2,$$

so

$$\mathbb{F}_4[\mathscr{A}, \mathscr{B}]\mathscr{A}\mathscr{B}\mathscr{C}^2\mathscr{D}^2 \oplus \mathbb{F}_4[\mathscr{E}, \mathscr{F}]\mathscr{E}\mathscr{F}\mathscr{C}^2\mathscr{D}^2 = (\mathbb{F}_4[\mathscr{A}, \mathscr{B}] \oplus \mathbb{F}_4[\mathscr{E}, \mathscr{F}])LM \oplus \mathscr{A}\mathscr{B}\mathscr{C}^2\mathscr{D}^2.$$

Putting these results together we have proved Benson's result correcting the original calculation of Yagita:

PROPOSITION 2.1. *The E_4-term of the spectral sequence, after tensoring with $\mathbb{F}_4$, is an algebra on nine generators, four in dimension 1, $\mathscr{E}$, $\mathscr{F}$, $\mathscr{A}$ and $\mathscr{B}$, two in dimension 3, L, M, two in dimension 4, $v = \mathscr{C}^4$, $w = \mathscr{D}^4$, and one in dimension 6, $\mathscr{A}\mathscr{B}\mathscr{C}^2\mathscr{D}^2$. The relations are $\mathscr{A}\mathscr{E} = \mathscr{B}\mathscr{F} = 0$, $\mathscr{F}\mathscr{A}^2 = \mathscr{B}\mathscr{E}^2$, $\mathscr{E}\mathscr{B}^2 = \mathscr{A}\mathscr{F}^2$, $\mathscr{A}\mathscr{F}M = \mathscr{B}\mathscr{E}L$, $\mathscr{A}\mathscr{F}L = \mathscr{B}\mathscr{E}M = 0$, and $L^2 = \mathscr{B}^2\mathscr{C}^4 + \mathscr{E}^2\mathscr{D}^4$, $M^2 = \mathscr{A}^2\mathscr{D}^4 + \mathscr{F}^2\mathscr{C}^4$, while $\mathscr{E}\mathscr{A}\mathscr{B}\mathscr{C}^2\mathscr{D}^2 = \mathscr{F}\mathscr{A}\mathscr{B}\mathscr{C}^2\mathscr{D}^2 = 0$ and $\mathscr{A}\mathscr{A}\mathscr{B}\mathscr{C}^2\mathscr{D}^2 = \mathscr{A}LM$, $\mathscr{B}\mathscr{A}\mathscr{B}\mathscr{C}^2\mathscr{D}^2 = \mathscr{B}LM$.*

Moreover, it is clear that there can be no further differentials since the only possible candidate is a d_7-differential on $\mathscr{A}\mathscr{B}\mathscr{C}^2\mathscr{D}^2$. However, its image can only lie on the bottom row, and there, can only lie in the subalgebra $\mathbb{F}_4[\mathscr{A}, \mathscr{B}]$. But this is carried by the group $\langle A, B\rangle = \mathbb{Z}/2 \times \mathbb{Z}/2$, and since this is a split summand of $\mathrm{Syl}_2(L_3(4))$, there cannot be any such differential. For explicitness and later use we give the cohomology ring expanded out:

$$(2.2) \quad H^*(\mathrm{Syl}_2(L_3(4))) \cong \mathbb{F}_4[v, w] \otimes \{(\mathbb{F}_4[\mathscr{A}, \mathscr{B}] \oplus \mathbb{F}_4[\mathscr{E}, \mathscr{F}])(1, L, M, LM) \oplus (\mathscr{A}\mathscr{F}, \mathscr{B}\mathscr{E}, \mathscr{A}\mathscr{F}^2, \mathscr{F}\mathscr{A}^2, \mathscr{A}\mathscr{F}M, \mathscr{A}\mathscr{B}\mathscr{C}^2\mathscr{D}^2)\}.$$

Note that the restriction map to the subgroup $\langle A, B, C, D\rangle = (\mathbb{Z}/2)^4$ injects on L, M, LM and the polynomial algebra $\mathbb{F}_4[\mathcal{A}, \mathcal{B}, \mathcal{C}^4, \mathcal{D}^4]$ since it does so in the E_4-terms of the respective spectral sequences. Of course this does not directly calculate the exact images of these elements. However, these images can easily be obtained from the next result, which can be found in **[14]**. However, for completeness, we give a proof as it is representative of several similar situations which we will encounter as we proceed.

PROPOSITION 2.3. *The group $\langle E, F\rangle$ acts on $\langle A, B, C, D\rangle$ and this action induces an action of $\langle E, F\rangle$ on $\mathbb{F}_2[a, b, c, c]$. Moreover, the subring of invariants has the form*

$$\mathbb{F}_2[a, b, c, d]^{\langle E,F\rangle} = \mathbb{F}_2[a, b, v_4, w_4](1, M_3, L_3, L_3M_3),$$

and thus the image of the restriction map is exactly the subring of $\langle E, F\rangle$ invariants.

Proof. The action is given explicitly in cohomology by $E(c) = c + a$, $E(d) = d + b$, $F(c) = c + b$, $F(d) = d + a + b$, while both E and F fix a, b. We proceed by first determining the E invariants, and then, in this subring, the F invariants. The determination of the E invariants is quite standard,

$$\mathbb{F}_2[a, b, c, d]^{\langle E\rangle} = \mathbb{F}_2[a, b, c(c + a), d(d + b)](m),$$

where $m = ad + cb$. For convenience, write $R = c(c + a)$, $S = d(d + b)$. Then we find that

$$F(R) = R + b(a + b),$$
$$F(S) = S + a(a + b),$$
$$F(m) = m + a^2 + b^2 + ab.$$

Set $v = R(R + b(a + b))$, $w = S(S + a(a + b))$; then we can rewrite the E invariants above as

$$\mathbb{F}_2[a, b, c, d]^{\langle E\rangle} = \mathbb{F}_2[v, w, a, b](1, R, S, RS) \oplus \mathbb{F}_2[v, w, a, b](m, mR, mS, mRS).$$

Note that the action of F takes the subalgebra $\mathbb{F}_2[v, w, a, b](1, R, S, RS)$ to itself and that if we factor out by this subalgebra, then the induced action of F on the quotient, $\mathbb{F}_2[v, w, a, b](m, mR, mS, mRS)$ make it into an isomorphic copy of the previous algebra, just shifted up by m. Consequently, if we write the subalgebra as $\mathcal{H}$, we have the exact sequence

$$\mathcal{H}^{\langle F\rangle} \to \mathbb{F}_2[a, b, c, d]^{\langle E,F\rangle} \to \mathcal{H}^{\langle F\rangle}m \xrightarrow{\delta} H^1(\langle F\rangle ; \mathcal{H}) \to \dots.$$

We proceed to determine $\mathcal{G} = \mathbb{F}_2[v, w, a, b](1, R, S, RS)^{\langle F\rangle}$. Note that $F + 1$ is a derivation. Consequently, we filter $\mathcal{G}$ by $\mathbb{F}_2[v, w, a, b] \subset \mathbb{F}_2[v, w, a, b](1, R, S) \subset \mathcal{G}$ obtaining

$$\mathrm{Ker}(F + 1)\colon\ \mathbb{F}_2[v, w, a, b](R, S) \to \mathbb{F}_2[v, w, a, b] = \mathbb{F}_2[v, w, a, b](aR + bS),$$

and $F + 1$: $\mathbb{F}_2[v, w, a, b]RS \to \mathbb{F}_2[v, w, a, b](R, S)$ is an injection since

$$(F + 1)(RS) = (a + b)(aR + bS).$$

It follows, on writing $L = aR + bS$, that the invariants here are $\mathbb{F}_2[v, w, a, b](1, L)$. Now, we apply the exact sequence above to get the entire set of invariants,

$$0 \to \mathbb{F}_2[v, w, a, b](1, L) \to \mathscr{G}^{\langle F\rangle} \to \mathbb{F}_2[v, w, a, b](1, L)m \xrightarrow{\delta} \mathbb{F}_2[v, w, a, b](1, L)/(\mathrm{im}(1+F)),$$

where $\delta(m) = \{a^2 + b(a+b)\} \sim a^2$. The image of $(1+F)$ is generated by $a(a+b)$, $b(a+b)$ and $(a+b)L$. Consequently,

$$\mathbb{F}_2[v, w, a, b](1, L)/(\mathrm{im}(1+F)) = \mathbb{F}_2[v, w, a] \oplus \mathbb{F}_2[v, w](a+b) \oplus \mathbb{F}_2[v, w, a]L.$$

As a direct result, the kernel of δ is exactly the submodule

$$\mathbb{F}_2[v, w, a, b](a+b)(1, L)m = \mathbb{F}_2[v, w, a, b](M, ML),$$

where $M = (a+b)m$. From this the result follows.

3. *The cohomology of* $2^4 : \mathscr{A}_4$, $2^4 : \mathscr{A}_5$, *and* $L_3(4)$

We consider the extension $(\mathbb{Z}/2)^4 : \mathscr{A}_4$, given as the subgroup of $L_3(4)$ consisting of all matrices of the form $\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & 1 \end{pmatrix}$, and $(\mathbb{Z}/2)^4 : \mathscr{A}_5$, the subgroup of $L_3(4)$ given as all matrices of the form $\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 1 \end{pmatrix}$. From (1.3) the cohomology of the latter is given as

$$H^*(\mathrm{Syl}_2(L_3(4)))^{\mathbb{Z}/3} \cap \mathrm{res}^{-1}(H^*(\langle A, B, C, D\rangle))^{\mathscr{A}_5}.$$

Similarly, the extension $(\mathbb{Z}/2)^4 : \mathscr{A}_4 = N_{L_3(4)}(\mathrm{Syl}_2(L_3(4)))$ has its cohomology determined as

$$H^*((\mathbb{Z}/2)^4 : \mathscr{A}_4) = H^*(\mathrm{Syl}_2(L_3(4)))^{\mathbb{Z}/3}.$$

From (2.2), the image of restriction from $H^*(\mathrm{Syl}_2(L_3(4)))$ is exactly the ring of invariants $H^*(\langle A, B, C, D\rangle)^{\langle E, F\rangle}$. From this it follows that

$$H^*(\langle A, B, C, D\rangle)^{\mathscr{A}_5} \subset \mathrm{im}(\mathrm{res}^*),$$

and $H^*((\mathbb{Z}/2)^4 : \mathscr{A}_5) \cong \mathrm{Ker}(\mathrm{res}^*)^{\mathbb{Z}/3} \oplus H^*(\langle A, B, C, D\rangle)^{\mathscr{A}_5}$. From (2.1) we see that the kernel of the restriction map can be expanded out, after tensoring with $\mathbb{F}_4$, as

$$\mathbb{F}_4[v, w] \otimes \{(\mathbb{F}_4[\mathscr{E}]\mathscr{E} \oplus \mathbb{F}_4[\mathscr{E}, \mathscr{F}]\mathscr{F})(1, L, M, LM) \oplus (\mathscr{A}\mathscr{F}, \mathscr{B}\mathscr{E}, \mathscr{A}\mathscr{F}^2, \mathscr{F}\mathscr{A}^2, \mathscr{A}\mathscr{F}M, LM + \mathscr{A}\mathscr{B}\mathscr{C}^2\mathscr{D}^2)\}.$$

Now, L, M, $\mathscr{A}\mathscr{F}$, $\mathscr{B}\mathscr{E}$, $\mathscr{A}\mathscr{F}M$, and $\mathscr{A}\mathscr{B}\mathscr{C}^2\mathscr{D}^2$ are invariant under T, while $\mathscr{A}\mathscr{F}^2$ is an eigenvector with eigenvalue ζ_3^2, and $\mathscr{F}\mathscr{A}^2$ is an eigenvector with eigenvalue ζ_3. We now determine the set of T-invariants in this kernel.

First, the invariants of $\mathbb{F}_4[v, w]$ under the action of T are given as $\mathbb{F}_4[v, w]^{\mathbb{Z}/3} = \mathbb{F}_4[v^3, w^3](1, vw, v^2w^2)$. Similarly, the elements which have eigenvalue ζ_3^2 are of the form $\mathbb{F}_4[v^3, w^3](v, v^2w, w^2)$, and those with eigenvalue ζ_3 are of the form $\mathbb{F}_4[v^3, w^3](w, w^2v, v^2)$. Thus the invariants involved in the part of the sum above

$$\mathbb{F}_4[v, w] \otimes \{(\mathscr{A}\mathscr{F}, \mathscr{B}\mathscr{E}, \mathscr{A}\mathscr{F}^2, \mathscr{F}\mathscr{A}^2, \mathscr{A}\mathscr{F}M, LM + \mathscr{A}\mathscr{B}\mathscr{C}^2\mathscr{D}^2)\}$$

have the form

$$\mathbb{F}_4[v^3, w^3](1, vw, v^2w^2)(\mathscr{A}\mathscr{F}, \mathscr{B}\mathscr{E}, \mathscr{A}\mathscr{F}M, LM + \mathscr{A}\mathscr{B}\mathscr{C}^2\mathscr{D}^2) \oplus \mathbb{F}_4[v^3, w^3](v, v^2w, w^2)\mathscr{F}\mathscr{A}^2 \oplus \mathbb{F}_4[v^3, w^3](w, vw^2, v^2)\mathscr{A}\mathscr{F}^2.$$

The invariants $\mathbb{F}_4[v, w, \mathscr{E}, \mathscr{F}]^{\mathbb{Z}/3}$ are generated as a free $\mathbb{F}_4[v^3, w^3, \mathscr{E}^3, \mathscr{F}^3]$-module by the 27 elements

$$1, vw, \mathscr{E}\mathscr{F}, v\mathscr{F}, \mathscr{E}w, v^2\mathscr{E}, w^2\mathscr{F}, \mathscr{E}^2v, \mathscr{F}^2w, v^2w^2, \mathscr{E}^2\mathscr{F}^2, v^2\mathscr{F}^2, \mathscr{E}^2w^2,$$
$$vw\mathscr{E}\mathscr{F}, v^2w\mathscr{F}, w^2v\mathscr{E}, \mathscr{F}\mathscr{E}^2w, \mathscr{E}\mathscr{F}^2v, v^2\mathscr{F}\mathscr{E}^2, v^2w\mathscr{E}^2, w^2v\mathscr{F}^2,$$
$$w^2\mathscr{E}\mathscr{F}^2, v^2w^2\mathscr{E}^2\mathscr{F}^2, vw\mathscr{E}^2\mathscr{F}^2, v^2w^2\mathscr{E}\mathscr{F}, vw^2\mathscr{F}\mathscr{E}^2, v^2w\mathscr{E}\mathscr{F}^2,$$

and this ring of invariants has Poincaré series

$$\frac{1 + x^2 + x^4 + 2x^5 + 2x^6 + 2x^7 + x^8 + 2x^9 + 3x^{10} + 2x^{11} + x^{12} + 2x^{13} + 2x^{14} + 2x^{15} + x^{16} + x^{18} + x^{20}}{(1 - x^3)^2(1 - x^{12})^2},$$

which simplifies to

$$p(x) = \frac{1 + x^2 + 2x^5 + x^6 + 2x^7 + x^{10} + x^{12}}{(1 - x^3)^2(1 - x^4)(1 - x^{12})}.$$

The Poincaré series for the invariants in $\mathbb{F}_4[v^4, w^4]$ is

$$q(x) = \frac{1 + x^8 + x^{16}}{(1 - x^{12})^2},$$

and consequently, the Poincaré series for the invariants in the kernel above becomes

$$(p(x) - q(x))(1 + x^3)^2 + (2x^2 + x^5 + x^6)q(x) + \left(\frac{1}{(1 - x^4)^2} - q(x)\right)x^3.$$

This expands out as

$$\mathscr{K}(x) = \frac{x^{16} - x^{15} - 2x^{14} + 6x^{13} - 6x^{12} + 5x^{11} + x^{10} - 4x^9 + 4x^8 - x^7 + x^6 + 3x^4 - 4x^3 + 3x^2}{(1 + x^2)(1 - x)^2(1 - x^3)(1 - x^{12})},$$

so the Poincaré series for $H^*((\mathbb{Z}/2)^4 : \mathscr{A}_4)$ is

(3.1) $$\mathscr{Q}(x) = (1 + x^3)^2p(x) + \mathscr{K}(x) = \frac{x^{16} - 4x^{14} + 9x^{13} - 7x^{12} + 3x^{11} + 8x^{10} - 11x^9 + 8x^8 + 3x^7 - 6x^6 + 7x^5 + x^4 - 5x^3 + 6x^2 - 2x + 1}{(1 + x^2)(1 - x)^2(1 - x^3)(1 - x^{12})},$$

and the Poincaré series for $H^*((\mathbb{Z}/2)^4 : \mathscr{A}_5)$ is $L(x) + \mathscr{K}(x)$ where $L(x)$ is the Poincaré series for $\mathbb{F}_4[\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{D}]^{\mathscr{A}_5}$.

THEOREM 3.2. *The series* $L(x)$ *is given as*

$$\frac{\begin{array}{c}1+2x^3+3x^6+x^8+6x^9+2x^{11}+9x^{12}+x^{14}+10x^{15}\\ +x^{16}+9x^{18}+2x^{19}+6x^{21}+x^{22}+3x^{24}+2x^{27}+x^{30}\end{array}}{(1-x^5)^2(1-x^{12})^2}.$$

Indeed, $\mathbb{F}_4[\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}]^{\mathcal{A}_5}$ *is generated by nine elements*

$$\mathcal{A}^4\mathcal{C}+\mathcal{A}\mathcal{C}^4,\quad \mathcal{B}^4\mathcal{D}+\mathcal{B}\mathcal{D}^4,\quad \mathcal{A}^{12}+\mathcal{A}^9\mathcal{C}^3+\mathcal{A}^6\mathcal{C}^6+\mathcal{A}^3\mathcal{C}^9+\mathcal{C}^{12},$$

$$\mathcal{B}^{12}+\mathcal{B}^9\mathcal{D}^3+\mathcal{B}^6\mathcal{D}^6+\mathcal{B}^3\mathcal{D}^9+\mathcal{D}^{12},$$

$$M=\mathcal{A}^2\mathcal{D}+\mathcal{C}^2\mathcal{B},\quad L=\mathcal{A}\mathcal{D}^2+\mathcal{C}\mathcal{B}^2,$$

$$C=\mathcal{C}^4\mathcal{D}^4+\mathcal{C}^4\mathcal{B}^3\mathcal{D}+\mathcal{A}\mathcal{C}^3\mathcal{B}^4+\mathcal{A}^2\mathcal{C}^2\mathcal{B}^2\mathcal{D}^2+\mathcal{A}^3\mathcal{C}\mathcal{D}^4+\mathcal{A}^4\mathcal{B}\mathcal{D}^3+\mathcal{A}^4\mathcal{B}^4,$$

$$D=\mathcal{A}\mathcal{D}^8+\mathcal{C}\mathcal{B}^8,\quad E=\mathcal{A}^8\mathcal{D}+\mathcal{C}^8\mathcal{D}.$$

The first four generate a polynomial algebra on four variables, $\mathcal{P}$, *and the entire ring of invariants is free over* $\mathcal{P}$ *on sixty generators. A complete list of these generators is given in Table* 2.

(We defer the proof to § 5. There are two main steps. The first is to reduce the determination of the invariants to the calculation of the invariants in 225 relatively small modules over $\mathbb{F}_4[\mathcal{A}_5]$. Then these invariants are determined via a computer calculation, and a great many additional sieve-type calculations are applied to reduce the answer to the form above.)

TABLE 2

Dimension	Module generators
0	1
3	L, M
6	L^2, LM, M^2
8	C
9	$D, L^3, L^2M, LM^2, M^3, E$
11	LC, MC
12	$LD, MD, L^4, L^3M, L^2M^2, LM^3, M^4, LE, ME$
14	LMC
15	$L^2D, LMD, M^2D, L^4M, L^3M^2, L^2M^3, LM^4, L^2E, LME, M^2E$
16	C^2
18	$L^3D, L^2MD, LM^2D, M^3D, DE, L^3E, L^2ME, LM^2E, M^3E$
19	LC^2, MC^2
21	$L^2M^2D, M^4D, L^4M^3, L^3M^4, L^4E, L^2M^2E$
22	LMC^2
24	L^5M^3, L^4M^4, L^3M^5
27	L^5M^4, L^4M^5
30	L^5M^5

COROLLARY 3.3. *The Poincaré series for $H^*((\mathbb{Z}/2)^4 : \mathcal{A}_5)$ is given as the series*

$$\frac{\begin{array}{c}(x^{21}+2x^{20}-2x^{19}+2x^{18}+3x^{17}-x^{16}+7x^{15}+7x^{14}+x^{13}+7x^{12}\\ +8x^{11}+x^{10}+9x^9+8x^8+x^7+6x^6+5x^5+3x^3+3x^2+1)\end{array}}{(x^3-1)(x^4-1)(x^5-1)(x^{12}-1)}.$$

Using (1.3), we obtain

THEOREM 3.4. *The Poincaré series for the cohomology of $L_3(4)$ is given by*

$$\frac{\begin{array}{c}(x^{21}+2x^{20}-2x^{19}+x^{18}+x^{17}-2x^{16}+4x^{15}+3x^{14}-x^{13}+2x^{12}\\ +3x^{11}-x^{10}+4x^9+3x^8-x^7+2x^6+2x^5-x^4+x^3+2x^2+1)\end{array}}{(x^3-1)(x^4-1)(x^5-1)(x^{12}-1)}.$$

We now describe the explicit cohomology ring $H^*(L_3(4);\mathbb{F}_4) \cong H^*(L_3(4);\mathbb{F}_2) \otimes_{\mathbb{F}_2} \mathbb{F}_4$ as far as we need it. Let

$$V_1 = \left\{ \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \middle| \; * \in \mathbb{F}_4 \right\} \cong (\mathbb{Z}/2)^4,$$

$$V_2 = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \; * \in \mathbb{F}_4 \right\} \cong (\mathbb{Z}/2)^4$$

be representatives for the two non-conjugate copies of $(\mathbb{Z}/2)^4$ in $L_3(4)$. We have

$$C = V_1 \cap V_2 = \left\{ \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} \cong (\mathbb{Z}/2)^2,$$

the centre of the Sylow 2-subgroup, $\mathrm{Syl}_2(L_3(4))$.

LEMMA 3.5. *The restriction maps*

$$H^*(L_3(4)) \xrightarrow{\mathrm{res}^{V_i}} H^*(V_i) \xrightarrow{\mathrm{res}^{C}} H^*(V_1 \cap V_2)$$

have image

$$\mathbb{F}_4[\mathscr{C}^4, \mathscr{D}^4]^{\mathbb{Z}/3} \cong \mathbb{F}_4[\mathscr{C}^{12}, \mathscr{D}^{12}](1, \mathscr{C}^4\mathscr{D}^4, \mathscr{C}^8\mathscr{C}^8)$$

when cohomology is taken with $\mathbb{Z}/4$ as coefficients.

Proof. This follows directly from Theorem 3.2. Indeed, the generator of dimenson 8, $\mathscr{C}^4\mathscr{D}^4 + \ldots$, restricts to $\mathscr{C}^4\mathscr{D}^4$, and the two generators of dimension 12 restrict to $\mathscr{C}^{12}$ and $\mathscr{D}^{12}$, respectively. Of course, the image must lie in the $(\mathbb{Z}/3)$-invariants since the $\mathbb{Z}/3$ which normalizes the Sylow 2-subgroup acts non-trivially on the centre.

Also from our discussion in the beginning of this section we have that the kernel $\mathcal{K}$ of $(\text{res}^{V_1})^* \oplus (\text{res}^{V_2})^*$: $H^*(L_3(4)) \to H^*(V_1) \oplus H^*(V_2)$ is

$$\mathbb{F}_4[v^3, w^3](w\mathcal{A}\mathcal{F}^2, w^2v\mathcal{A}\mathcal{F}^2, v^2\mathcal{A}\mathcal{F}^2, v\mathcal{F}\mathcal{A}^2, v^2w\mathcal{F}\mathcal{A}^2, w^2\mathcal{F}\mathcal{A}^2) \oplus \mathbb{F}_4[v, w]^{\mathbb{Z}/3}(\mathcal{A}\mathcal{F}, \mathcal{B}\mathcal{E}, \mathcal{A}\mathcal{F}M).$$

Finally, there is clearly a *zero* sequence

$$0 \to \mathcal{K} \to H^*(L_3(4)) \xrightarrow{\text{res}^{V_1} \oplus \text{res}^{V_2}} H^*(V_1)^{\mathcal{A}_5} \oplus H^*(V_2)^{\mathcal{A}_5} \xrightarrow{\text{res}^C + \text{res}^C} \mathbb{F}_4[v^3, w^3](1, vw, v^2w^2) \to 0$$

but it is not exact. The reason is the fact that $\mathbb{F}_4[v, w]L$ and $\mathbb{F}_4[v, w]M$ inject diagonally into the two rings, while $\mathcal{A}\mathcal{F}^2$, $\mathcal{B}\mathcal{E}^2$ both map to zero. On the other hand, there are elements in $H^*(L_3(4))$ which independently hit both copies of $\mathbb{F}_4[v, w]LM$ due to the element $\mathcal{A}\mathcal{B}\mathcal{C}^2\mathcal{D}^2$. When we take this into account we find the exact sequence

$$(3.6) \quad 0 \to \mathcal{K} \to H^*(L_3(4)) \xrightarrow{\text{res}^{V_1} \oplus \text{res}^{V_2}} H^*(V_1)^{\mathcal{A}_5} \oplus H^*(V_2)^{\mathcal{A}_5} \to \mathbb{F}_4[v^3, w^3](1, vw, v^2w^2, L, M, Lvw, Mvw, Lv^2w^2, Mv^2w^2) \to 0,$$

which effectively describes the ring $H^*(L_3(4))$.

4. *The groups* $2^4 : \mathcal{A}_4 : 2_2$, $2^4 : \mathcal{A}_5 : 2_2$ *and* $L_3(4) : 2_2$

We now study $L_3(4) : 2_2$. Here, as mentioned in the introduction, 2_2 is the automorphism $2_2 : g \mapsto g^\phi$, where ϕ: $\mathbb{F}_4 \to \mathbb{F}_4$ is the Galois automorphism $(\phi(\zeta) = \zeta^2)$ and if

$$g = \left\{\begin{pmatrix} a & b & c \\ d & e & f \\ h & i & j \end{pmatrix}\right\} \quad \text{then} \quad g^\phi = \left\{\begin{pmatrix} a^\phi & b^\phi & c^\phi \\ d^\phi & e^\phi & f^\phi \\ h^\phi & i^\phi & j^\phi \end{pmatrix}\right\}.$$

Now $L_3(4) : 2_2$ is maximal and has odd index in the sporadic groups M_{23} and $\text{M}^\text{c}\text{L}$. In M_{22} the group $L_3(4)$ is maximal (it is M_{21}) but not of odd index. There are two maximal 2-locals in M_{22} which are of odd index. They are $2^4 : \mathcal{A}_6$ and $2^4 : \mathcal{A}_5 : 2_2 = 2^4 : \mathcal{S}_5$ with intersection $2^4 : \mathcal{A}_4 : 2_2 = 2^4 : \mathcal{S}_4$. We determine $H^*(2^4 : \mathcal{S}_4)$ and $H^*(2^4 : \mathcal{S}_5)$ completely here but leave $H^*(2^4 : \mathcal{A}_6)$ to a sequel since it is not a subgroup of $L_3(4) : 2_2$. Additionally, we determine $H^*(L_3(4) : 2_2)$. The procedure is to first determine $H^*(\text{Syl}_2(M_{22}))$. This is done using the Lyndon spectral sequence for the extension which we show collapses at E_2. Then the rings $H^*(2^4 : \mathcal{S}_4)$ and $H^*(2^4 : \mathcal{S}_5)$ are studied using double coset decompositions. Finally, using similar techniques we determine $H^*(L_3(4) : 2_2)$.

The 2-Sylow subgroup of $L_3(4) : 2_2$

The action of 2_2 in $H^*(\text{Syl}_2(L_3(4))) \otimes \mathbb{F}_4$ is determined by $\mathcal{A} \leftrightarrow \mathcal{B}$, $\mathcal{C} \leftrightarrow \mathcal{D}$, $\mathcal{E} \leftrightarrow \mathcal{F}$. In particular, $L \leftrightarrow M$, $\mathcal{A}\mathcal{F}M$, $\mathcal{A}\mathcal{B}\mathcal{C}^2\mathcal{D}^2$ are fixed and $\mathcal{A}\mathcal{F} \leftrightarrow \mathcal{B}\mathcal{E}$, $\mathcal{A}\mathcal{F}^2 \leftrightarrow \mathcal{F}\mathcal{A}^2$, while $v \leftrightarrow w$. Consequently, the Lyndon spectral sequence for

$\mathrm{Syl}_2(M_{22}) = \mathrm{Syl}_2(L_3(4)) : 2_2$ has E_2-term

$$E_2^{i,j} \otimes \mathbb{F}_4 = H^i_{2_2}(\mathbb{Z}/2\,; H^*(\mathrm{Syl}_2(L_3(4))) \otimes \mathbb{F}_4)$$

and this has the explicit form

$$\begin{aligned}
(4.1)\quad & \mathbb{F}_4[v+w, vw]\{\mathbb{F}_4[\mathscr{A}+\mathscr{B}, \mathscr{A}\mathscr{B}](\mathscr{A}v+\mathscr{B}w) \\
& \oplus \mathbb{F}_4[\mathscr{E}+\mathscr{F}, \mathscr{E}\mathscr{F}](\mathscr{E}v+\mathscr{F}w)\}(1, LM) \\
& \oplus \mathbb{F}_4[v, w]\{\mathbb{F}_4[\mathscr{A}, \mathscr{B}] \oplus \mathbb{F}_4[\mathscr{E}, \mathscr{F}]\}L \\
& \oplus \mathbb{F}_4[v, w](\mathscr{A}\mathscr{F}, \mathscr{A}\mathscr{F}^2) \\
& \oplus \{\mathbb{F}_4[v+w, vw, h]/(h(v+w)=0)\}(LM + \mathscr{A}\mathscr{B}\mathscr{C}^2\mathscr{D}^2, \mathscr{A}\mathscr{F}M) \\
& \oplus \{\mathbb{F}_4[vw, \mathscr{A}\mathscr{B}, \mathscr{E}\mathscr{F}, h]/(\mathscr{A}\mathscr{B}\cdot\mathscr{E}\mathscr{F}=0)\}(h, LMh).
\end{aligned}$$

Here h is dual to the element of order 2 representing the automorphism 2_2 (which we usually write 2_2) and h has filtration $(1,0)$ while every other generator θ has filtration $(0, \dim(\theta))$. Consequently, the only possible differentials occur on elements not annihilated by h. The generators of this subalgebra are vw, $\mathscr{A}\mathscr{B}$, $\mathscr{E}\mathscr{F}$, $\mathscr{A}\mathscr{F}M$, $\mathscr{A}\mathscr{B}\mathscr{C}^2\mathscr{D}^2$ and LM. It is not hard to see that vw restricts to the cohomology of the centre $C(\mathrm{Syl}_2(M_{22})) = \mathbb{Z}/2$ as κ^8, which, by standard Stiefel–Whitney class calculations for the representation ring, can be shown to be actually in the image. From this it follows that vw must be an infinite cycle. Also, since the extension is split, it follows that $\mathbb{F}_2[h] \subset H^*(\mathrm{Syl}_2(M_{22}))$ and this implies that $\mathscr{A}\mathscr{B}$, $\mathscr{E}\mathscr{F}$ are also infinite cycles. Hence, the first possible differentials occur on $\mathscr{A}\mathscr{F}M$, $\mathscr{A}\mathscr{B}\mathscr{C}^2\mathscr{D}^2$ or LM. Moreover, if these are infinite cycles then the spectral sequence must collapse.

LEMMA 4.2. *In the Lyndon spectral sequence above we have $E_2 = E_\infty$.*

Proof. The proof is based on the inclusions of the index-2 subgroup,

$$V = \left\langle \mu_2, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\rangle,$$

and its reflection through the anti-diagonal,

$$V' = \left\langle \mu_2, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle,$$

in $\mathrm{Syl}_2(M_{22})$. The groups V and V' are isomorphic, so we concentrate on V. Let a basis for $(\mathbb{Z}/2)^4 = \left\{ \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\}$ be given as 3×3 matrices with third columns

$$e_1 = \begin{pmatrix} 0 \\ \zeta_3 \\ 1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} \zeta_3 \\ \zeta_3 \\ 1 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ \zeta_3^2 \\ 1 \end{pmatrix}, \quad e_4 = \begin{pmatrix} \zeta_3^2 \\ \zeta_3^2 \\ 1 \end{pmatrix}$$

while the block matrix $\begin{pmatrix} I \\ 0 \end{pmatrix}$ is the first two columns in each case. Then $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ acts as $e_1 \leftrightarrow e_2$, $e_3 \leftrightarrow e_4$, while μ_2 acts as $e_1 \leftrightarrow e_3$, $e_2 \leftrightarrow e_4$. Consequently, we have constructed an embedding $V \hookrightarrow D_8 \wr \mathbb{Z}/2$ sending μ_2 to $(1, \tau)$ and $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ to $((t, t), 1)$.

Any index-2 subgroup V of a group π is determined by and determines a homomorphism $\phi_V\colon \pi \to \mathbb{Z}/2$ and passing to cohomology ϕ_V determines a cohomology class $e_V \in H^1(\pi)$ as $\phi_V^*(e)$ where e is the non-zero class in $H^1(\mathbb{Z}/2)$. In turn there is the Gysin sequence connecting $H^*(\pi)$ and $H^*(V)$,

$$\cdots \xrightarrow{\cup e_V} H^*(\pi) \xrightarrow{\text{res}} H^*(V) \xrightarrow{\text{tr}} H^*(\pi) \xrightarrow{\cup e_V} H^{*+1}(\pi) \longrightarrow \ldots,$$

which allows us to determine $H^*(V)$ from the action of e_V in $H^*(\pi)$.

We have $H^*(D_8) = \mathbb{F}_2[f, x, y]/(xy = 0)$, with $\dim(f) = 2$, $\dim(x) = \dim(y) = 1$ and

$$H^*(D_8 \wr \mathbb{Z}/2) = (H^*(D_8) \otimes H^*(D_8))^{\mathbb{Z}/2} \oplus_W \mathbb{F}_2[\Gamma(f), \Gamma(x), \Gamma(y), h']/(\Gamma(x)\Gamma(y) = 0),$$

where $W = \mathbb{F}_2[f \otimes f, x \otimes x, y \otimes y]/(x \otimes x \cdot y \otimes y = 0)$ and the restriction of $\Gamma(f)$ is $f \otimes f$, that of $\Gamma(x)$ is $x \otimes x$, and that of $\Gamma(y)$ is $y \otimes y$.

The class e_V is $x \otimes 1 + 1 \otimes x$ and since no element of W is in the image of $\cup e_V$, it follows that res: $\mathbb{F}_2[\Gamma(f), \Gamma(x), \Gamma(y), h']/(\Gamma(x)\Gamma(y) = 0)$ injects into $H^*(V)$. Clearly $\text{res}(h') = h$. Now consider the diagram of inclusions

$$\begin{array}{ccc} (\mathbb{Z}/2)^4 & \hookrightarrow & V \\ \downarrow & & \downarrow \\ \text{Syl}_2(L_3(4)) & \hookrightarrow & \text{Syl}_2(M_{22}) \end{array}$$

where $(\mathbb{Z}/2)^4$ is the subgroup above $\langle e_1, e_2, e_3, e_4 \rangle$. A direct calculation shows that

$$a(e_1) = a(e_2) = 0, \quad a(e_3) = a(e_4) = 1,$$
$$b(e_1) = b(e_2) = b(e_3) = b(e_4) = 1,$$
$$c(e_1) = c(e_2) = c(e_3) = 0, \quad c(e_4) = 1,$$
$$d(e_1) = d(e_3) = 0, \quad d(e_2) = d(e_4) = 1.$$

Also we can expand $\mathscr{A}\mathscr{B} = (\zeta a + \zeta^2 b)(\zeta^2 a + \zeta b) = a^2 + ab + b^2$. Consequently, the restriction of $\mathscr{A}\mathscr{B}$ to $(\mathbb{Z}/2)^4$ is the same as that of $y_1 y_2 + (y_1 + y_2)^2$ since it is easy to check that the restriction of y_1 is $a + b$ while the restriction of y_2 is a. It follows from this that $h^i(\mathscr{A}\mathscr{B})^j \neq 0$ in $H^{i+2j}(\text{Syl}_2(M_{22})) \otimes \mathbb{F}_4$ for any $i, j > 0$, and we may obtain a similar result using V' for $h^i(\mathscr{E}\mathscr{F})^j$.

This gives an embedding

$$\mathbb{F}_4[h](\mathscr{A}\mathscr{B}, \mathscr{E}\mathscr{F}, (\mathscr{A}\mathscr{B})^2, (\mathscr{E}\mathscr{F})^2) \to H^*(V) \otimes \mathbb{F}_4.$$

We can, in fact, extend this embedding to an embedding

$$\mathbb{F}_4[h](\mathscr{A}\mathscr{B}, \mathscr{E}\mathscr{F}, (\mathscr{A}\mathscr{B}^2, (\mathscr{E}\mathscr{F})^2, (\mathscr{A}\mathscr{F}M)) \hookrightarrow H^*(V) \otimes \mathbb{F}_4,$$

which clearly will complete the proof of Lemma 4.2. However, to do this we must analyse the entire ring $H^*(V)$. The main step is to show that $H^*(V)$ is detected by restriction to elementary 2-subgroups, and then construct an explicit element θ_5 in $H^*(V)$ which has the form $(1 + T^*)(\gamma_5)$, where T is the automorphism of V induced by $\begin{pmatrix} 1 & \zeta_3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. It follows that $\theta \in \mathrm{im}(\mathrm{res}^*)$. Moreover, by the construction, one sees by inspection that $h^i\theta \neq 0$ for all i and is, moreover, independent of the classes constructed previously. Finally, by looking at the E_2-term in (4.1), we see that the only candidate for the class which restricts to θ is $\mathscr{A}\mathscr{F}M$.

The details are rather intricate, so will be deferred to a forthcoming paper on M_{22}.

REMARK. The first (partial) calculation of $H^*(\mathrm{Syl}_2(M_{22}))$ that we are aware of was obtained by J. Maginnis in private correspondence. He used the fact that

$$\mathrm{Syl}_2(M_{22})/C(\mathrm{Syl}_2(M_{22})) \cong \mathrm{Syl}_2(L_4(2))$$

and studied the spectral sequence of this central extension together with the known structure of $H^*(\mathrm{Syl}_2(L_4(2)))$.

REMARK. The group $\mathrm{Syl}_2(M_{22})$ can be written as a semidirect product $2^4 : D_8$ where

$$2^4 = \langle e_1, e_2, e_3, e_4 \rangle \quad \text{and} \quad H^*(2^4)^{D_8} \subset H^*(2^4)^{\langle E,F \rangle}.$$

This last invariant subring is determined in Proposition 2.3. From this one can check directly that this particular action of D_8 gives the ring of invariants,

$$(4.3)\quad H^*(2^4)^{D_8} = \mathbb{F}_2[v + w, vw, \lambda + \tau, \lambda\tau]\{1, \lambda v + \tau w, L + M, vL + wM, \lambda L + \tau M, \lambda vL + \tau wM, LM, (\lambda v + \tau w)LM\}.$$

In particular, the collapsing of the Lyndon spectral sequence above gives

COROLLARY 4.4. *The restriction map* res: $H^*(\mathrm{Syl}_2(M_{22})) \to H^*(2^4)$ *has image exactly the ring of D_8-invariants above.*

Double coset decompositions

The rings $H^*(2^4 : \mathscr{A}_4 : 2_2)$ and $H^*(2^4 : \mathscr{A}_5 : 2_2)$ are easily determined from (4.1) and Lemma 4.2, using double coset decompositions. Specifically, we have

PROPOSITION 4.5. (1) *There is a double coset decomposition*

$$2^4 : \mathscr{A}_4 : 2_2 = \mathrm{Syl}_2(M_{22}) \sqcup \mathrm{Syl}_2(M_{22})T\, \mathrm{Syl}_2(M_{22})$$

and $\mathrm{Syl}_2(M_{22}) \cap T\, \mathrm{Syl}_2(M_{22})T^{-1} = \mathrm{Syl}_2(L_3(4))$ *where T is represented by the matrix* $\begin{pmatrix} \zeta_3 & 0 \\ 0 & \zeta_3^2 \end{pmatrix} \in L_2(4)$.

(2) *There is a double coset decomposition*

$$2^4 : \mathcal{A}_5 : 2_2 = 2^4 : \mathcal{A}_4 : 2_2 \sqcup (2^4 : \mathcal{A}_4 : 2_2)\tau(2^4 : \mathcal{A}_4 : 2_2)$$

where τ *is represented by* $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} 2_2 \in L_2(4) : 2_2$. *Moreover,*

$$\tau(2^4 : \mathcal{A}_4 : 2_2)\tau^{-1} \cap 2^4 : \mathcal{A}_4 : 2_2 = V : \mathcal{S}_3 \cong 2^4 : \mathcal{S}_3.$$

(This is direct. The only thing to note is that 2_2 commutes with $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ since the coefficients are in the fixed field of ϕ.)

As a consequence we have

COROLLARY 4.6. *The ring* $H^*(2^4 : \mathcal{A}_5 : 2_2)$ *injects into* $H^*(2^4 : \mathcal{A}_4 : 2_2)$, *and* $H^*(2^4 : \mathcal{A}_4 : 2_2)$ *injects into* $H^*(\mathrm{Syl}_2(M_{22}))$ *under restriction since they all contain* $\mathrm{Syl}_2(M_{22})$ *as Sylow 2-subgroup. Moreover, the images are determined as follows.*

(1) *For* $\alpha \in H^*(\mathrm{Syl}_2(M_{22}))$ *we have that* $\alpha \in \mathrm{im}(H^*(2^4 : \mathcal{A}_4 : 2_2))$ *if and only if, under the restriction map*

$$\mathrm{res}\colon H^*(\mathrm{Syl}_2(M_{22})) \to H^*(\mathrm{Syl}_2(L_3(4))),$$

we have $\mathrm{res}(\alpha) \in H^*(\mathrm{Syl}_2(L_3(4)))^{\mathbb{Z}/3}$ *where the* $\mathbb{Z}/3$ *action is induced by* T.

(2) *Similarly* $\beta \in H^*(2^4 : \mathcal{A}_4 : 2_2)$ *is in the image of restriction from* $H^*(2^4 : \mathcal{A}_5 : 2_2)$ *if and only if, under the restriction map*

$$\mathrm{res}'\colon H^*(2^4 : \mathcal{A}_4 : 2_2) \to H^*(2^4 : \mathcal{S}_3),$$

the image of β *is invariant under* τ^*.

REMARK. As a consequence of Corollary 4.6, we claim that the Lyndon spectral sequence which converges to $H^*(2^4 : \mathcal{A}_4 : 2_2)$ with E_2-term $H^*(\mathbb{Z}/2 ; H^*(2^4 : \mathcal{A}_4))$ has $E_2 = E_\infty$. To see this, note that differentials must originate on the fibre $H^*(2^4 : \mathcal{A}_4)^{2_2}$. On the other hand, Lemma 4.2 shows that the restriction map for

$$H^*(\mathrm{Syl}_2(M_{22})) \to H^*(\mathrm{Syl}_2(L_3(4)))$$

has image $H^*(\mathrm{Syl}_2(L_3(4)))^{2_2}$, and since the projector $e = (1 + T^* + T^{*2})$ commutes with 2_2, it follows that

$$(H^*(\mathrm{Syl}_2(L_3(4)))^{\mathbb{Z}/3})^{2_2} = e(H^*(\mathrm{Syl}_2(L_3(4)))^{2_2}).$$

But from this the entire edge term survives to E_∞ and the claim follows.

Next, note that the group $2^4 : \mathcal{S}_3$ has cohomology

$$H^*(2^4 : \mathcal{S}_3) \otimes \mathbb{F}_4 = \mathbb{F}_4[\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}]^{\mathcal{S}_3} \oplus \mathbb{F}_4[\mathcal{A}\mathcal{B}, \mathcal{C}\mathcal{D}, h]h$$

and the element τ^* in Corollary 4.6(2) fixes h, while $\mathcal{A} \leftrightarrow \mathcal{C}$ and $\mathcal{B} \leftrightarrow \mathcal{D}$. Thus we have

COROLLARY 4.7. *An element* β *in* $H^*(2^4 : \mathcal{S}_4)$ *is in the image from* $H^*(2^4 : \mathcal{S}_5)$ *if and only if, in* (4.1), *if it comes from line* 1, *then it restricts to* $H^*(V)^{\mathcal{S}_5}$, *or if it comes from line* 5 *it restricts to* $\mathbb{F}_4[\mathcal{A}\mathcal{B} + \mathcal{C}\mathcal{D}, \mathcal{A}\mathcal{B}\mathcal{C}\mathcal{D}, h]h$.

Finally, we consider the double coset decomposition of $L_3(4):2_2$ using $2^4:\mathcal{S}_5$. Let

$$\mu=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}.$$

Then there is a double coset decomposition

$$L_3(4)=2^4:\mathcal{S}_5\sqcup(2^4:\mathcal{S}_5\mu 2^4:\mathcal{S}_5)$$

with

$$\mu(2^4:\mathcal{S}_5)\mu\cap 2^4:\mathcal{S}_5=V'\times_T\mathcal{S}_3.$$

Here

$$H^*(V'\times_T\mathcal{S}_3)\otimes\mathbb{F}_4\cong\mathbb{F}_4[\mathscr{C},\mathscr{D},\mathscr{E},\mathscr{F}]^{\mathcal{S}_3}\oplus\mathbb{F}_4[\mathscr{C}\mathscr{D},\mathscr{E}\mathscr{F},h]h \tag{4.8}$$

and μ^* fixes h while $\mathscr{C}\leftrightarrow\mathscr{E}$, $\mathscr{D}\leftrightarrow\mathscr{F}$.

The extension $2^4:\mathcal{S}_4=2^4:\mathcal{A}_4:2_2$

We now determine these rings explicitly. First we consider $H^*(2^4:\mathcal{S}_4)$. The last three lines of (4.1) give the terms

$$\begin{aligned}&\mathbb{F}_4[v^3,w^3](1,vw,v^2w^2)(\mathcal{A}\mathcal{F},\mathcal{A}\mathcal{F}^2)\\&\quad\oplus\mathbb{F}_4[v^3+w^3,vw,h]/(h(v^3+w^3)=0)(LM+\mathcal{A}\mathcal{B}\mathscr{C}^2\mathscr{D}^2,\mathcal{A}\mathcal{F}M)\\&\quad\oplus\mathbb{F}_4[vw,\mathcal{A}\mathcal{B},\mathscr{E}\mathscr{F},h]/(\mathcal{A}\mathcal{B}\cdot\mathscr{E}\mathscr{F}=0)(h,LMh).\end{aligned}$$

The second line gives the term

$$(\mathbb{F}_4[v,w]\{\mathbb{F}_4[\mathcal{A},\mathcal{B}]\oplus\mathbb{F}_4[\mathscr{E},\mathscr{F}]\})^{\mathbb{Z}/3}L,$$

and the first line gives

$$(\mathbb{F}_4[v,w]\{\mathbb{F}_4[\mathcal{A},\mathcal{B}]\oplus\mathbb{F}_4[\mathscr{E},\mathscr{F}]\})^{\mathcal{S}_3}(1,LM)=\mathcal{P}^{\mathcal{S}_3}(1,LM).$$

Only this last term needs to be clarified. There is an exact sequence

$$\begin{aligned}0\to\mathbb{F}_4[v,w]^{\mathcal{S}_3}\to\mathbb{F}_4[v,w,\mathcal{A},\mathcal{B}]^{\mathcal{S}_3}\oplus\mathbb{F}_4[v,w,\mathscr{E},\mathscr{F}]^{\mathcal{S}_3}\\\to\mathcal{P}^{\mathcal{S}_3}\xrightarrow{\delta}\operatorname{Ext}^1_{\mathcal{S}_3}(\mathbb{F}_4,\mathbb{F}_4[v,w])\to\dots.\end{aligned} \tag{4.9}$$

Now $\operatorname{Ext}^1_{\mathcal{S}_3}(\mathbb{F}_4,\mathbb{F}_4[v,w])\subset\operatorname{Ext}^1_{\mathbb{Z}/2}(\mathbb{F}_4,\mathbb{F}_4[v,w])=\mathbb{F}_4[vw]$, and this injects into

$$\operatorname{Ext}^1_{\mathbb{Z}/2}(\mathbb{F}_4,\mathbb{F}_4[v,w,\mathcal{A},\mathcal{B}]).$$

From this it follows that in (4.9) the map δ is zero and $\mathcal{P}^{\mathcal{S}_3}$ is simply the amalgamated sum of the two subrings preceding it in (4.9).

REMARK. Note that only the contributions from lines 1 and 2 in (4.1) restrict non-trivially to $H^*(2^4)$ and this restriction can be written

$$\mathbb{F}_4[v,w,\mathcal{A},\mathcal{B}]^{\mathcal{S}_3}(1,LM)\oplus(1+\tau)\mathbb{F}_4[v,w,\mathcal{A},\mathcal{B}]^{\mathbb{Z}/3}L, \tag{4.10}$$

which is clearly just $H^*(2^4)^{\mathcal{S}_4}$ where $\mathcal{S}_4$ is the extension of $\langle E,F\rangle$ by $\mathcal{S}_3$.

For any $\mathcal{S}_3$-module $\mathcal{V}$ we have $\mathcal{V}^{\mathcal{S}_3} = (\mathcal{V}^{\mathbb{Z}/3})^{\mathbb{Z}/2}$, so we find that

$$\begin{aligned}\mathbb{F}_4[v, w, \mathcal{A}, \mathcal{B}]^{\mathcal{S}_3} &= \mathbb{F}_4[v^3+w^3, vw, \mathcal{A}^3+\mathcal{B}^3, \mathcal{A}\mathcal{B}](v^3\mathcal{A}^3+w^3\mathcal{B}^3) \oplus (1+\tau)\mathbb{F}_4[v^3, w^3, \mathcal{A}^3, \mathcal{B}^3] \\ &\quad \times \{v\mathcal{A}, v^2\mathcal{B}, \mathcal{B}^2 v, v^2\mathcal{A}^2, v^2 w\mathcal{A}, w\mathcal{A}\mathcal{B}^2, v^2\mathcal{B}\mathcal{A}^2, v^2 w\mathcal{B}^2, vw^2\mathcal{B}\mathcal{A}^2\},\end{aligned}$$

while $\mathbb{F}_4[v, w]^{\mathcal{S}_3} = \mathbb{F}_4[vw, v^3+w^3]$. The Poincaré series for the first of these rings is

$$\begin{aligned}&\frac{1+x^{15}}{(1-x^{12})(1-x^8)(1-x^3)(1-x^2)} + \frac{x^5+x^6+x^7+x^9+x^{10}+x^{11}+x^{13}+x^{14}+x^{15}}{(1-x^{12})^2(1-x^3)^2} \\ &\quad = \frac{1+x^5+x^6+x^9+x^{10}+x^{15}}{(1-x^{12})(1-x^8)(1-x^3)(1-x^2)} \\ &\quad = \mathcal{W}(x),\end{aligned}$$

while the Poincaré series for the second is

$$\frac{1}{(1-x^8)(1-x^{12})}.$$

Consequently, the Poincaré series for $\mathcal{P}^{\mathcal{S}_3}$ is given as

$$2\mathcal{W}(x) - \frac{1}{(1-x^8)(1-x^{12})} = \frac{1+x^2+x^3+x^5+2x^6+2x^9+2x^{10}+2x^{15}}{(1-x^{12})(1-x^8)(1-x^3)(1-x^2)}.$$

The second line has Poincaré series

$$x^3\left(2p(x) - \frac{1+x^8+x^{16}}{(1-x^{12})^2}\right),$$

which equals

$$\frac{x^{12}-x^{11}+2x^{10}+x^9-2x^8+x^7+2x^6-5x^5+2x^4-x^3-2x^2+x-1}{x^{20}-x^{19}-x^{17}+x^{15}+x^{13}-x^{12}-x^8+x^7+x^5-x^3-x+1},$$

while the third line has Poincaré series

$$\frac{x^2+x^7+x^{10}+x^{11}+x^{15}+x^{18}}{(1-x^3)^2},$$

and the fourth line has series

$$\frac{(x^5+x^6)(1+x+x^2+x^{11}+x^{12})}{(1-x^{12})(1-x^8)}.$$

Finally, the last line has series

$$\frac{(x+x^7)(1+x^2)}{(1-x^8)(1-x^2)(1-x)}.$$

Putting these together, we find that the Poincaré series for $H^*(2^4 : \mathcal{A}_4 : 2_2)$ is given as

$$(4.11)\qquad \frac{\begin{array}{r}x^{21}-4x^{19}+x^{18}+x^{17}-x^{16}+3x^{15}-x^{13}+3x^{12}+5x^{11} \\ -x^{10}+2x^9+3x^8+4x^6+4x^5-x^4+x^3+3x^2+1\end{array}}{(1-x^{12})(1-x^8)(1-x^3)(1-x)}.$$

The extension $2^4 : \mathcal{S}_5 = 2^4 : \mathcal{A}_5 : 2_2$

We apply Corollary 4.7 to (4.10). Note that the restriction to $H^*(V \times_\alpha \mathcal{S}_3)$ of $LM + \mathcal{A}\mathcal{B}\mathcal{C}^2\mathcal{D}^2$ is 0, as are the restrictions of $\mathcal{A}\mathcal{F}$, $\mathcal{A}\mathcal{F}^2$ and $\mathcal{A}\mathcal{F}M$. The first two lines of (4.1), taking account of (4.10), give a term $\mathcal{Q}$ obtained from the exact sequence

(4.12) $$0 \to \mathcal{Q} \hookrightarrow \mathbb{F}_4[\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}]^{\mathcal{S}_5} \oplus \mathbb{F}_4[\mathcal{C}, \mathcal{D}, \mathcal{E}, \mathcal{F}]^{\mathcal{S}_4} \to \{\mathbb{F}_4[v, w](1, L, M)\}^{\mathcal{S}_3} \to 0.$$

The remaining lines now give the terms

$$\begin{aligned}\mathbb{F}_4[v^4, w^3](1, vw, v^2w^2)(\mathcal{A}\mathcal{F}, \mathcal{A}\mathcal{F}^2) &\oplus \mathbb{F}_4[v^3 + w^3, vw](\mathcal{A}\mathcal{F}M)\\ &\oplus \mathbb{F}_4[vw, h](h(LM + \mathcal{A}\mathcal{B}\mathcal{C}^2\mathcal{D}^2), h\mathcal{A}\mathcal{F}M)\\ &\oplus \mathbb{F}_4[vw + (\mathcal{A}\mathcal{B})^4, (\mathcal{A}\mathcal{B})^4 vw, h](h, hLM)\\ &\oplus \mathbb{F}_4[vw, \mathcal{E}\mathcal{F}, h](\mathcal{E}\mathcal{F}h, \mathcal{E}\mathcal{F}hLM).\end{aligned}$$

This completes the description of $H^*(2^4 : \mathcal{S}_5)$.

To make it effective we need only describe the structure of $H^*(2^4)^{\mathcal{S}_5}$. The action of 2_2 on $H^*(2^4)^{\mathcal{A}_5}$ interchanges the generators m_5, $\bar{m}_5$, and the generators m_{12}, $\bar{m}_{12}$ of the polynomial algebra. It also interchanges L, M, and D, E, but it leaves C fixed. Thus setting

$$\begin{aligned}\mathcal{G} &= \mathbb{F}_4[m_5, \bar{m}_5, m_{12}, \bar{m}_{12}]^{2_2}\\ &= \mathbb{F}_4[m_5 + \bar{m}_5, m_5\bar{m}_5, m_{12} + \bar{m}_{12}, m_{12}\bar{m}_{12}](1, m_5m_{12} + \bar{m}_5\bar{m}_{12}),\end{aligned}$$

we have, from Theorem 3.2,

$$\begin{aligned}H^*(2^4)^{\mathcal{S}_5} = \mathbb{F}_4[m_5, \bar{m}_5, m_{12}, \bar{m}_{12}](&L, L^2, D, L^3, L^2M, LC, LD, MD, L^4, L^3M, L^2D,\\ &LMD, M^2D, L^4M, L^3M^2, L^3D, L^2MD, LM^2D,\\ &M^3D, LC^2, L^2M^2D, M^4D, L^4M^3, L^5M^3, L^5M^4)\\ \oplus\, \mathcal{G}(1, LM, C, L^2M^2, LMC, &C^2, DE, LMC^2, L^4M^4, L^5M^5).\end{aligned}$$

The group $L_3(4) : 2_2$

Here we must add the conditions implied by (4.8) to those above for $H^*(2^4 : \mathcal{S}_5)$. This changes (4.12) to give $\mathcal{R} \subset H^*(L_3(4) : 2_2)$ defined as

$$0 \to \mathcal{R} \hookrightarrow H^*(V)^{\mathcal{S}_5} \oplus H^*(V')^{\mathcal{S}_5} \to \{\mathbb{F}_4[v, w](1, L, M)\}^{\mathcal{S}_3} \to 0.$$

The remaining terms then become

$$\begin{aligned}\mathbb{F}_4[v^3, w^3](1, vw, v^2w^2)(\mathcal{A}\mathcal{F}, \mathcal{A}\mathcal{F}^2) \oplus \mathbb{F}_4[v^3 + w^3, vw, h]/\{(v^3 + w^3)h = 0\}(\mathcal{A}\mathcal{F}M)\\ \oplus \mathbb{F}_4[vw + \mathcal{A}^4\mathcal{B}^4 + \mathcal{E}^4\mathcal{F}^4, \mathcal{A}^4\mathcal{B}^4vw, \mathcal{E}^4\mathcal{F}^4vw, h](h, hLM, h\mathcal{A}\mathcal{B}\mathcal{C}^2\mathcal{D}^2)\end{aligned}$$

with relations

$$\mathcal{A}^4\mathcal{B}^4vw \cdot \mathcal{E}^4\mathcal{F}^4vw = 0, \quad \mathcal{A}^4\mathcal{B}^4vwh(LM + \mathcal{A}\mathcal{B}\mathcal{C}^2\mathcal{D}^2) = 0,$$
$$\mathcal{E}^4\mathcal{F}^4vwh(\mathcal{A}\mathcal{B}\mathcal{C}^2\mathcal{D}^2) = 0,$$

and this gives a complete description of $H^*(L_3(4) : 2_2)$.

5. The $\mathcal{A}_5$-invariant subring of $\mathbb{F}_2[x, y, z, w]$

The action of $\mathcal{A}_5$ on

$$\mathcal{M} = \{\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}\}$$

is given by $T(\mathcal{C}) = \zeta^2\mathcal{C}$, $T(\mathcal{D}) = \zeta\mathcal{D}$, $T(\mathcal{A}) = \zeta\mathcal{A}$, $T(\mathcal{B}) = \zeta^2\mathcal{B}$, while, setting $E = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, we have $E(\mathcal{A}) = \mathcal{A}$, $E(\mathcal{B}) = \mathcal{B}$, $E(\mathcal{C}) = \mathcal{C} + \mathcal{A}$, $E(\mathcal{D}) = \mathcal{D} + \mathcal{B}$, and $\sigma(\mathcal{A}) = \mathcal{C}$, $\sigma(\mathcal{B}) = \mathcal{D}$. This shows that the module $\mathcal{M}$ splits over $\mathcal{A}_5$ as the direct sum $\mathbb{V} \oplus \mathbb{V}^c$ where $\mathbb{V} = \{\mathcal{B}, \mathcal{D}\}$, and $\mathbb{V}^c = \{\mathcal{C}, \mathcal{A}\}$ and $\mathbb{V} = \mathbb{F}_4^2$ is the usual module for the action of $\mathcal{A}_5 = \mathrm{SL}_2(4)$. Consequently $\mathbb{F}_4[\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}] = \mathbb{F}_4[\mathbb{V}] \otimes_{\mathbb{F}_4} \mathbb{F}_4[\mathbb{V}^c]$.

LEMMA 5.1. (1) *We have* $\mathbb{F}_4[\mathbb{V}]^{\mathcal{A}_5} = \mathbb{F}_4[M_5, M_{12}]$ *where* $M_5 = \mathcal{B}\mathcal{D}^4 + \mathcal{D}\mathcal{B}^4$ *and* $M_{12} = \mathcal{B}^{12} + \mathcal{B}^9\mathcal{D}^3 + \mathcal{B}^6\mathcal{D}^6 + \mathcal{B}^3\mathcal{D}^9 + \mathcal{D}^{12}$. *Note that we may also write* $M_{12} = (\mathcal{B}(\mathcal{B}^3 + \mathcal{D}^3))^3 + \mathcal{D}^{12}$.

(2) *Let* $(\mathbb{Z}/2)^2 = \mathrm{Syl}_2(\mathcal{A}_5)$ *act via* $E(x) = x + y$, $F(x) = x + \zeta_3 y$, $E(y) = F(y) = y$. *Then* $\mathbb{F}_4[x, y]^{\langle E, F\rangle} = \mathbb{F}_4[r_4, y]$ *where* $r_4 = x(x^3 + y^3)$.

(3) *The module* $\mathbb{F}_4[y, r_4] \supset \mathbb{F}_4[x, y]^{\mathcal{A}_5}$ *and is a free* $\mathbb{F}_4[M_5, M_{12}]$*-module with basis*

$$\{1, y, y^2, \dots, y^{11}, r_4, r_4^2, r_4^3\}.$$

Proof. (1) This is a theorem of L. E. Dickson [7]. The exact invariant generators are

$$M_5 = \begin{vmatrix} \mathcal{B}^4 & \mathcal{D}^4 \\ \mathcal{B} & \mathcal{D} \end{vmatrix}, \quad M_{12} = \begin{vmatrix} \mathcal{B}^{16} & \mathcal{D}^{16} \\ \mathcal{B} & \mathcal{D} \end{vmatrix} \Big/ M_5.$$

(2) The element $r_4 = x(x + y)(x + \zeta_3 y)(x + \zeta_3^2 y)$ is evidently invariant under $\langle E, F\rangle$. Also $\mathbb{F}_4[x, y]^{\langle E\rangle} = \mathbb{F}_4[x(x + y), y]$ and

$$F(x(x + y)) = (x + \zeta_3 y)(x + \zeta_3^2 y) = x(x + y) + y^2.$$

From this (2) follows directly.

(3) First note the two relations

$$y^{12} = r^3 + M_{12}, \quad yr = M_5.$$

Next, we establish without difficulty, using the above relations and their consequences,

$$\begin{aligned}
r^4 &= M_5 y^{11} + rM_{12}, \\
r^5 &= M_5^2 y^{10} + r^2 M_{12}, \\
r^6 &= M_5^3 y^9 + r^3 M_{12}, \\
y^{12+k} &= \begin{cases} M_5^k r^{3-k} + y^k M_{12} & \text{if } k \leqslant 3, \\ M_5^3 y^{k-3} + y^k M_{12} & \text{if } 11 \geqslant k > 3, \end{cases}
\end{aligned}$$

that if we can write A, B in the polynomial ring in the form

$$A = \sum_{i=0}^{11} y^i p_i(M_5, M_{12}) + \sum_{j=1}^{3} r^j q_j(M_5, M_{12}),$$

$$B = \sum_{i=0}^{11} y^i v_i(M_5, M_{12}) + \sum_{j=1}^{3} r^j w_j(M_5, M_{12}),$$

then the same is true for their product,

$$AB = \sum_{i=0}^{11} y^i \alpha(M_5, M_{12}) + \sum_{j=1}^{3} r^j \beta(M_5, M_{12}).$$

Next we establish by an easy induction that for all $n \geqslant 1$ and $m \geqslant 1$ the elements y^n and r^m can be written in the form above. From this it follows that every element has such an expansion. Finally note that

$$\frac{(1-x^5)(1-x^{12})}{(1-x)(1-x^4)} = (1+x+x^2+x^3+x^4)(1+x^4+x^8).$$

From this we see that the Poincaré series for $\mathbb{F}_4[r, y]$ has the form

$$\frac{(1+x+x^2+x^3+x^4)(1+x^4+x^8)}{(1-x^5)(1-x^{12})} = \frac{1+x+x^2+x^3+2x^4+x^5+x^6+x^7+2x^8+x^9+x^{10}+x^{11}+x^{12}}{(1-x^5)(1-x^{12})},$$

so it follows that the expansion above

$$\mathbb{F}_4[r, y] = \mathbb{F}_4[M_5, M_{12}](1, y, \ldots, y^{11}, r, r^2, r^3)$$

must be free and the proof is complete.

Now $\mathbb{F}_4[x, y]$ as an $\mathscr{A}_5$-module is graded and

$$\mathbb{F}_4[x, y]_n = \langle x^n, x^{n-1}y, \ldots, xy^{n-1}, y^n \rangle \cong \mathbb{F}_4^{n+1}.$$

We denote this $\mathscr{A}_5$-module as $\langle n+1 \rangle$. For example, $\langle 2 \rangle = \langle x, y \rangle \cong \mathbb{F}_4^2$ is just given via the usual $\mathscr{A}_5 = \mathrm{SL}_2(4)$ action. We have

LEMMA 5.2. *The map induced by multiplication*

$$\mu\colon\ \mathbb{F}_4[\mathbb{V}]^{\mathscr{A}_5}\{\langle 2 \rangle, \langle 3 \rangle, \ldots, \langle 16 \rangle\} \to \mathbb{F}_4[\mathbb{V}]$$

is surjective.

Proof. Indeed, $\mathbb{F}_4[x, y]$ is free and generated by $(1, x, x^2, x^3)$ as a module over $\mathbb{F}_4[r, y]$, and consequently $\mathbb{F}_4[x, y]$ is free over $\mathbb{F}_4[M_5, M_{12}]$ on the sixty generators

$$(1, x, x^2, x^3)(1, y, \ldots, y^{11}, r, r^2, r^3)$$

which are all in the image of $\langle 2 \rangle, \ldots, \langle 16 \rangle$.

REMARK. In particular, μ allows us to decompose the general module $\langle n \rangle$ for $n > 16$ in terms of $\langle 2 \rangle, \ldots, \langle 16 \rangle$. For example,

$$\langle 12 \rangle M_5 + \langle 5 \rangle M_{12} + \langle 7 \rangle M_5^2 \to \langle 17 \rangle$$

is surjective, but $\langle 7 \rangle M_5 \subset \langle 12 \rangle$, so we see that $\langle 17 \rangle = \langle 12 \rangle M_5 \oplus \langle 5 \rangle M_{12}$. Similarly,

$$\langle 18 \rangle = \langle 13 \rangle M_5 + \langle 6 \rangle M_{12},$$

but the sum is not direct since $M_5 M_{12}$ is contained in each term above. We also have $\langle 19 \rangle = M_4 \langle 14 \rangle + M_{12} \langle 7 \rangle$ with intersection $\{M_5 M_{12} \langle 2 \rangle\}$. The first time three summands occur is

$$\langle 29\rangle = (M_5^3\langle 14\rangle + M_{12}M_5\langle 12\rangle) \oplus M_{12}^5\langle 5\rangle$$

where the intersection of the first two summands is $\{M_5^3 M_{12}\langle 2\rangle\}$.

We now turn to the polynomial algebra on four generators $\mathbb{F}_4[\mathbb{V}\oplus\mathbb{V}^c] = \mathbb{F}_4[\mathbb{V}]\otimes_{\mathbb{F}_4}\mathbb{F}_4[\mathbb{V}^c]$. The following is immediate from our previous discussion.

Corollary 5.3. *We have*

$$\mathbb{F}_4[M_5\otimes 1, M_{12}\otimes 1, 1\otimes M_5, 1\otimes M_{12}] = \mathcal{P} \subset \mathbb{F}_4[\mathbb{V}\oplus\mathbb{V}^c]$$

and the multiplication map

$$\mu\colon \mathcal{P}(\langle 2\rangle\otimes\langle 2\rangle^c, \langle 2\rangle\otimes\langle 3\rangle^c, \ldots, \langle 16\rangle\otimes\langle 16\rangle^c) \to \mathbb{F}_4[\mathbb{V}\oplus\mathbb{V}^c]$$

is surjective.

We now turn to the determination of the rings of invariants

$$\mathbb{F}_4[\mathbb{V}\oplus\mathbb{V}^c]^{\langle E,F\rangle}, \quad \mathbb{F}_4[\mathbb{V}\oplus\mathbb{V}^c]^{\mathcal{A}_4}, \quad \mathbb{F}_4[\mathbb{V}\oplus\mathbb{V}^c]^{\mathcal{A}_5}.$$

Lemma 5.4. *We have*

$$\mathbb{F}_4[\mathbb{V}\oplus\mathbb{V}^c]^{\langle E,F\rangle} \cong \mathbb{F}_4[r\otimes 1, y\otimes 1, 1\otimes r, 1\otimes y](1, L_3, M_3, L_3M_3),$$

where $L_3 = x^2\otimes y + y^2\otimes x \in \langle 3\rangle\otimes\langle 2\rangle$, $M_3 = x\otimes y^2 + y\otimes x^2 \in \langle 2\rangle\otimes\langle 3\rangle$, *and* L_3, M_3 *are* $\mathcal{A}_5$*-invariant.*

Proof. In view of the final proposition in the first section the only thing to verify is that L_3 and M_3 are $\mathcal{A}_5$-invariant. Note that

$$E(x\otimes 1) = (x+y)\otimes 1, \qquad T(1\otimes x) = 1\otimes(x+y),$$
$$F(x\otimes 1) = (x+\zeta_3 y)\otimes 1, \quad F(1\otimes x) = 1\otimes(x+\zeta_3^2 y),$$

so L_3, M_3 are invariant under $\langle E, F\rangle$. Next note that T acts as $T(x\otimes 1) = \zeta_3(x\otimes 1)$, $T(y\otimes 1) = \zeta_3^2(y\otimes 1)$, $T(1\otimes x) = \zeta_3^2(1\otimes x)$, and $T(1\otimes y) = \zeta_3(1\otimes y)$. Hence L_3 and M_3 are invariant under T. Finally, note that $\sigma\colon x\otimes 1 \leftrightarrow y\otimes 1$, $\sigma\colon 1\otimes x \leftrightarrow 1\otimes y$, and L_3, M_3, are invariant under σ. But E, F, T, and σ generate $L_2(4)$, so the invariance of L_3, M_3 under $\mathcal{A}_5$ is established.

Corollary 5.5. (1) *The multiplication map on* $\langle E, F\rangle$-invariants

$$\mathcal{P}(\langle 2\rangle\otimes\langle 2\rangle^c, \ldots, \langle 16\rangle\otimes\langle 16\rangle^c)^{\langle E,F\rangle} \to \mathbb{F}_4[\mathbb{V}\oplus\mathbb{V}^c]^{\langle E,F\rangle}$$

is surjective.

(2) *The multiplication map on* $\mathcal{A}_4$*-invariants*

$$\mathcal{P}(\langle 2\rangle\otimes\langle 2\rangle^c, \ldots, \langle 16\rangle\otimes\langle 16\rangle^c)^{\mathcal{A}_4} \to \mathbb{F}_4[\mathbb{V}\oplus\mathbb{V}^c]^{\mathcal{A}_4}$$

is surjective.

(3) *The same statement holds for the* $\mathcal{A}_5$*-invariants:*

$$\mathcal{P}(\langle 2\rangle\otimes\langle 2\rangle^c, \ldots, \langle 16\rangle\otimes\langle 16\rangle^c)^{\mathcal{A}_5} \to \mathbb{F}_4[\mathbb{V}\oplus\mathbb{V}^c]^{\mathcal{A}_5}$$

is surjective.

Proof. The previous calculation and the preceding lemmas verify the statement of the corollary in Case (1). Next, $N_{\mathscr{A}_5}(\langle E, F\rangle) = \mathscr{A}_4$, the semidirect product $\langle E, F\rangle \times_T (\mathbb{Z}/3)$. Consequently, T acts to take $\mathbb{F}_2[\mathbb{V} \oplus \mathbb{V}^c]^{\langle E,F\rangle}$ to itself and $e = 1 + T + T^2$ is an idempotent with

$$\operatorname{im}(e)|\mathbb{F}_4[\mathbb{V} \oplus \mathbb{V}^c]^{\langle E,F\rangle} = \mathbb{F}_4[\mathbb{V} \oplus \mathbb{V}^c]^{\mathscr{A}_4}.$$

Given any $l \in \mathbb{F}_4[\mathbb{V} \oplus \mathbb{V}^c]^{\mathscr{A}_4}$ there is an element

$$w \in \mathscr{P}(\langle 2\rangle \otimes \langle 2\rangle^c, \ldots, \langle 16\rangle \otimes \langle 16\rangle^c)^{\langle E,F\rangle}$$

with $\mu(w) = l$. Then $\mu(ew) = el = l$ and ew is also $\mathscr{A}_4$-invariant. This shows (2).

It remains to prove (3). Let K be the kernel of μ, so we have the short exact sequence of $\mathbb{F}_4[\mathscr{A}_5]$-modules

$$0 \to K \to \mathscr{P}(\langle 2\rangle \otimes \langle 2\rangle^c, \ldots, \langle 16\rangle \otimes \langle 16\rangle^c) \xrightarrow{\mu} \mathbb{F}_4[\mathbb{V} \oplus \mathbb{V}^c] \to 0.$$

Passing to Ext-groups gives the long exact sequence

$$K^{\mathscr{A}_5} \hookrightarrow \mathscr{P}(\langle 2\rangle \otimes \langle 2\rangle^c, \ldots, \langle 16\rangle \otimes \langle 16\rangle^c)^{\mathscr{A}_5} \xrightarrow{\mu_*} \mathbb{F}_4[\mathbb{V} \oplus \mathbb{V}^c]^{\mathscr{A}_5}$$

$$\xrightarrow{\partial} \operatorname{Ext}^1_{\mathbb{F}_4[\mathscr{A}_5]}(\mathbb{F}_4, K) \xrightarrow{j_1} \cdots.$$

For $i \geqslant 1$ we know that the restriction map

$$\text{res}^*\colon \operatorname{Ext}^i_{\mathbb{F}_4[\mathscr{A}_5]}(\mathbb{F}_4, \mathscr{M}) \to \operatorname{Ext}^i_{\mathbb{F}_4[\mathscr{A}_4]}(\mathbb{F}_4, \mathscr{M})$$

is an isomorphism for any $\mathbb{F}_4[\mathscr{A}_5]$-module $\mathscr{M}$. Since μ is surjective for $\mathscr{A}_4$-invariants, it follows that j_1 is injective for $\operatorname{Ext}^1_{\mathbb{F}_4[\mathscr{A}_4]}$, and from the isomorphism above, j_1 is also injective for $\operatorname{Ext}^1_{\mathbb{F}_4[\mathscr{A}_5]}$. But from the exactness of the sequence above, this implies that μ is surjective on invariants for $\mathscr{A}_5$. The result follows.

This last result reduces the determination of $\mathbb{F}_4[\mathbb{V} \oplus \mathbb{V}^c]^{\mathscr{A}_5}$ to the determination of the invariants

$$(\langle n\rangle \otimes \langle m\rangle)^{\mathscr{A}_5}, \quad \text{with } 2 \leqslant n, m \leqslant 16.$$

This was done with the aid of a computer. The technique is clear. First the determination of $(\langle n\rangle \otimes \langle m\rangle)^{\mathscr{A}_4}$ is direct. But an $\mathscr{A}_4$-invariant is an $\mathscr{A}_5$-invariant if and only if it is invariant under the action of the matrix σ. Consequently, the program ran over the list of $\mathscr{A}_4$-invariants and checked the image under $\sigma + 1$ of these elements. A list of independent images was maintained, and each time $(\sigma + 1)(\alpha)$ was found to be independent of the elements already in the list, this vector was added to the list. If, however, it was dependent on the list, then α plus a linear combination of preceding vectors were invariant. This linear combination was determined and output.

Afterwards, these invariants were sieved to obtain multiplicative generators. The process was to construct all the products of known generators, particularly with the elements in the polynomial subalgebra, which would land in one of the tensor-product modules discussed above, and compare with the set of invariants coming from that module. In this way the generators for the algebra were determined, and some of the relations (but not all) were also determined.

6. *Two extensions connected with the O'Nan group*

The O'Nan group contains a configuration of two maximal, odd index, 2-locals, $(4\cdot L_3(4)):2_1$ and $4^3\cdot L_3(2)$ with intersection $(4\cdot 2^4:\mathcal{A}_4):2_1$ where

$$4\cdot 2^4\cong \mathbb{Z}/4 * D_8 * D_8,$$

the central product. A good description of $4^3\cdot L_3(2)$ is given by Griess in **[8]** where he constructs the group as a subgroup of $\mathrm{Aff}_3(\mathbb{Z}/8)\subset GL_4(\mathbb{Z}/8)$. Also, Alperin **[4]** has described $\mathrm{Syl}_2(4^3\cdot L_3(2))$ as a 2-group with five generators s, t, v_1, v_2, v_3, so that $\langle v_1, v_2, v_3\rangle = (\mathbb{Z}/4)^3$ and

$$\mathrm{Syl}_2(\mathrm{O'N}) = \langle s, t, v_1, v_2, v_3 |\ s^4 = v_1v_3,\ t^2=1,\ tst=s^{-1},\ tv_2t=v_2^{-1},\ tv_1t=v_3^{-1},$$
$$sv_1s^{-1}=v_2,\ sv_2s^{-1}=v_3,\ sv_3s^{-1}=v_1v_2^{-1}v_3\rangle.$$

Griess's embedding can be chosen so that

$$t\leftrightarrow\begin{pmatrix}1&0&0&0\\1&7&1&2\\2&0&1&4\\7&0&0&7\end{pmatrix},\quad s\leftrightarrow\begin{pmatrix}1&0&0&0\\2&1&7&6\\2&6&1&7\\5&4&4&7\end{pmatrix}$$

$$v_1\leftrightarrow\begin{pmatrix}1&0&0&0\\2&1&0&0\\6&0&1&0\\6&0&0&1\end{pmatrix},\quad v_2\leftrightarrow\begin{pmatrix}1&0&0&0\\0&1&0&0\\4&0&1&0\\2&0&0&1\end{pmatrix},\quad v_3\leftrightarrow\begin{pmatrix}1&0&0&0\\0&1&0&0\\2&0&1&0\\6&0&0&1\end{pmatrix},$$

and the intersection, $(4\cdot 2^4:\mathcal{A}_4):2_1$ is generated by the elements above together with

$$\beta=\begin{pmatrix}1&0&0&0\\0&1&4&0\\7&0&4&3\\6&2&1&3\end{pmatrix}.$$

Here, the automorphism 2_1 is represented by ts^3 so $\langle ts^3,\beta\rangle\cong\mathcal{S}_3$ and $(4\cdot 2^4:\mathcal{A}_4)=\langle\beta, t, s^2, v_1, v_2, v_3\rangle$. This group can also be described as the centralizer of $\langle v_1v_2^2v_3^{-1}\rangle\cong(\mathbb{Z}/4)$ in $4^3\cdot L_3(2)$. The explicit action of β on $\langle t, s^2, v_1, v_2, v_3\rangle$ is given by

$$\beta t\beta^{-1}=v_3^{-1}ts^2,\quad \beta s^2\beta^{-1}=(v_2v_3)^{-1}t,$$

$$\beta v_1\beta^{-1}=v_1v_2^{-1},\quad \beta v_2\beta^{-1}=v_3^{-1}t,\quad \beta v_3\beta^{-1}=v_2v_3^{-1}.$$

Also, from **[11]**, the group $(4\cdot L_3(4)):2_1$ is the centralizer in O'N of $v_2^2v_3^2$. In particular, there is an explicit homomorphism

$$\langle\beta, t, s^2, v_1, v_2, v_3\rangle\to L_3(4)$$

given by

$$v_3^2 \mapsto \begin{pmatrix} 1 & 0 & \zeta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad v_2 \mapsto \begin{pmatrix} 1 & 0 & \zeta^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\beta \to \begin{pmatrix} \zeta & 0 & 0 \\ 0 & \zeta^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad v_2 t \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \zeta \\ 0 & 0 & 1 \end{pmatrix}, \qquad v_3^2 ts^2 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \zeta^2 \\ 0 & 0 & 1 \end{pmatrix},$$

$$(v_1 v_3)^{-1} v_2^2 t \mapsto \begin{pmatrix} 1 & \zeta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; v_1^{-2} v_3^2 ts^2 \mapsto \begin{pmatrix} 1 & \zeta^2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

with kernel exactly $\langle v_1 v_2^2 v_3^{-1} \rangle$.

We now discuss the structure of the group $2 \cdot L_3(4)$ obtained by factoring out the centre $\langle v_1^2 v_3^2 \rangle$. Let π: $2 \cdot L_3(4) \to L_3(4)$ be the projection and $U^1 = \mathrm{Syl}_2(2 \cdot L_3(4))$, $B^1 = \pi^{-1}(B)$ where $B \subset L_3(4)$ is the Borel subgroup $2^4 : \mathscr{A}_4$, and $P^1 = \pi^{-1}(P_1)$ where P_1 is the maximal parabolic given as all elements of the form $\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 1 \end{pmatrix}$. The following lemma is a direct calculation.

LEMMA 6.1. (i) *The group U^1 is given as a semidirect product*

$$(\mathbb{Z}/2)^5 \to U^1 \to (\mathbb{Z}/2)^2$$

where the

$$(\mathbb{Z}/2)^5 = \langle v_1 v_3, v_2^2, v_3^2, v_2 t, s^2 t \rangle$$

and the

$$(\mathbb{Z}/2)^2 = \langle (v_1 v_3)^{-1} v_2^2 t, v_1^{-1} v_3^2 ts^2 \rangle.$$

(ii) *The group B^1 is given as a semidirect product*

$$B^1 \cong 2^5 : \mathscr{A}_4.$$

(iii) *The group P^1 is given as a semidirect product*

$$P^1 \cong 2^5 : \mathscr{A}_5.$$

(iv) *There is a central extension*

$$\mathbb{Z}/2 \to U^1 \to D_8 \times D_8$$

where $(\mathbb{Z}/2) = \langle v_1 v_3 \rangle$, the first dihedral group D_8 is $\langle v_2 t, t \rangle$, and the second is $\langle v_1 s^2 t, s^2 t \rangle$.

We can write $D_8 \cong \{T, S \mid T^2 = S^2 = (TS)^4 = 1\}$. It has exactly two elementary 2-groups $(\mathbb{Z}/2)^2$, $I = \langle T, (TS)^2 \rangle$ and $II = \langle S, (TS)^2 \rangle$. Also, the centre of U^1 is seen to be $(\mathbb{Z}/2)^3 = \langle v_1 v_3, v_2^2, v_3^2 \rangle$ and is given as the intersection of the two $(\mathbb{Z}/2)^5$ contained in U^1,

$$\langle v_1 v_3, v_2^2, v_3^2, v_2 t, s^2 t \rangle = \langle v_1 v_2^2 v_3^{-1} \rangle \times I \times II$$

and

$$\langle v_1 v_3, v_2^2, v_3^2, t, v_3 ts^2 \rangle = \langle v_1 v_2^2 v_3^{-1} \rangle \times II \times I.$$

The determination of $H^*(U^1)$

It seems easiest to calculate $H^*(U^1)$ using Lemma 6.1(iv). Recall that

$$H^*(D_8) = \mathbb{F}_2[x, y, f]/(xy = 0),$$
$$\dim(x) = \dim(y) = 1, \quad \dim(f) = 2, \quad Sq^1(f) = (x + y)f.$$

The restriction map $(\mathrm{res}^{I}_{D_8})^*$ in cohomology has the form $x \mapsto t$, $y \mapsto 0$, $f \mapsto c^2 + ct$, where t is dual to A and c is dual to $(TS)^2$ in $H^1(I) = \mathrm{Hom}(I\,; \mathbb{F}_2)$. Similarly, $(\mathrm{res}^{II}_{D_8})^*$ satisfies $x \mapsto 0$, $y \mapsto s$, $f \mapsto c^2 + cs$ where s is dual to S. These two restriction maps are injective in cohomology, and by checking the inverse images, in U^1 of the groups $I \times I$, $I \times II$, $II \times I$, and $II \times II$, we find that the k-invariant for the extension of Lemma 6.1(iv) is $x_1x_2 + y_1y_2$.

Write the E_2-term of the spectral sequence for the extension as

$$E_2 \cong \mathbb{F}_2[\kappa^2]\{\kappa \circ H^*(D_8 \times D_8) \oplus H^*(D_8 \times D_8)\}.$$

The differential d^2 takes κ to $x_1x_2 + y_1y_2$, so after a short calculation we find that E_3 has the form

(6.3) $\mathbb{F}_2[\kappa^2] \otimes \mathbb{F}_2[f_1, f_2]\{\mathbb{F}_2[x_1, y_2](1, \kappa x_1 y_2) \oplus \mathbb{F}_2[x_2, y_1](1, \kappa x_2 y_1) \oplus x_1x_2\}.$

Moreover, since $Sq^1(x_1x_2 + y_1y_2) = (x_1 + x_2 + y_1 + y_2)(x_1x_2 + y_1y_2)$, we see that $E_3 = E_\infty$. In particular, the restriction maps to the two subgroups $(\mathbb{Z}/2)^5$ described above have images $\mathbb{F}_2[\kappa^2] \otimes \mathbb{F}_2[f_1, f_2, x_1, y_2](1, \kappa x_1 y_2)$ (at least) and the same with 1's interchanged with 2's for the other group. Hence the kernel of restriction to these two groups is (at most) $\mathbb{F}_2[f_1, f_2]x_1x_2$. Of course, more exactly, these are the images up to filtration. In order to get more precise knowledge of the cohomology of U^1 we should get the exact images. But, as before, this is obtained by looking at the invariants in $H^*((\mathbb{Z}/2)^5)$.

For the next lemma define $\kappa \in H^1((\mathbb{Z}/2)^5)$ as dual to v_1v_3, λ dual to v_2t, τ dual to s^2t, α dual to v_2^2, and β dual to v_3^2. In terms of these classes, and related to the description above of $H^*(D_8)$ set $f_1 = \alpha^2 + \lambda\alpha$, $f_2 = \beta^2 + \tau\beta$, while $k = \kappa^2 + \kappa(\alpha + \beta) + \beta\lambda + \alpha\tau$.

LEMMA 6.4. *The invariant subalgebra* $H^*((\mathbb{Z}/2)^5)^{\langle t, v_1s^2t\rangle}$ *is given as* $\mathbb{F}_2[k, f_1, f_2, \lambda, \tau](L)$ *where* $L = \kappa\lambda\tau + \alpha\lambda^2 + \beta\tau^2$ *and* L *satisfies a quadratic equation over* $\mathbb{F}_2[k, f_1, f_2, \lambda, \tau]$. *As a consequence* $\mathrm{im}(\mathrm{res}^{(\mathbb{Z}/2)^5}_{U^1}) = H^*((\mathbb{Z}/2)^5)^{\langle t, v_1s^2t\rangle}$ *and* L *may be regarded as a representative for* $\{\kappa x_1 y_2\} \in H^*(U^1)$.

Proof. The action of $\langle t, v_1s^2t\rangle$ on $(\mathbb{Z}/2)^5$ is given as shown in Table 3. So, in cohomology the action of $\langle t, v_1s^2t\rangle$ is given as in Table 4.

TABLE 3

t-action	v_1s^2t-action
$tv_2tt = v_2^2v_2t$	$v_1s^2t(v_2t)ts^{-2}v_1^{-1} = (v_1v_3)v_2t$
$tv_2^2t = v_2^2$	$v_2s^2tv_2^2ts^{-2}v_1^{-1} = v_2^2$
$tv_3^2t = v_3^2$	$v_1s^2tv_3^2ts^{-2}v_1^{-1} = v_3^2$
$ts^2tt = (v_1v_3)s^2t$	$v_1s^2t(s^2t)ts^{-2}v_1^{-1} = v_3^2s^2t$
$tv_1v_3t = v_1v_3$	$v_2s^2t(v_1v_3)ts^{-2}v_1^{-1} = (v_1v_3)$

TABLE 4

t-action in cohomology	v_1s^2t-action in cohomology
$\beta \mapsto \beta + \lambda$	$\alpha \mapsto \alpha + \tau$
$\alpha \mapsto \alpha$	$\beta \mapsto \beta$
$\tau \mapsto \tau$	$\tau \mapsto \tau$
$\lambda \mapsto \lambda$	$\lambda \mapsto \lambda$
$\kappa \mapsto \kappa + \tau$	$\kappa \mapsto \kappa + \lambda$

Consequently, as is standard,

$$\mathbb{F}_2[\kappa, \alpha, \beta, \lambda, \tau]^{\langle t\rangle} = \mathbb{F}_2[\lambda, \tau, \alpha, f_2 = \beta(\beta+\lambda), \theta = \kappa(\kappa+\tau)](\beta\tau + \kappa\lambda = V),$$

where V satisfies a quadratic equation over the polynomial algebra $\mathbb{F}_2[\lambda, \tau, \alpha, f_2, \theta]$. We next note that the action of v_1s^2t on these elements is given as in Table 5.

TABLE 5

v_1s^2t-action on $\mathbb{F}_2[\lambda, \tau, \alpha, f_2, \theta](V)$
$V \mapsto V + \lambda^2$
$\theta \mapsto \theta + \lambda(\lambda+\tau)$
$f_2 \mapsto f_2$
$\lambda \mapsto \lambda$
$\tau \mapsto \tau$
$\alpha \mapsto \alpha + \tau$

Next, note that $\mathbb{F}_2[\lambda, \tau] = \mathbb{F}_2[\lambda(\lambda+\tau), \tau](1, \lambda)$, so we have

$$\mathbb{F}_2[\lambda, \tau, \alpha, \beta, \kappa]^{\langle t\rangle} = \mathbb{F}_2[\lambda(\lambda+\tau), \tau, f_2, \theta, \alpha](1, \lambda, V, \lambda V)$$

and there is an exact sequence of v_2s^2t-modules:

$$\mathbb{F}_2[\lambda(\lambda+\tau), \tau, f_2, \theta, \alpha](1, \lambda) \to \mathbb{F}_2[\lambda, \tau, \alpha, \beta, \kappa]^{\langle t\rangle} \to \mathbb{F}_2[\lambda(\lambda+\tau), \tau, f_2, \theta, \alpha](1, \lambda)V.$$

This gives rise to a long exact sequence

$$\mathbb{F}_2[\lambda(\lambda+\tau), \tau, f_2, \theta, \alpha]^{v_1s^2t}(1, \lambda) \to \mathbb{F}_2[\lambda, \tau, \alpha, \beta, \kappa]^{\langle t, v_1s^2t\rangle} \to \mathbb{F}_2[\lambda(\lambda+\tau), \tau, f_2, \theta, \alpha]^{v_1s^2t}(1, \lambda)V \xrightarrow{\delta} \mathrm{Ext}^1_{v_1s^2t}(\mathbb{F}_2, \mathbb{F}_2[\lambda(\lambda+\tau), \tau, \alpha, f_2, \theta](1, \lambda)) \to \dots$$

and

$$\begin{aligned}
&\mathbb{F}_2[\lambda(\lambda+\tau), \tau, \alpha, f_2, \theta](1, \lambda)^{v_1s^2t} \\
&= \mathbb{F}_2[(\lambda(\lambda+\tau)+\theta)\theta, f_1, f_2, \lambda(\lambda+\tau), \tau](1, \lambda)(\theta\tau + \alpha\lambda(\lambda+\tau) = L') \\
&= \mathbb{F}_2[(\lambda(\lambda+\tau)+\theta)\theta, f_1, f_2, \lambda, \tau](L'),
\end{aligned}$$

where L' satisfies a quadratic equation over $\mathbb{F}_2[(\lambda(\lambda+\tau)+\theta)\theta, f_1, f_2, \lambda, \tau]$. Also,

$$V + \theta + \alpha\lambda = \kappa^2 + \kappa(\lambda+\tau) + \beta\lambda + \alpha\tau = k$$

is $v_1 s^2 t$-invariant, and since this differs from V only by adding elements in the kernel, it follows that δ in the exact sequence above is identically 0 and we have shown that

$$\mathbb{F}_2[\lambda, \tau, \alpha, \beta, \kappa]^{\langle t, v_1 s^2 t\rangle} = \mathbb{F}_2[(\lambda(\lambda+\tau)+\theta)\theta, f_1, f_2, \lambda, \tau](1, k, L', kL').$$

We have the equation

$$(\lambda(\lambda+\tau)+\theta)\theta = k(k+\lambda\tau) + f_1\lambda^2 + f_2\tau^2,$$

so $\mathbb{F}_2[(\lambda(\lambda+\tau)+\theta)\theta, f_1, f_2, \lambda, \tau](k) = \mathbb{F}_2[k, f_1, f_2, \lambda, \tau]$. Finally,

$$L' = \tau\kappa^2 + \kappa\tau^2 + \alpha\lambda^2 + \alpha\lambda\tau.$$

We can thus write

$$L' + \tau k = L = \kappa\tau\lambda + \alpha\lambda^2 + \beta\tau^2$$

and the determination of the invariant subring is complete.

A direct counting argument now shows that the image of the restriction map is exactly the ring of invariants above, since it must certainly lie in this subring.

We have determined the image of $H^*(U^1)$ in $H^*(I \times II \times \langle v_1 v_3\rangle)$, and with a little more effort we even determine the action of the Steenrod algebra. The result is

PROPOSITION 6.5. (a) *The image of $H^*(U^1)$ in $H^*((\mathbb{Z}/2)^5 = I \times II \times \langle v_1 v_3\rangle)$ is a quadratic extension of a polynomial algebra on five variables*

$$\mathbb{F}_2[k_2, A_2, B_2, \lambda_1, \tau_1](L_3),$$

where $A = \alpha^2 + \alpha\tau$, $B = \beta^2 + \beta\lambda$, and $L = \kappa\lambda\tau + \alpha\lambda^2 + \beta\tau^2$. Moreover, L satisfies the relation

$$L^2 + \lambda\tau(\lambda+\tau)L + \lambda^2\tau^2 k + \lambda^4 A + \tau^4 B = 0.$$

(b) *The Steenrod algebra acts on this image as $Sq^1(A) = \tau A$, $Sq^1(B) = \lambda B$, $Sq^1 k = (\lambda+\tau)k + A\lambda + B\tau$, while $Sq^1(L) = \tau\lambda k + \lambda^2 A + \tau^2 B$, and*

$$Sq^2(L) = (\lambda+\tau)\lambda\tau k + (\lambda^2 + \tau\lambda + \tau^2)L.$$

(Here (a) is clear from the proposition above, and (b) is a direct verification using the explicit description of the generators given above.)

COROLLARY 6.6. *The groups*

$$\langle v_1 v_3\rangle \times I \times II, \quad \langle v_1 v_3\rangle \times II \times I \quad \text{and} \quad \langle v_1 v_3, v_1, v_2\rangle = \mathbb{Z}/2 \times (\mathbb{Z}/4)^2$$

detect $H^*(U^1)$ under restriction.

Proof. The only thing to check is the restriction to $\langle v_1 v_3, v_1, v_2\rangle$. Here we have the diagram

$$\begin{array}{ccccc}
\mathbb{Z}/2 & \longrightarrow & \langle v_1 v_3, v_1, vb_2\rangle & \longrightarrow & \mathbb{Z}/4 \times \mathbb{Z}/4 \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{Z}/2 & \longrightarrow & \tilde{\Gamma}_2^1 & \longrightarrow & D_8 \times D_8
\end{array}$$

and, as is well known, for the inclusion i: $\mathbb{Z}/4 \hookrightarrow D_8$ we have $i^*(x) = i^*(y) = e$, $i^*(f) = b$ where $H^*(\mathbb{Z}/4) = E(e) \otimes \mathbb{F}_2[b]$. Thus, $x_1x_2 \mapsto e_1e_2$, $y_1y_2 \mapsto e_1e_2$, and

$$\mathbb{F}_2[f_1, f_2] \xrightarrow{\cong} \mathbb{F}_2[b_1, b_2],$$

so the result follows.

7. *The extensions* $2^5 : \mathscr{A}_5$ *and* $2 \cdot L_3(4)$ *associated to* O'N

We have $L_2(4) = \mathrm{SL}_2(4) = \mathscr{A}_5$ and, if we use the usual presentation of $\mathscr{A}_5$,

$$\mathscr{A}_5 = \{A, B \mid A^3 = B^2 = (AB)^5 = 1\},$$

we can construct an explicit isomorphism by

$$r_1\colon A \mapsto \begin{pmatrix} \zeta_3 & \zeta_3 \\ 0 & \zeta_3^2 \end{pmatrix}, \quad r_1\colon B \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Now we consider the subgroup $\mathscr{L} \subset \mathrm{Syl}_2(\mathrm{O'N})$, $\mathscr{L} = \langle v_2^2, v_3^2, v_2t, v_3^2ts^2 \rangle = (\mathbb{Z}/2)^4$. An action of $\mathbb{Z}/3$ on $\mathscr{L}$ is given by

$$v_2^2 \to v_3^2 \to v_2^2v_3^2 \to v_2^2,$$
$$v_2t \to v_2tv_3^2ts^2 \to v_3^2ts^2 \to v_2t,$$

and this gives $\mathscr{L}$ the structure of a vector space $\mathbb{F}_4^2$. Choose v_2^2, v_2t as basis elements. Then the element $\beta \in \mathrm{Alp}_2^2$ acts on $\mathscr{L}$ as the matrix $\begin{pmatrix} \zeta_3 & 0 \\ 0 & \zeta_3^2 \end{pmatrix}$. Moreover, the subgroup $A_3 \subset U^1 = \langle (v_1v_3)^{-1}v_2^2t, v_1v_2^2s^2 \rangle \cong (\mathbb{Z}/2)^2$ commutes with the action of ζ_3 and is represented by the matrices

$$(v_1v_3)^{-1}v_2^2t \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$
$$v_1v_2^2s^2 \mapsto \begin{pmatrix} 1 & \zeta_3 \\ 0 & 1 \end{pmatrix}.$$

Thus we obtain an embedding

$$\mathscr{A}_4 = \langle \beta(v_1v_3)^{-1}v_2^2t, v_1v_2^2s^2 \rangle \hookrightarrow L_2(4)$$

as the subgroup of upper triangular matrices. This action extends to an action on

$$\langle v_1v_2^2v_3^{-1}, v_2^2, v_3^2, v_2t, v_3^2ts^2 \rangle = (\mathbb{Z}/2)^5$$

in $\langle U^1, \beta \rangle$ according to the easily-verified formulae described in Table 6.

TABLE 6

Action of $(v_1v_3)^{-1}v_2^2t$	Action of $v_1v_2^2s^2$	Action of β
$v_2^2 \to v_2^2$	$v_2^2 \to v_2^2$	$v_2^2 \to v_3^2$
$v_3^2 \to v_3^2$	$v_3^2 \to v_3^2$	$v_3^2 \to v_2^2v_3^2$
$v_2t \to v_2^2v_2t$	$v_2t \to (v_1v_2^2v_3^{-1})v_3^2v_2t$	$v_2t \to v_3^2ts^2$
$v_2tv_3^2ts^2 \to (v_1v_2^2v_3^{-1})v_3^2v_2tv_3^2ts^2$	$v_2tv_3^2ts^2 \to v_2^2v_3^2v_2tv_3^2ts^2$	$v_2tv_3^2ts^2 \to v_2t$

PROPOSITION 7.1. *There is a unique action of $\mathcal{A}_5$ on*

$$(\mathbb{Z}/2)^5 = \langle v_1 v_2^2 v_3^{-1},\ v_2^2,\ v_3^2,\ v_2 t,\ v_3^2 t s^2 \rangle$$

extending the action of $\mathcal{A}_4 = \langle \beta, (v_1 v_3)^{-1} v_2^2 t, v_1 v_2^2 t s^2 \rangle$. For this action we can represent β by the matrix $\begin{pmatrix} 1 & 0\ 0 & 0\ 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^2 \end{pmatrix}$, *while*

$$(v_1 v_3)^{-1} v_2^2 t \mapsto \begin{pmatrix} 1 & 0\ 0 & 0\ 1 \\ 0 & I & I \\ 0 & 0 & I \end{pmatrix}, \quad v_1 v_2^2 s^2 \mapsto \begin{pmatrix} 1 & 0\ 0 & 1\ 0 \\ 0 & I & \zeta \\ 0 & 0 & I \end{pmatrix}$$

where $\zeta = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Under these choices, it follows that

$$A \mapsto \begin{pmatrix} 1 & 0\ 0 & 0\ 1 \\ 0 & \zeta & \zeta \\ 0 & 0 & \zeta^2 \end{pmatrix}, \quad B \mapsto \begin{pmatrix} 1 & 0\ 0 & 0\ 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{pmatrix}.$$

Proof. Any extension of the action of $\mathcal{A}_5$ on $\mathcal{L}$ by a trivial action on a 1-dimensional $\mathbb{F}_2$-vector space is given by

$$r_2\colon A \mapsto \begin{pmatrix} 1 & v & w \\ 0 & \zeta & \zeta \\ 0 & 0 & \zeta^2 \end{pmatrix}, \quad r_2\colon B \mapsto \begin{pmatrix} 1 & x & x \\ 0 & 0 & I \\ 0 & I & 0 \end{pmatrix},$$

where v, w, and x are arbitrary (1×2) row vectors with coefficients in $\mathbb{F}_2$. (In fact it is easily checked that $(r_2(A))^3 = I$ and $(r_2(A) r_2(B))^5 = I$ for any values of v, w, x.) We find that $r_1(BABA^2B) = \begin{pmatrix} 1 & \zeta_3 \\ 0 & 1 \end{pmatrix}$ while $r_1(ABABA^2BA^2) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Applying r_2 to these words gives

$$r_2(BABA^2B) = \begin{pmatrix} 1 & 0 & x + (v+w)\zeta^2 \\ 0 & I & \zeta \\ 0 & 0 & I \end{pmatrix},$$

$$r_2(ABABA^2BA^2) = \begin{pmatrix} 1 & 0 & x\zeta + v\zeta + w \\ 0 & I & I \\ 0 & 0 & I \end{pmatrix}.$$

Now, using the explicit representations of β, $(v_1 v_3)^{-1} v_2^2 t$, and $v_1 v_2^2 s^2$ given in the statement of the proposition, we find that

$$\beta \cdot (v_1 v_3)^{-1} v_2^2 t \mapsto \begin{pmatrix} 1 & 0\ 0 & 0\ 1 \\ 0 & \zeta & \zeta \\ 0 & 0 & \zeta^2 \end{pmatrix},$$

which must thus be equal to $r_2(A)$. It follows that in this particular case $w = (0, 1)$, $v = (0, 0)$ while

$$x + (v+w)\zeta^2 = x + (0\ \ 1)\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = (1\ \ 0)$$

so $x = (0, 0)$ and the proof is complete.

REMARK. If we conjugate r_2 by elements of the form $v = \begin{pmatrix} 1 & m & n \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$ then it is easy to see that there are choices of m, n, such that v, w both become 0. Consequently, the set of central extensions

$$\mathbb{Z}/2 \to E \to 2^4 : \mathcal{A}_5$$

which can be identified with $H^2(2^4 : \mathcal{A}_5 ; \mathbb{Z}/2) = (\mathbb{Z}/2)^2$ are exactly determined by the x in our expression for $r_2(B)$ under the assumption that $v = w = 0$ and the extensions associated to the four different choices for x are all distinct.

The action of $\mathcal{A}_5$ on $H^*((\mathbb{Z}/2)^5)$

Let x_1, x_2, x_3, x_4, x_5, be dual to $v_1 v_3^{-1} v_2^2$, v_2^2, v_3^2, $v_2 t$, and $v_2 t v_3^2 t s^2$. Then the action of $\langle (v_1 v_3)^{-1} v_2^2 t,\ v_1 v_2^2 s^2,\ \beta \rangle$ is given by the adjoint matrices of their actions on the original coordinates, and similarly for the action of B. Thus we have Table 7.

TABLE 7

Action of $(v_1 v_3)^{-1} v_2^2 t$	Action of $v_1 v_2^2 s^2$	Action of β	Action of B
$x_1 \mapsto x_1 + x_5$	$x_1 \mapsto x_1 + x_4$	$x_1 \mapsto x_1$	$x_1 \mapsto x_1$
$x_2 \mapsto x_2 + x_4$	$x_2 \mapsto x_2 + x_5$	$x_2 \mapsto x_3$	$x_2 \mapsto x_4$
$x_3 \mapsto x_3 + x_5$	$x_3 \mapsto x_3 + x_4 + x_5$	$x_3 \mapsto x_2 + x_3$	$x_3 \mapsto x_5$
$x_4 \mapsto x_4$	$x_4 \mapsto x_4$	$x_4 \mapsto x_5 + x_4$	$x_4 \mapsto x_2$
$x_5 \mapsto x_5$	$x_5 \mapsto x_5$	$x_5 \mapsto x_4$	$x_5 \mapsto x_3$

We begin by considering the action of $(\mathbb{Z}/2)^2 = \langle (v_1 v_3)^{-1} v_2^2 t,\ v_1 v_2^2 s^2 \rangle$. To do this we make a change of variables as shown in Table 8.

TABLE 8

New variables	Action of $(v_1 v_3)^{-1} v_2^2 t$	Action of $v_1 v_2^2 s^2$
$\kappa = x_1$	$\kappa \mapsto \kappa + x_5$	$\kappa \to \kappa + x_4$
$a = x_1 + x_3$	$a \mapsto a$	$a \mapsto a + x_5$
$b = x_1 + x_2 + x_3$	$b \mapsto b + x_4$	$b \mapsto b$
$x_4 = x_4$	$x_4 \mapsto x_4$	$x_4 \mapsto x_4$
$x_5 = x_5$	$x_5 \mapsto x_5$	$x_5 \mapsto x_5$

Thus, with this change of basis the action agrees with that used in Lemma 6.4 and

$$\mathbb{F}_2[x_1, x_2, x_3, x_4, x_5]^{(\mathbb{Z}/2)^2} = \mathbb{F}_2[k, f_1', f_2', x_4, x_5](L'),$$

where

$$\begin{aligned} k &= \kappa^2 + \kappa(x_4 + x_5) + bx_5 + ax_4 \\ &= x_1^2 + (x_2 + x_3)x_5 + x_3x_4, \\ f_1' &= a(a + x_5) \\ &= x_1^2 + x_3^2 + x_1x_5 + x_3x_5, \\ f_2' &= b(b + x_4) \\ &= x_1^2 + x_2^2 + x_3^2 + x_1x_4 + x_2x_4 + x_3x_4, \\ L' &= \kappa x_4x_5 + bx_5^2 + ax_4^2 \\ &= x_1(x_4^2 + x_4x_5 + x_5^2) + x_3(x_5^2 + x_4^2) + x_2x_5^2. \end{aligned}$$

The action of β on these generators is given by $\beta(k) = k$, $\beta(f_1') = f_2'$, $\beta(f_2') = f_1' + f_2' + k$, $\beta(L') = L'$. To simplify somewhat we now replace f_1' by $f_1' + k$ and f_2' by $f_2' + k$. We will call these new generators f_1 and f_2, so we now have the expressions

$$\begin{aligned} f_1 &= x_3^2 + x_1x_5 + x_2x_5 + x_3x_4, \\ f_2 &= x_2^2 + x_3^2 + x_1x_4 + x_2x_4 + x_2x_5 + x_3x_5, \end{aligned}$$

and $\beta(f_1) = f_2$, $\beta(f_2) = f_1 + f_2$.

A further simplification occurs when we replace L' by

$$\begin{aligned} L &= x_4f_2 + (x_4 + x_5)f_1 + L' \\ &= x_3x_5^2 + x_3^2x_5 + x_2^2x_4 + x_2x_4^2. \end{aligned}$$

Clearly L is invariant under both β and B, so it is invariant under $\mathcal{A}_5$.

REMARK. The element

$$\begin{aligned} M &= x_4f_1 + x_5f_2 \\ &= x_4x_3^2 + x_3x_4^2 + x_5(x_2^2 + x_3^2) + x_5^2(x_2 + x_3) \end{aligned}$$

is also $\mathcal{A}_5$-invariant. This element and L above are the invariants already discussed (Proposition 2.3) in $\mathbb{F}_2[x_2, x_3, x_4, x_5]^{(\mathbb{Z}/2)^2}$.

Similarly, we have the other two $(\mathbb{Z}/2)^2$-invariant generators in $\mathbb{F}_2[x_2, x_3, x_4, x_5]$,

$$\begin{aligned} v &= f_1^2 + x_5^2k \\ &= x_3^4 + (x_2^2 + x_3x_4)x_5^2 + (x_2 + x_3)x_5^3 + x_3^2x_4^2, \\ w &= f_2^2 + x_4^2k \\ &= x_2^4 + x_3^4 + (x_2 + x_3)x_4^2x_5 + x_3x_4^3 + x_2^2x_4^2 + (x_2^2 + x_3^2)x_5^2. \end{aligned}$$

Consequently, we see that

$$\mathbb{F}_2[x_1, x_2, x_3, x_4, x_5]^{(\mathbb{Z}/2)^2} \cap \mathbb{F}_2[x_1, x_2, x_3, x_4] = \mathbb{F}_2[v, w, x_4, x_5](1, L, M, LM)$$

and the entire $(\mathbb{Z}/2)^2$-invariant subalgebra can be written in the form

$$\mathbb{F}_2[k, v, w, x_4, x_5](L)(1, f_1, f_2, f_1f_2).$$

The next proposition shows that we can replace f_1f_2 by an element θ which is invariant under $\mathcal{A}_5$.

PROPOSITION 7.2. *The element* $\theta = f_1f_2 + kx_4x_5 + v + w + x_4^4 + x_5^4 + x_4^2x_5^2$ *is an* $\mathscr{A}_5$*-invariant.*

Proof. We consider

$$\begin{aligned}(1+\beta+\beta^2)(f_1f_2 + kx_4x_5 + x_5^2x_4^2) &= f_1f_2 + kx_4x_5 + x_5^2x_4^2 \\ &\quad + (f_1+f_2)f_2 + k(x_4+x_5)x_4 + (x_4^2+x_5^2)x_4^2 \\ &\quad + f_1(f_1+f_2) + kx_5(x_4+x_5) + (x_4^2+x_5^2)x_5^2 \\ &= f_1f_2 + kx_4x_5 + v + w + x_4^4 + x_5^4 + x_4^2x_5^2,\end{aligned}$$

so θ is an $\mathscr{A}_4$-invariant. We expand it out in terms of $x_1, \ldots, x_5$ to verify that it is invariant under the action of B. The expansion is

$$\begin{aligned}x_1(x_5(x_2^2+x_3^2) + (x_2+x_3)x_5^2 + x_4x_3^2 + x_3x_4^2) + x_4^4 + x_5^4 + x_4^2x_5^2 \\ + x_2^4 + x_3^4 + x_2^2x_3^2 + x_2x_5^2x_4 + x_2x_4^2x_5 + x_2x_3^2x_4 + x_2^2x_3x_4 \\ + x_3^3x_5 + x_2^3x_5 + x_3^3x_4 + x_2x_5^3 + x_3x_5^3 + x_3x_4^3 \\ + x_2^2x_4^2 + x_3^2x_5^2 + x_3^2x_4^2 + x_2^2x_5^2 + x_2^2x_4x_5 + x_2x_3x_4^2 \\ + x_2x_3x_5^2 + x_3^2x_4x_5 + x_2x_3x_4x_5,\end{aligned}$$

and, by observation, interchanging x_2 with x_4 and x_3 with x_5 while fixing x_1 leaves the above expression invariant.

The $\mathscr{A}_5$*-invariant subring of* $H^*((\mathbb{Z}/2)^5)$

We can write

$$H^*((\mathbb{Z}/2)^5) = \mathbb{F}_2[k, x_2, x_3, x_4, x_5](1, x_1) = \coprod_{i=0}^{\infty} k^i\mathbb{F}_2[x_2, x_3, x_4, x_5](1, x_1),$$

and we have

LEMMA 7.3. (i) *The decomposition above decomposes* $H^*((\mathbb{Z}/2)^5)$ *as a direct sum of* $\mathscr{A}_5$*-modules. Consequently,*

$$\mathbb{F}_2[x_1, x_2, x_3, x_4, x_5]^{\mathscr{A}_5} = \coprod_{i=0}^{\infty} k^i\{\mathbb{F}_2[x_2, x_3, x_4, x_5](1, x_1)^{\mathscr{A}_5}\}.$$

(ii) *We have* $(\mathbb{F}_2[x_1, x_2, x_3, x_4, x_5]/(k))^{\mathscr{A}_5} \cong \mathbb{F}_2[x_2, x_3, x_4, x_5](1, x_1)^{\mathscr{A}_5}$ *as* $\mathbb{Z}_2$ *vector spaces. In particular, the natural projection*

$$\mathbb{F}_2[x_1, x_2, x_3, x_4, x_5]^{\mathscr{A}_5} \to (\mathbb{F}_2[x_1, x_2, x_3, x_4, x_5]/(k))^{\mathscr{A}_5}$$

is surjective.

Proof. As we have seen $\mathbb{F}_2[x_2, \ldots, x_5]$ is a sub-$\mathscr{A}_5$-module in $\mathbb{F}_2[x_1, \ldots, x_5]$. Moreover, each element $g \in \mathscr{A}_5$ acts on x_1 as $g(x_1) = x_1 + p_g(x_1)$ where $p_g(x_1) \in \langle x_2, x_3, x_4, x_5\rangle$. Consequently, the graded $\mathbb{F}_2$ vector space $\mathbb{F}_2[x_2, x_3, x_4, x_5](1, x_1)$ is a sub-$\mathscr{A}_5$-module of $\mathbb{F}_4[x_1, \ldots, x_5]$. Finally, since k is $\mathscr{A}_5$-invariant, and since, in each degree, the decomposition above is finite, (i) follows.

To see (ii) note that restricting the projection to $\mathbb{F}_2[x_2, x_3, x_4, x_5](1, x_1)$ gives an isomorphism of $\mathbb{F}_2$ graded vector spaces, but since the projection is an $\mathscr{A}_5$-map, it follows that it is an $\mathscr{A}_5$-isomorphism and the proof is complete.

COROLLARY 7.4. *Let $\mathscr{L}(t)$ be the Poincaré series for $\mathbb{F}_2[x_2, \ldots, x_5]^{\mathscr{A}_4}$ and $\mathscr{P}(t)$ the Poincaré series for $\mathbb{F}_2[x_2, \ldots, x_5]^{\mathscr{A}_5}$. Then the Poincaré series $\mathscr{V}(t)$ for $\mathbb{F}_2[x_1, \ldots, x_5]^{\mathscr{A}_5}$ is given as*

$$\mathscr{V}(t) = \frac{1}{1-t^2}\left((1+t)\mathscr{P}(t) + \frac{t^2+t^5}{(1-t)^2(1-t^4)^2} - \frac{(t+t^2+t^3)}{(1+t^3)}\mathscr{L}(t)\right)$$

and $(1-t^2)\mathscr{V}(t)$ is the Poincaré series for $(\mathbb{F}_2[x_1, \ldots, x_5]/(k))^{\mathscr{A}_5}$.

Proof. Let us write $\mathscr{M} = \mathbb{F}_2[x_2, \ldots, x_5](1, x_1)$. Then, as usual, there is a long exact sequence

$$0 \to \mathbb{F}_2[x_2, \ldots, x_5]^{\mathscr{W}} \to \mathscr{M}^{\mathscr{W}} \to \mathbb{F}_2[x_2, \ldots, x_5]^{\mathscr{W}} x_1$$
$$\xrightarrow{\delta} \operatorname{Ext}^1_{\mathbb{F}_2[\mathscr{W}]}(\mathbb{F}_2, \mathbb{F}_2[x_2, \ldots, x_5]) \to \operatorname{Ext}^1_{\mathbb{F}_2[\mathscr{W}]}(\mathbb{F}_2, \mathscr{M}) \to \ldots$$

where $\mathscr{W} = \mathscr{A}_4$ or $\mathscr{A}_5$. Moreover, from the $\operatorname{Ext}^1_{\mathbb{F}_2[\mathscr{W}]}$-terms onwards, the two exact sequences are the same. Thus, if we set $\mathscr{D} = \operatorname{im}(\delta)$, $\mathscr{D}$ is the same for both cases.

We now use the results of the last section to calculate the Poincaré series for $\mathscr{M}^{\mathscr{A}_4}$. We see that it is given by the formula

$$(1+t^4)\frac{\mathscr{L}(t)}{(1+t^3)} + t^2\left(\frac{1+t^3}{(1-t)^2(1-t^4)^2} - \frac{\mathscr{L}(t)}{(1+t^3)}\right),$$

and writing this as $(1+x)\mathscr{L}(t) - \mathscr{D}$ gives that

$$\mathscr{D} = (t+t^2+t^3)\frac{\mathscr{L}(t)}{(1+t^3)} - \frac{t^2+t^5}{(1-t)^2(1-t^4)^2}.$$

The result now follows by direct substitution.

COROLLARY 7.5. *The Poincaré series for the ring of invariants $\mathbb{F}_2[x_1, x_2, x_3, x_4, x_5]^{\mathscr{A}_5}$ is given by*

$$\mathscr{V}(t) = \frac{1+t^5+2t^6+t^7+t^{12}}{(1-t^2)(1-t^3)^2(1-t^4)(1-t^5)}.$$

The cohomology of $2 \cdot L_3(4)$.

Recall from the results in § 6 that $U^1 = \operatorname{Syl}_2(2 \cdot L_3(4))$ and that its cohomology can be expressed additively as

$$\mathbb{F}_2[\kappa^2] \otimes \mathbb{F}_2[f_1, f_2]\{\mathbb{F}_2[x_1, y_2](1, \kappa x_1 y_2) \oplus \overline{\mathbb{F}_2[x_2, y_1](1, \kappa x_2 y_1)} \oplus x_1 x_2\}.$$

Now U^1 fits into an extension

$$1 \to U^1 \to 2^5 : \mathscr{A}_4 \to \mathbb{Z}/3 \to 1$$

and this action can be seen to induce the cohomology action

$$\kappa^2 \mapsto \kappa^2, \quad \kappa x_1 y_2 \mapsto \kappa x_1 y_2, \quad \kappa x_2 y_1 \mapsto \kappa x_2 y_1,$$
$$f_1 \mapsto f_2, \quad f_2 \mapsto f_1 + f_2, \quad x_1 x_2 \mapsto x_1 x_2.$$

Let $r(t)$ be the Poincaré series for $H^*(2^5)^{\mathscr{A}_4}$, which from §3 and the proof of Corollary 7.4 is given as

$$\frac{1}{(1-t^2)}\left(\frac{(1+t^3)(1-t^2+t^4)(1+t^2+2t^5+t^6+2t^7+t^{10}+t^{12})}{(1-t^3)^2(1-t^4)(1-t^{12})}+\frac{t^2+t^5}{(1-t)^2(1-t^4)^2}\right),$$

and $s(t)$ be the series for $\mathbb{F}_2[f_1, f_2]^{\mathbb{Z}/3}$,

$$\frac{1+t^4+t^8}{(1-t^6)^2}.$$

Then our discussion in Lemmas 6.1 and 6.4 of the restriction maps $H^*(U^1)\rightarrow H^*(2^5)$ for the two subgroups $\langle I \times II, v_1v_3\rangle$ and $\langle II \times I, v_1v_3\rangle$ implies that the Poincaré series for $H^*(U^1)^{\mathbb{Z}/3}$ is given as

$$2r(t)-\frac{s(t)}{(1-t^2)}+\frac{t^2s(t)}{(1-t^2)}=2r(t)-s(t).$$

We have proved

PROPOSITION 7.6. *The Poincaré series for the group* $2^5:\mathscr{A}_4$ *is given as*

$$\begin{aligned}2r(t)-s(t)&=P_{2^5:\mathscr{A}_4}(t)\\&=\frac{t^9-4t^7+5t^6+6t^5-8t^4+3t^3+4t^2-2t+1}{(1-t)^2(1-t^2)(1-t^3)(1-t^6)},\end{aligned}$$

which can also be written as

$$\frac{\begin{array}{r}t^{17}+2t^{16}+3t^{14}+6t^{13}+6t^{12}+21t^{11}+22t^{10}+21t^9\\+31t^8+18t^7+19t^6+20t^5+8t^4+9t^3+4t^2+1\end{array}}{(1-t^3)(1-t^4)^3(1-t^6)}$$

in order to get strictly positive terms in the numerator.

If, as in Corollary 7.5, we denote the Poincaré series for $H^*(2^5)^{\mathscr{A}_5}$ as $\mathscr{V}(t)$ then the discussion in § 1 gives

$$P_{2^5:\mathscr{A}_5}(t)=P_{2^5:\mathscr{A}_4}(t)+\mathscr{V}(t)-r(t)$$

yielding

PROPOSITION 7.7. *The Poincaré series for* $2^5:\mathscr{A}_5$ *is* $\mathscr{V}(t)+r(t)-s(t)$ *which is equal to*

$$\frac{t^{13}+t^{12}-3t^{11}+t^{10}+6t^9-t^8+t^7+6t^6+2t^5-2t^4+4t^3+2t^2-t+1}{(1-t)(1-t^2)(1-t^3)(1-t^5)(1-t^6)}.$$

Finally, using the formula

$$P_{2\cdot L_3(4)}(t)=2P_{2^5:\mathscr{A}_5}(t)-P_{2^5:\mathscr{A}_4}(t)$$

and simplifying we obtain

THEOREM 7.8. *The Poincaré series for the cohomology of* $2 \cdot L_3(4)$ *is given by* $P_{2 \cdot L_3(4)}(t) = 2\mathscr{V}(t) - s(t)$ *which is equal to*

$$\frac{t^{13} + t^{12} - 3t^{11} + 4t^9 - t^8 + 2t^6 + t^5 - 2t^4 + 2t^3 + t^2 - t + 1}{(1-t)(1-t^2)(1-t^3)(1-t^5)(1-t^6)}.$$

In a sequel [3] we will use the preceding results to calculate the mod 2 cohomology of $4 \cdot L_3(4)$ and more importantly that of its extension, $4 \cdot L_3(4) \cdot 2_1$, which is an important factor in the cohomological analysis of the sporadic simple group O'N.

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