

ON DECOMPOSING SUSPENSIONS OF SIMPLICIAL SPACES

A. ADEM, A. BAHRI, M. BENDERSKY, F. R. COHEN, AND S. GITLER

ABSTRACT. Let X_\bullet denote a simplicial space. The purpose of this note is to record a decomposition of the suspension of the individual spaces X_n occurring in X_\bullet in case the spaces X_n satisfy certain mild topological hypotheses and where these decompositions are natural for morphisms of simplicial spaces. In addition, the summands of X_n which occur after one suspension are stably equivalent to choices of filtration quotients of the geometric realization $|X_\bullet|$. The purpose of recording these decompositions is that they imply decompositions of the single suspension of certain spaces of representations [1, 2] as well as other varieties and are similar to decompositions of suspensions of moment-angle complexes [4] which appear in a different context.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let X_\bullet denote a simplicial space. The purpose of this note is to give a decomposition of the suspension of the individual spaces X_n occurring in X_\bullet in case the spaces X_n satisfy certain mild topological hypotheses. These decompositions are natural for morphisms of simplicial spaces. In addition, the summands of X_n which occur after one suspension are stably equivalent to choices of filtration quotients of the geometric realization $|X_\bullet|$.

These structures occur in several contexts in useful ways and the following spaces admit decompositions of the type discussed above.

- (1) The suspension of the the loop space for a (path-connected) suspension of a CW-complex Y is homotopy equivalent to a bouquet of the suspension of smash products of Y [7, 11].
- (2) Spaces of ordered commuting n -tuples in a Lie group G , $Hom(\bigoplus_n \mathbb{Z}, G)$, assemble to give a simplicial space denoted $Hom(\mathbb{Z}_\bullet, G)$. If G is a closed subgroup of $GL_r(\mathbb{C})$, there are natural homotopy equivalences

Date: February 24, 2009.

A.Adem was partially supported by the NSF and NSERC. A.Bahri was partially supported by the award of a research leave from Rider University. F.Cohen was partially supported by NSF grant number 0340575 and DARPA grant number 2006-06918-01. S.Gitler was partially supported by the Department of Mathematics at Princeton University.

$$\Sigma \text{Hom}(\oplus_n \mathbb{Z}, G) \rightarrow \bigvee_{1 \leq k \leq n} \Sigma \bigvee^{\binom{n}{k}} \text{Hom}(\oplus_k \mathbb{Z}, G) / S_k(G)$$

where $S_k(G)$ denotes the *singular subspace* defined as those commuting k -tuples where at least one entry is equal to 1 (see [1]). The associated spaces of representations

$$\text{Rep}(\oplus_n \mathbb{Z}, G) = \text{Hom}(\oplus_n \mathbb{Z}, G) / G^{\text{ad}}$$

where G acts by conjugation also assemble into a simplicial space, with similar decompositions (see [3]). For G a finite group, the simplicial spaces $\text{Hom}(\mathbb{Z}_\bullet, G)$ and $\text{Rep}(\mathbb{Z}_\bullet, G)$ have natural connections with the cohomology of finite groups [2].

- (3) The suspension of moment-angle complexes as well as their generalizations are homotopy equivalent to a bouquet of *smash product moment-angle complexes* (see [4]).
- (4) A compact real algebraic variety as given in §5 is homeomorphic to the geometric realization of a simplicial space V_\bullet for which each space V_n decomposes after a single suspension.

It is the purpose of this note to show that many of these decompositions carry over into the context of simplicial spaces which satisfy a mild cofibration condition and to put these in a coherent picture. Recall that a simplicial space X_\bullet is a set of topological spaces X_n , $n \geq 0$, together with continuous maps $d_i : X_n \rightarrow X_{n-1}$ and $s_j : X_n \rightarrow X_{n+1}$ which satisfy the simplicial identities. A natural filtration of each space X_n is defined next.

Definition 1.1. Define subspaces $S^t(X_n) = \cup_{s_{i_1} s_{i_2} \cdots s_{i_t}} (X_{n-t}) \subset X_n$ with $S^0(X_n) = X_n$ and $S^1(X_n) = S(X)$ (for notational convenience). This defines a natural *decreasing filtration* for the spaces X_n in a simplicial space X_\bullet , where

$$s_0^n(X_0) = S^n(X_n) \subset \cdots \subset S^t(X_n) \subset \cdots \subset S(X_n) \subset S^0(X_n) = X_n$$

and $S^{n+1}(X_n)$ is empty by convention.

The following concepts appear in [9], Definition 11.2.

Definition 1.2. A pair of spaces (X, A) is said to be a *strong NDR pair* provided that there are maps $u : X \rightarrow [0, 1]$ and a homotopy $h : X \times [0, 1] \rightarrow X$ such that (X, A) is an NDR pair, namely

- (1) $A = u^{-1}(0)$,
- (2) $h(0, x) = x$ for all $x \in X$,
- (3) $h(a, t) \in A$ for all $(a, t) \in A \times [0, 1]$,

(4) if $u(x) < 1$ then $h(x, 1) \in A$.

and if $u(x) < 1$ then $u(h(x, t)) < 1$.

Definition 1.3. A simplicial space X_\bullet is said to be *proper* if each pair $(X_n, S(X_n))$ is a strong NDR-pair for all n .

Definition 1.4. A simplicial space X_\bullet is said to be *simplicially NDR* if each

$$(S^{t-1}(X_n), S^t(X_n))$$

is an NDR pair for all $t - 1 \geq 0$ and all n .

Note that every degenerate element x in X_n has a unique decomposition as

$$x = s_{j_r} s_{j_{r-1}} \cdots s_{j_1}(y)$$

where y is in X_{n-r} with y non-degenerate and $j_r > j_{r-1} > \cdots > j_1$. Given any sequence $I = (i_r, i_{r-1}, \dots, i_1)$, write $s_I(X_{n-r}) = s_{i_r} s_{i_{r-1}} \cdots s_{i_1}(X_{n-r})$ with $|I| = r$.

Definition 1.5. The sequence $I = (i_r, i_{r-1}, \dots, i_1)$ is said to be *admissible* provided $i_r > i_{r-1} > \cdots > i_1$. In case I is admissible, define $s_I(X_{n-r}) = s_I(X_{n-r})/s_I S(X_{n-r})$.

The point-set topological properties of X_\bullet are basic in these results. One instance is illustrated by the natural inclusion $\iota : S(X_n) \rightarrow X_n$ with mapping cone denoted $K(\iota)$. The proof of the main Theorem 1.6 implies the suspension of $K(\iota)$ is a retract of the suspension $\Sigma(X_n)$. On the other-hand, the quotient space $X_n/S(X_n)$ sometimes has independent useful features such as the case in [1] where these spaces are sometimes identified as natural Spanier-Whitehead duals of certain choices of Lie groups. To ensure that the properties of $X_n/S(X_n)$ are reflected in the structure of Theorem 1.6, it is useful to know that the inclusion $S(X_n) \rightarrow X_n$ is a cofibration.

The precise point-set topology for $|X_\bullet|$ admits several natural choices. Milnor originally topologized $|X_\bullet|$ by the natural quotient topology [12]. Milgram topologized BG , the geometric realization of a simplicial space similarly [10]. Subsequently, Steenrod topologized BG by the natural compactly generated topology [17]. Finally, May topologized $|X_\bullet|$ with the natural compactly generated, weak Hausdorff topology which is both elegant and convenient. This topology is used throughout the current article.

Theorem 1.6. *Assume that the simplicial space X_\bullet is simplicially NDR. Then the spaces X_n in the simplicial space X_\bullet are naturally filtered where*

$$s_0^n(X_0) = S^n(X_n) \subset \cdots \subset S^r(X_n) \subset \cdots \subset S(X_n) \subset S^0(X_n) = X_n.$$

Furthermore, these filtrations are split up to homotopy after suspending once. Thus there are homotopy equivalences which are natural for morphisms of simplicial spaces

$$(1) \quad \Theta(n) : \Sigma(X_n) \longrightarrow \bigvee_{0 \leq r \leq n} \Sigma(S^r(X_n)/S^{r+1}(X_n)),$$

$$(2) \quad H(n) : \Sigma(X_n) \longrightarrow \bigvee_{0 \leq r \leq n} \bigvee_J \Sigma(s_J(\widehat{X_{n-r}}))$$

where

$$J = (j_r, j_{r-1} \cdots, j_1)$$

is admissible with $|J| = r$ and $0 \leq r \leq n$ and

(3) the map $H(n)$ restricts to a homotopy equivalence

$$H(n)|_t : \Sigma(S^t(X_n)) \longrightarrow \bigvee_{t \leq r \leq n} \bigvee_J \Sigma(s_J(\widehat{X_{n-r}})).$$

Remarks:

(1) The splitting maps above in Theorem 1.6 are induced by the natural transformation from the identity to the decomposition maps $\Theta(n)$ regarded as functors from simplicial spaces to spaces.

(2) The finer decompositions obtained using the maps $H(n)$ arise from spaces $s_{J_t}(\widehat{X_{n-t}})$ with fixed $t = |J_t|$. In case t is fixed, the spaces $s_{J_t}(\widehat{X_{n-t}})$ are homeomorphic, but not equal.

(3) The notation in the proof and statement of Theorem 1.6 simplifies considerably if for fixed t , these differences of the $s_{J_t}(\widehat{X_{n-t}})$ are not addressed. Since the proof that moment-angle complexes admit stable decompositions in [4] uses an analogous proof which keeps track of these differences, the more technically complicated statement as well as proof are retained here.

The following was proved by J. P. May as Lemma 11.3 [9].

Proposition 1.7. *Assume that the simplicial space X_\bullet is proper. Then the geometric realization $|X_\bullet|$ is naturally filtered by $F_j|X_\bullet|$ with induced homeomorphisms*

$$\Sigma^j(X_j/S(X_j)) \rightarrow F_j|X_\bullet|/F_{j-1}|X_\bullet|.$$

The next corollary follows from Theorem 1.6 and Proposition 1.7. Notice that Corollary 1.8 implies that the stable summands in Theorem 1.6 are all given in terms of the filtration quotients $F_j|X_\bullet|/F_{j-1}|X_\bullet|$. In addition, the natural d^1 -differential in homology arising from the natural spectral sequence first investigated by G. Segal [15] then admits a geometric interpretation in terms of this decomposition, a point not developed here.

Corollary 1.8. *Assume that the simplicial space X_\bullet is proper and simplicially NDR. Then there are natural homotopy equivalences*

$$K(n, t) : \Sigma^{n+1}(S^t(X_n)/S^{t+1}(X_n)) \longrightarrow \bigvee_{J_t} \Sigma^{t+1}(F_{n-t}|X_\bullet|/F_{n-t-1}|X_\bullet|)$$

where J_t runs over all admissible sequences with $t = |J_t|$ and t is a fixed integer such that $0 \leq t \leq n$. Thus by Theorem 1.6, there are natural homotopy equivalences

$$\Theta(n) : \Sigma^{n+1}(X_n) \longrightarrow \bigvee_{0 \leq t \leq n} \bigvee_{J_t} \Sigma^{t+1}(F_{n-t}|X_\bullet|/F_{n-t-1}|X_\bullet|)$$

where J_t runs over all admissible sequences with $t = |J_t|$.

The authors would like to thank Tom Baird for discussions concerning real algebraic sets and Phil Hirschhorn for discussions concerning simplicial spaces.

2. SIMPLICIAL SPACES

The purpose of this section is to recall standard properties of simplicial spaces to be used in the proof of the main theorem. Throughout this article X_\bullet is assumed to be a simplicial space which is proper. Standard properties are stated in the next lemma.

Lemma 2.1. *If X_\bullet is a simplicial space which is simplicially NDR then it satisfies the following properties.*

(1) *The pair*

$$(s_{i_r} s_{i_{r-1}} \cdots s_{i_1} S^{t-1}(X_n), s_{i_r} s_{i_{r-1}} \cdots s_{i_1} S^t(X_n))$$

is an NDR pair for all sequences $(i_r, i_{r-1}, \dots, i_1)$ and all n and t with $r \leq n - t$. Thus the maps

$$s_{j_r} s_{j_{r-1}} \cdots s_{j_1} S^t(X_n) \rightarrow s_{j_r} s_{j_{r-1}} \cdots s_{j_1} S^{t-1}(X_n)$$

are cofibrations for all sequences $(i_r, i_{r-1}, \dots, i_1)$ and all n and t with $r \leq n - t$.

(2) *If $0 \leq r \leq n - 1$, there are homeomorphisms which are natural for morphisms of simplicial spaces*

$$\gamma(n, r) : \bigvee_J \widehat{s_J(X_{n-r})} \longrightarrow S^r(X_n)/S^{r+1}(X_n)$$

where

- (a) $J = (j_r, j_{r-1}, \dots, j_1)$ is admissible with $|J| = r$,
- (b) $\widehat{s_J(X_{n-r})} = s_J(X_{n-r})/s_J S(X_{n-r})$ and
- (c) $S^n(X_n)$ is equal to $s_0^n(X_0)$.

Proof. The pair $(S^{t-1}(X_n), S^t(X_n))$ is an NDR pair for all n and t with $t \leq n$ by hypothesis. Let $I = (i_r, i_{r-1} \cdots, i_1)$. Since the map $s_I = s_{i_r} s_{i_{r-1}} \cdots, s_{i_1}$ is a homeomorphism onto its image with one choice of inverse $d_{i_1} \cdots d_{i_{r-1}} d_{i_r}$, the pair

$$(s_{i_r} s_{i_{r-1}} \cdots s_{i_1} S^{t-1}(X_n), s_{i_r} s_{i_{r-1}} \cdots s_{i_1} S^t(X_n))$$

is an NDR pair for all sequences. The first part of Lemma 2.1 follows.

To prove the second part of Lemma 2.1, observe that $S^r(X_n) = \cup_I s_{i_r} s_{i_{r-1}} \cdots s_{i_1}(X_{n-r})$ for $I = (i_r, i_{r-1} \cdots, i_1)$ admissible with $|I| = r$. Now consider $J = (j_r, j_{r-1} \cdots, j_1)$ admissible for $I \neq J$. Since $I \neq J$, let t denote the largest integer for which the entries i_t and j_t are not equal; without loss of generality we can assume that $i_t > j_t$. Applying $d_{i_t} d_{i_{t+1}} \cdots d_{i_{r-1}} d_{i_r}$ and using the simplicial identities gives that $s_I(X_{n-r}) \cap s_J(X_{n-r}) \subset S^{r+1}(X_n)$.

Now if $J = (j_r, j_{r-1} \cdots, j_1)$ is admissible with $|J| = r$, then the inclusions give rise to a relative homeomorphism

$$F(n, r) : (\sqcup_I s_J(X_{n-r}), \sqcup_J s_J(S(X_{n-r}))) \rightarrow (S^r(X_n), S^{r+1}(X_n))$$

which thus induces a natural map

$$\gamma(n, r) : \vee_J s_J(\widehat{X_{n-r}}) \rightarrow S^r(X_n)/S^{r+1}(X_n)$$

that is a continuous bijection. Hence, it suffices to check that the map $\gamma(n, r)$ is open. Note that there is a commutative diagram

$$\begin{array}{ccc} \sqcup_J s_J(X_{n-r}) & \xrightarrow{F(n,r)} & S^r(X_n) \\ \pi_1 \downarrow & & \downarrow \pi_2 \\ \vee_J s_J(\widehat{X_{n-r}}) & \xrightarrow{\gamma(n,r)} & S^r(X_n)/S^{r+1}(X_n). \end{array}$$

where π_1 and π_2 are the natural projection maps. $F(n, r)$ is an open map as it is a local homeomorphism, and so it follows that the induced map $\gamma(n, r)$ is also open. The second part of Lemma 2.1 follows. \square

3. THE PROOF OF THEOREM 1.6

Theorem 1.6 gives two different decompositions:

- (1) One decomposition arises from the equivalence

$$\Theta(n) : \Sigma(X_n) \longrightarrow \bigvee_{0 \leq r \leq n} \Sigma(S^r(X_n)/S^{r+1}(X_n)).$$

(2) The second decomposition arises by using the equivalences

$$\widehat{H(n)}|_t : \Sigma(S^t(X_n)/S^{t+1}(X_n)) \longrightarrow \Sigma\left(\bigvee_{J_t} s_{J_t}(\widehat{X_{n-t}})\right)$$

induced by the $H(n)|_t$.

One direct proof arises from giving both splittings at once as given below. This proof is the precise setting of the analogue of classical James-Hopf invariant maps and how they fit into a simplicial setting as well as a splitting of simplicial spaces.

The details of proof come from a construction of the maps $\widehat{H(n)}|_t$ together with some tedious verifications using simplicial identities. The main work requires definitions of the analogue of James-Hopf invariants.

Let

$$D(n, r) = \bigvee_{|J|=r} s_J(\widehat{X_{n-r}})$$

where

- (1) $J = (j_r, j_{r-1}, \dots, j_1)$ is admissible,
- (2) $s_J(\widehat{X_{n-r}}) = s_J(X_{n-r})/s_J S(X_{n-r})$ and
- (3) $S^n(X_n)$ is equal to $s_0^n(X_0)$.

The method of proof is to exhibit a map

$$H(n) : \Sigma(X_n) \rightarrow \bigvee_{0 \leq r \leq n} \Sigma(D(n, r))$$

as described next with the following properties.

Lemma 3.1. *Assume that the simplicial space X_\bullet is simplicially NDR. Then there is a map*

$$H(n) : \Sigma(X_n) \rightarrow \bigvee_{0 \leq r \leq n} \Sigma(D(n, r))$$

with the following properties.

- (1) *The map $H(n)$ restricts to a map*

$$H(n)|_t : \Sigma(S^t(X_n)) \rightarrow \bigvee_{t \leq r \leq n} \Sigma(D(n, r)).$$

(2) *There is a morphisms of cofibrations*

$$\begin{array}{ccc}
 \Sigma(S^{t+1}(X_n)) & \xrightarrow{H(n)|_{t+1}} & \bigvee_{t+1 \leq r \leq n} \Sigma(D(n, r)) \\
 i(n, t+1) \downarrow & & \downarrow \bar{i}(n, t+1) \\
 \Sigma(S^t(X_n)) & \xrightarrow{H(n)|_t} & \bigvee_{t \leq r \leq n} \Sigma(D(n, r)) \\
 q(t) \downarrow & & \downarrow \bar{q}(t) \\
 \Sigma(S^t(X_n)/S^{t+1}(X_n)) & \xrightarrow{\widehat{H(n)}|_t} & \Sigma(D(n, t))
 \end{array}$$

where $i(n, t+1)$, $\bar{i}(n, t+1)$, $q(t)$ and $\bar{q}(t)$ are the natural inclusions and projections.

(3) *The map $\widehat{H(n)}|_t : \Sigma(S^t(X_n)/S^{t+1}(X_n)) \rightarrow \Sigma(D(n, t))$ is induced by $H(n)$.*

(4) *The map $\widehat{H(n)}|_t$ is a homotopy equivalence, and*

(5) *the map $H(n)|_{t+1} : \Sigma(S^{t+1}(X_n)) \rightarrow \bigvee_{t+1 \leq r \leq n} \Sigma(D(n, r))$ is a homotopy equivalence by downward induction on t starting with $t = n$ which is the equivalence $\widehat{H(n)}|_n : \Sigma(S^n(X_n) = s_0^n(X_0) \rightarrow \Sigma(D(n, n))$.*

Notice that Theorem 1.6 is an immediate consequence of Lemma 3.1. The key step is to define the map $H(n)$; the verification of its properties will be left to the reader.

Consider admissible sequences $I = (i_r, i_{r-1}, \dots, i_1)$ with $n \geq i_r > i_{r-1} > \dots > i_1 \geq 0$. Thus $s_I(X_{n-r})$ is a subspace of X_n . Define $\chi(I) = (i_1, i_2, \dots, i_{r-1}, i_r)$. Thus

$$d_{\chi(I)} = d_{i_1} d_{i_2} \dots d_{i_{r-1}} d_{i_r} \text{ and } d_{\chi(I)} \circ s_I(x) = x.$$

The natural lexicographical total ordering on such admissible sequences is obtained next from a partial ordering on all such sequences (not necessarily admissible).

Definition 3.2. If $I = (i_r, i_{r-1}, \dots, i_1)$ and $J = (j_r, j_{r-1}, \dots, j_1)$ are sequences with $|I| = |J| = r$ and $I \neq J$, define $I < J$ provided there exists a $p \leq r$ such that $i_k = j_k$ if $p < k \leq r$ with $i_p < j_p$.

Since an admissible sequence $I = (i_r, i_{r-1}, \dots, i_1)$ satisfies $n \geq i_r > i_{r-1} > \dots > i_1 \geq 0$, there are exactly $\binom{n+1}{r}$ choices of admissible sequences with $|I| = r$. Furthermore, the partial ordering in Definition 3.2 restricts to a total ordering on admissible sequences which satisfy $|I| = r$. The next lemma is a direct verification with details omitted.

Lemma 3.3. *Assume that X_\bullet is a simplicial space, both I and J are admissible with $|I| = |J| = r$ and that $x \in X_{n-r}$. Then*

$$d_{\chi(I)s_J}(x) = \begin{cases} x & \text{if } I = J \text{ and} \\ s_m(y) & \text{for some } s_m \text{ and some } y \text{ if } I < J. \end{cases}$$

Definition 3.4. Restrict to admissible sequences I_s with $|I_s| = r$. Define

$$\delta(r) : X_n \rightarrow (X_{n-r})^{\binom{n+1}{r}}$$

by

$$\delta(r) = d_{\chi(I_1)} \times d_{\chi(I_2)} \times \cdots \times d_{\chi(I_{\alpha(n,r)})}$$

where $I_s < I_{s+1}$ for all $1 \leq s$ and $\alpha(n, r) = \binom{n+1}{r}$.

If $r = 0$, then

$$\delta(0) : X_n \rightarrow X_n$$

is the identity map by convention.

The next lemma follows at once from the definitions.

Lemma 3.5. *Assume that X_\bullet is a simplicial space. Then the map*

$$\delta(r) : X_n \rightarrow (X_{n-r})^{\binom{n+1}{r}}$$

restricts to a map

$$\delta(r) : S^{r+1}(X_n) \rightarrow (S(X_{n-r}))^{\binom{n+1}{r}}$$

for all $0 \leq r \leq n$. Thus there is a commutative diagram

$$\begin{array}{ccc} S^{r+1}(X_n) & \xrightarrow{\delta(r)} & (S(X_{n-r}))^{\binom{n+1}{r}} \\ \downarrow i(n,r+1) & & \downarrow i(n-r,1)^{\binom{n+1}{r}} \\ S^r(X_n) & \xrightarrow{\delta(r)} & (X_{n-r})^{\binom{n+1}{r}} \\ \downarrow i(n,r) & & \downarrow 1 \\ X_n & \xrightarrow{\delta(r)} & (X_{n-r})^{\binom{n+1}{r}}. \end{array}$$

The definition of the map $H(n)$ is given next. Recall the maps

$$\delta(r) : X_n \rightarrow (X_{n-r})^{\binom{n+1}{r}}$$

given by

$$\delta(r) = d_{\chi(I_1)} \times d_{\chi(I_2)} \times \cdots \times d_{\chi(I_{\alpha(n,r)})}$$

where $I_s < I_{s+1}$ for all $s \geq 1$. The coordinates in $(X_{n-r})^{\binom{n+1}{r}}$ are indexed by $\chi(I)$ for I admissible with $|I| = r$. Let

$$P_{\chi(I)} : (X_{n-r})^{\binom{n+1}{r}} \rightarrow X_{n-r}$$

denote the projection map to the $\chi(I)$ -th coordinate.

Recall that

$$D(n, r) = \bigvee_{|J|=r} \widehat{s_J(X_{n-r})}$$

where

- (1) $J = (j_r, j_{r-1}, \dots, j_1)$ is admissible,
- (2) $s_J(\widehat{X_{n-r}}) = s_J(X_{n-r})/s_J S(X_{n-r})$ and
- (3) $S^n(X_n)$ is equal to $s_0^n(X_0)$.

Definition 3.6. (1) Let

$$\nu(n, J) : \Sigma((X_{n-r})^{\binom{n+1}{r}}) \longrightarrow s_J(\widehat{X_{n-r}})$$

denote the composite $\nu(n, J) = \bar{\sigma}_J \circ P_{\chi(J)}$.

(2) Let

$$\lambda(n, J) : \Sigma(X_n) \longrightarrow s_J(\widehat{X_{n-r}})$$

denote the composite $\lambda(n, J) = \nu(n, J) \circ \Sigma(\delta(r))$.

(3) Let

$$\Phi(n, r) : \Sigma(X_n) \longrightarrow \bigvee_{|J|=r} s_J(\widehat{X_{n-r}})$$

denote the sum

$$\Phi(n, r) = \sum_{|J|=r} \lambda(n, J)$$

where the index is over all admissible sequence J with a fixed order of summation.

(4) Define

$$H(n) : \Sigma(X_n) \rightarrow \bigvee_{0 \leq r \leq n} \Sigma(D(n, r))$$

as the sum

$$H(n) = \sum_{0 \leq r \leq n} \Phi(n, r)$$

with a fixed order of summation.

Therefore this defines the desired map

$$H(n) : \Sigma(X_n) \rightarrow \bigvee_{0 \leq r \leq n} \Sigma(D(n, r))$$

which when restricted to $S^t(X_n)$ makes the following diagram commute (up to homotopy)

$$\begin{array}{ccc} \Sigma(S^t(X_n)) & \xrightarrow{H(n)|_t} & \bigvee_{t \leq r \leq n} \Sigma(D(n, r)) \\ i(n, t) \downarrow & & \downarrow \bar{i}(n, t) \\ \Sigma(X_n) & \xrightarrow{H(n)} & \bigvee_{0 \leq r \leq n} \Sigma(D(n, r)). \end{array}$$

The desired properties of $H(n)$ can be readily verified and are left to the reader.

4. SIMPLICIAL SETS AND REAL ALGEBRAIC SETS

The purpose of this section is to (i) recall standard properties of simplicial complexes as well as (ii) the way in which Theorem 1.6 can be applied to real algebraic varieties.

Let K denote an abstract simplicial complex on m vertices labeled by the set

$$[m] = \{1, 2, \dots, m\}.$$

Thus a simplex σ of K is given by an ordered sequence

$$\sigma = (i_1, \dots, i_k)$$

with $1 \leq i_1 < \dots < i_k \leq m$ such that if $\tau \subset \sigma$, then τ is a simplex of K . Recall the simplicial set $\Delta(K)$ obtained from an abstract simplicial complex as defined in [6] page 234.

Definition 4.1. The simplicial set $\Delta(K)$ is defined as follows.

- $\Delta(K)$ has n -simplices given by the $(n+1)$ -tuples of vertices (v_0, v_1, \dots, v_n) for which $v_0 \leq v_1 \leq \dots \leq v_n$
- the face and degeneracy operators are given by

$$d_i(v_0, v_1, \dots, v_n) = (v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_n)$$

and

$$s_i(v_0, v_1, \dots, v_n) = (v_0, \dots, v_i, v_i, \dots, v_n).$$

As pointed out in [6], the following result follows from the two paragraphs before Theorem 15 on page 111 in [16].

Theorem 4.2. *The geometric realization $|\Delta(K)|$ is homeomorphic to $|K|$.*

The benefits of this construction are expressed in the following result, which is left to the reader for verification.

Proposition 4.3. *The simplicial space $\Delta(K)$ is proper and simplicially NDR. Thus Theorem 1.6 applies to $\Delta(K)$.*

Definitions and basic properties of real algebraic and semi-algebraic varieties are listed next with main reference [5].

Definition 4.4. An affine real algebraic set is the the common zero set of a finite number of real polynomials in $\mathbb{R}[x_1, \dots, x_k]$ for some $k \geq 1$. A real algebraic set has a topology induced from \mathbb{R}^k equipped with the Euclidean topology which is called the classical topology.¹ A real semi-algebraic set is a subset of \mathbb{R}^k for some k , which is a finite union of sets each determined by a finite number of polynomial inequalities.

The following theorem was proven in [5] section 9.4.1.

Theorem 4.5. *If X be a compact semi-algebraic set then it is triangulable, i.e. it is homeomorphic to the geometric realization of a finite simplicial complex.*

Combining the fact that a compact semi-algebraic set is triangulable with Theorem 1.6 above, the next corollary follows at once.

Corollary 4.6. *Let X be a compact semi-algebraic set which is the geometric realization of a finite simplicial complex K with order complex $\Delta(K)$. Then $\Delta(K)$ is a simplicial space for which the n -space $\Delta(K)_n$ admits a decomposition after suspension given in Theorem 1.6.*

5. EXAMPLES AND PROBLEMS

This section gives a list of examples of decompositions which arise from the above method.

Example 5.1. Let G denote a closed subgroup of $GL_r(\mathbb{C})$ and consider the space of ordered commuting n -tuples in G , denoted $Hom(\oplus_n \mathbb{Z}, G)$ and topologized as a subspace of the product G^n . These spaces can be assembled to form a simplicial space which is proper and simplicially NDR, and thus Theorem 1.6 can be applied, yielding in particular the decompositions described in §1 (see [1] and [2] for details).

Example 5.2. The group G acts on $Hom(\oplus_n \mathbb{Z}, G)$ by conjugation with orbit space denoted $Rep(\oplus_n \mathbb{Z}, G)$. These spaces also form a simplicial space satisfying the hypotheses of 1.6 and thus admit analogous stable decompositions in case G is a compact Lie group (see [3]).

¹The Zariski topology, for which closed sets are given by algebraic subsets, will not be used here.

Example 5.3. A product of spaces $X_1 \times X_2 \times \cdots \times X_n$ decomposes as a wedge of suspensions after suspending once. These decompositions induce related decompositions of moment-angle complexes, in special cases, homotopy equivalent to various varieties obtained from complements of coordinate planes in Euclidean space (see [4]). The decompositions of simplicial spaces given here are more general than the decompositions of generalized moment-angle complexes given in [4]; however they are also coarser.

It seems useful to conclude by formulating some problems associated to the topics in this paper:

Problems:

- (1) Give interesting examples for which Corollary 4.6 can be used to provide useful information concerning semi-algebraic sets.
- (2) Identify useful conditions for which the splittings of Theorem 1.6 imply decompositions for the suspension of $|X_\bullet|$ for more general simplicial spaces X_\bullet .
- (3) Identify useful conditions when $Tot_\bullet(X)$ can be stably split.

REFERENCES

- [1] A. Adem and F. R. Cohen, *Commuting elements and spaces of homomorphisms*, Mathematische Annalen, **338**(2007), 587-626.
- [2] A. Adem, F. R. Cohen, E. Torres-Giese *Commuting elements, simplicial spaces and filtrations of classifying spaces*, arXiv:0901.0137v2 [math.AT].
- [3] A. Adem, F. R. Cohen, J. M. Gomez *Stable splittings and almost commuting elements in Lie groups*, in preparation.
- [4] A. Bahri, M. Bendersky, F. R. Cohen, and S. Gitler, *The polyhedral product functor: a method of computation for moment-angle complexes, arrangements and related spaces*, arXiv:0711.4689v1 [math.AT].
- [5] J. Bochnak, M. Coste, M. Roy, *Real Algebraic Geometry*, Springer-Verlag Berlin Heidelberg, (1998).
- [6] A. Bousfield and D. Kan, *Homotopy limits, completions and localizations*, Springer-Verlag Lecture Notes in Math., **304**, (1972).
- [7] I. James, *Reduced product spaces*, Ann. of Math., 62, No. 1 (1955), 170-197.
- [8] P. Hirschhorn, *Model categories and their localizations*, AMS Mathematical Surveys and Monographs, No. 99.
- [9] J.P. May, *The geometry of iterated loop spaces* Lecture Notes in Mathematics **171**, 1972.
- [10] R. J. Milgram, *The bar construction and abelian H-spaces*, Illinois J. Math. 11 (1967),. 242-250.
- [11] J. Milnor, *Universal Bundles I, II*, Ann. of Math., **63**(1956), 272-284 and **65**(1956), pp. 430-436
- [12] ———, *The geometrical realization of a semi-simplicial complex*, Ann. of Math., **65**(1957), 357–362.
- [13] ———, *On the construction $F[K]$* , In: A student’s Guide to Algebraic Topology, J.F. Adams, editor, Lecture Notes of the London Mathematical Society, **4**(1972), 119-136.
- [14] M. Rothenberg, and N. E. Steenrod, *The cohomology of classifying spaces of H-spaces*, Bull. A.M.S., **71**(1965), 872-875.

- [15] G. Segal, *Classifying spaces and spectral sequences*, Pub. Math. des I. H. E. S., **34**(1968), 105-112.
- [16] E. Spanier, **Algebraic Topology**, McGraw-Hill, New York, 1966.
- [17] N. E. Steenrod, *A convenient category of topological spaces*, Michigan Math. Journal **14** (1967), 133-152.

DEPARTMENT OF MATHEMATICS UNIVERSITY OF BRITISH COLUMBIA, VANCOUVER, B. C., CANADA
E-mail address: adem@math.ubc.ca

DEPARTMENT OF MATHEMATICS, RIDER UNIVERSITY, LAWRENCEVILLE, NJ 08648, U.S.A.
E-mail address: bahri@rider.edu

DEPARTMENT OF MATHEMATICS CUNY, EAST 695 PARK AVENUE NEW YORK, NY 10065, U.S.A.
E-mail address: mbenders@xena.hunter.cuny.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ROCHESTER, ROCHESTER, NY 14625, U.S.A.
E-mail address: cohf@math.rochester.edu

DEPARTMENT OF MATHEMATICS, CINVESTAV, SAN PEDRO ZACATENCO, MEXICO, D.F. CP 07360
APARTADO POSTAL 14-740, MEXICO
E-mail address: sgitler@math.cinvestav.mx