SÉMINAIRE N. BOURBAKI

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Séminaire N. Bourbaki, 2001-2002, exp. n° 894, p. 1-17.
<http://www.numdam.org/item?id=SB_2001-2002__44__1_0>

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http://www.numdam.org/

Séminaire BOURBAKI
54e année, 2001-2002, n° 894, p. 1 à 17

Novembre 2001

FINITE GROUP ACTIONS ON ACYCLIC 2-COMPLEXES

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1. A BRIEF HISTORY AND MOTIVATION

A simple consequence of the Brouwer fixed point theorem is that any cyclic group acting on a closed disk $\mathbb{D}^n$ must have a *fixed point*. The classical work of P.A. Smith [18] shows that if P is a finite p-group, then any action of P on $\mathbb{D}^n$ must have a fixed point. From this there arises a very evident question: is there a group of *composite order* which can act on some $\mathbb{D}^n$ *without* any fixed points? This was settled in the affirmative by Floyd and Richardson in 1959 (see [7]), when they constructed fixed point free actions of the alternating group A_5 on disks.

These examples stood out as special exceptions for several years — indeed no other such actions were known to exist until Oliver (see [13]) obtained a complete characterization of those finite groups which can act on disks without stationary points. To explain it we first need to introduce some group-theoretic concepts.

DEFINITION 1.1. — *For p and q primes, let $\mathcal{G}_p^q$ be the class of finite groups G with normal subgroups $P \lhd H \lhd G$, such that P is of p-power order, G/H is of q-power order, and H/P is cyclic; and let $\mathcal{G}_p = \cup_q \mathcal{G}_p^q$, $\mathcal{G} = \cup_p \mathcal{G}_p$.*

THEOREM 1.2. — *A finite group G has a fixed point free action on a disk if and only if $G \notin \mathcal{G}$. In particular, any non-solvable group has a fixed point free action on a disk, and an abelian group has such an action if and only if it has three or more non-cyclic Sylow subgroups.*

The smallest group with a fixed point free action on a disk is in fact the alternating group A_5; the smallest *abelian* group with such an action is $C_{30} \times C_{30}$. Oliver also proved that a group G will have a fixed point free action on a finite $\mathbb{Z}_p$-acyclic(1)

(1) Recall that a complex X is said to be $\mathbb{Z}_p$-*acyclic* if its reduced mod p homology is identically zero; if its reduced integral homology vanishes it is said to be *acyclic*.

complex if and only if $G \notin \mathcal{G}_p$. Note that a group G will act without fixed points on a contractible complex if and only if it acts without fixed points on an *acyclic* complex.

Taking into account Oliver's result, an obvious problem is that of constructing fixed point free actions on contractible or acyclic complexes of small dimension. A well-known theorem by J.-P. Serre states that any finite group acting on a tree must have a fixed point (see [17]). However, the situation for contractible 2-dimensional complexes is much more complicated — in fact it is an open question whether or not it is possible for a finite group to act on such a complex without fixed points. We will restrict our attention from now on to the case of acyclic 2-dimensional complexes.

Our starting point is the classical example of an A_5-action on an acyclic 2-dimensional complex without fixed points, which we now briefly recall. In fact it is an essential ingredient in the construction due to Floyd and Richardson which we discussed above. This example is constructed by considering the left A_5 action on the Poincaré sphere $\Sigma^3 = \mathrm{SO}(3)/\mathrm{A}_5$; as the action has a single fixed point (corresponding to the fact that A_5 is self-normalizing in $\mathrm{SO}(3)$) we may remove an open 3-disk U around it to obtain an acyclic compact 3-manifold $\Sigma^3 - U$ with a fixed point free action of A_5. This in turn can be collapsed to a 2-dimensional subcomplex $X \simeq \Sigma^3 - U$ upon which A_5 still acts without fixed points. Equivalently we could identify Σ^3 with the space obtained by identifying opposite faces of the solid dodecahedron in an appropriate way[(2)] and consider the A_5 action induced by the usual action on the dodecahedron. The fixed point is the center of D and by collapsing to its boundary we obtain an explicit 2-dimensional complex $X = \partial D/\simeq$ with a fixed point free action of A_5 which has 6 pentagonal 2-cells, 10 edges and 5 vertices. Note that if we take the join $A = \mathrm{A}_5 * X$ with the induced diagonal action of A_5, then we obtain a simply connected and acyclic complex, hence a contractible complex with a fixed point free action. From this we can obtain a fixed point free A_5 action on a disk via regular neighborhoods (as explained in [4], p. 57). This is the basic step in the construction of the Floyd-Richardson examples.

Now an obvious question arises from all of this: can we characterize those finite groups which can act without fixed points on acyclic 2-dimensional complexes? Indeed, are there even other examples of such actions? Remarkably it turns out that these actions are only possible for a small class of *simple* groups, and their precise determination and description will require using the classification of finite simple groups.

[(2)] To be precise: identify opposite faces of the dodecahedron by the map which pushes each face through the dodecahedron and twists it by $2\pi/10$ about the axis of the push in the direction of a right hand screw (see [12]).

2. STATEMENT OF RESULTS

In this note we will report on recent work of Oliver and Segev (see [15]) where they provide a complete description of the finite groups which can act on a 2-dimensional acyclic complex without fixed points. Their work builds on previous contributions by Oliver [13], [14], Segev [16] and Aschbacher-Segev [2]. To state their main result we need to introduce a useful technical condition for G-CW complexes. From now on we will use the term G-complex to refer to a G-CW-complex, however these results also hold for simplicial complexes with an admissible G-action[(3)].

DEFINITION 2.1. — *A G-complex X is said to be* essential *if there is no normal subgroup $1 \neq N \lhd G$ with the property that for each $H \subseteq G$, the inclusion $X^{HN} \to X^H$ induces an isomorphism on integral homology.*

If there were such a normal subgroup N, then the G-action on X is 'essentially' the same as the G-action on X^N, which factors through a G/N-action. For 2-dimensional complexes we have:

THEOREM 2.2. — *Let G be any finite group and let X be any 2-dimensional acyclic G-complex. Let N denote the subgroup generated by all normal subgroups $N' \lhd G$ such that $X^{N'} \neq \varnothing$. Then X^N is acyclic, X is essential if and only if $N = 1$, and if $N \neq 1$ then the action of G/N on X^N is essential.*

Based on this we restrict our attention to essential complexes, and we can now state the main result in [15]:

THEOREM 2.3. — *Given a finite group G, there is an essential fixed point free 2-dimensional acyclic G-complex if and only if G is isomorphic to one of the simple groups $\mathrm{PSL}_2(2^k)$ for $k \geqslant 2$, $\mathrm{PSL}_2(q)$ for $q \equiv \pm 3 \pmod 8$ and $q \geqslant 5$ or $\mathrm{Sz}(2^k)$ for odd $k \geqslant 3$. Furthermore the isotropy subgroups of any such G-complex are all solvable.*

Among the groups listed above, only the *Suzuki groups* $\mathrm{Sz}(q)$ are not commonly known; we will provide a precise definition for them as subgroups of $\mathrm{GL}_4(\mathbb{F}_q)$ in § 5. Note that the theorem is stated for arbitrary acyclic 2-dimensional complexes; there is no need to require that the complexes be finite.

Our main goal will be to explain the proof of this result. This naturally breaks up into a number of different steps. We begin in § 3 by explaining how the theorem can be reduced to simple groups, based mostly on a theorem due to Segev [16]. Next in § 4 we describe techniques for constructing the desired actions, using methods derived from Oliver's original work on group actions on acyclic complexes as well as a more detailed analysis of the associated subgroup lattices. This is then applied in § 5 to

(3) A simplicial complex X with a G action is called admissible if the action permutes the simplices linearly and sends a simplex to itself only via the identity.

provide explicit descriptions of fixed point free actions on an acyclic 2-complex for the simple groups listed in the main theorem. In § 6 we sketch conditions which imply the *non-existence* of fixed point free actions on acyclic 2-complexes for most simple groups; this part requires detailed information about the intricate subgroup structure for the finite simple groups. Finally in § 7 we use the classification of finite simple groups and the previous results to outline the proof of the main theorem, which has been previously reduced to verification for simple groups. We also make a few concluding remarks.

Remark 2.4. — The background required to understand these results and their proofs includes: (1) very basic equivariant algebraic topology; (2) familiarity with subgroup complexes and related constructions; and (3) a very detailed knowledge of the subgroup structure of the finite simple groups. As many of the arguments in the proofs depend on the particular properties of these groups, our synopsis cannot hope to contain complete details. However the original paper by Oliver and Segev [15] is written in a clear style accessible to a broad range of mathematicians and hence those interested in a deeper understanding of the results presented here should consult it directly.

3. REDUCTION TO SIMPLE GROUPS

The goal of this section will be to explain how we can restrict our attention to finite simple groups. This is based on the following key result due to Segev [16] :

THEOREM 3.1. — *Let X be any 2-dimensional acyclic G-complex. Then the subcomplex of fixed points X^G is either acyclic or empty. If G is solvable then X^G is acyclic.*

Proof. — Although Segev's original proof uses the Odd Order Theorem, it can be proved more directly. One can show that if X is an acyclic G-complex, then $H_1(X^G, \mathbb{Z}) = H_2(X^G, \mathbb{Z}) = 0$. Hence we are reduced to establishing that there is only one connected component (provided X^G is non-empty). For solvable groups this can be proved directly using induction and Smith Theory. Otherwise we consider a minimal group G for which a counterexample exists. If X^G has k components then in fact it can be shown that X looks roughly like the join of an acyclic fixed point free G-complex Y with a set of k points. However as X is 2-dimensional, Y would have to be 1-dimensional, in other words a tree, and this cannot hold. □

Remark 3.2. — The reader should keep in mind that Theorem 3.1 is a basic tool in many of our subsequent arguments and it will be used explicitly and implicitly on several occasions.

COROLLARY 3.3. — *Let X be any 2-dimensional acyclic G-complex. Assume that $A, B \subset X$ are G-invariant acyclic subcomplexes such that $X^G \subset A \cup B$; then $A \cap B \neq \varnothing$.*

Proof. — Assume that $A \cap B = \varnothing$ and let Z denote the G-complex obtained by identifying A and B each to a point. Then Z is acyclic since A and B both are, and Z^G consists of two points, thus contradicting Theorem 3.1. □

As an immediate consequence of Corollary 3.3 we obtain

LEMMA 3.4. — *Let X be a 2-dimensional acyclic G-complex. Then if $H, K \subset G$ are such that $H \subset N_G(K)$ and X^H, X^K are both non-empty, then $X^{HK} \neq \varnothing$. Moreover, if $H \subset G$ is such that $X^H = \varnothing$, then $X^{C_G(H)} \neq \varnothing$.*

Proof. — Since H normalizes K, both X^H and X^K are H-invariant acyclic subcomplexes of X. Hence we conclude from Corollary 3.3 that $\varnothing \neq X^H \cap X^K = X^{HK}$. For the second part, it suffices to prove it when H is minimal among subgroups without fixed points. Fix a pair $M, M' \subset H$ of distinct maximal subgroups (note that by Theorem 3.1, H is non-solvable). Then X^M and $X^{M'}$ are non-empty, but $X^M \cap X^{M'} = X^{\langle M, M' \rangle} = X^H = \varnothing$. Hence X^M and $X^{M'}$ are disjoint $C_G(H)$-invariant acyclic subcomplexes of X, meaning (by Corollary 3.3) that their union cannot contain $X^{C_G(H)}$, whence it must be non-empty. □

We can now prove one of the main reduction results, which allows us to restrict our attention to essential complexes.

THEOREM 3.5. — *Let G be any finite group, and let X be any 2-dimensional acyclic G-complex. Let N be the subgroup generated by all normal subgroups $N' \lhd G$ such that $X^{N'} \neq \varnothing$. Then X^N is acyclic; X is essential if and only if $N = 1$ and if $N \neq 1$ then the action of G/N on X^N is essential.*

Proof. — If $X^{N_1} \neq \varnothing$ and $X^{N_2} \neq \varnothing$ for $N_1, N_2 \lhd G$, then $X^{\langle N_1, N_2 \rangle} \neq \varnothing$ by Lemma 3.4. So we infer that X^N is non-empty, hence acyclic (by Theorem 3.1). Note that the action of any non-trivial normal subgroup of G/N on X^N has empty fixed point set, hence the action of G/N on X^N is always essential. Finally, assume that $N \neq 1$; by Theorem 3.1 we have that for all $H \subset G$, X^H and X^{NH} are acyclic or empty; and $X^{NH} \neq \varnothing$ if $X^H \neq \varnothing$, by Lemma 3.4. Hence the inclusion $X^{NH} \to X^H$ is always an equivalence of integral homology, and hence X is not essential. □

This result will allow us to focus our attention on actions of simple groups.

THEOREM 3.6. — *If G is a non-trivial finite group for which there exists an essential 2-dimensional acyclic G-complex X, then G is almost simple. In fact there is a normal simple subgroup $L \lhd G$ such that $X^L = \varnothing$ and such that $C_G(L) = 1$.*

Proof. — We know from Theorem 3.5 that $X^N = \varnothing$ for all normal subgroups $1 \neq N \lhd G$, including the case $N = G$. Now fix a minimal normal subgroup $1 \neq L \lhd G$; we know from Theorem 3.1 that L is not solvable, as $X^L = \varnothing$. Hence L is a direct product of isomorphic non-abelian simple groups (see [8], Thm 2.1.5). Assume that L is not simple; by Lemma 3.4, $X^H \neq \varnothing$ for some simple factor $H \lhd L$. Also, note that $L = \langle gHg^{-1} \mid g \in G \rangle$ since it is a minimal normal subgroup. Now we have that $X^{gHg^{-1}} = gX^H \neq \varnothing$ for all $g \in G$, hence applying the same lemma once again, but now to the L-action on X, we infer that $X^L \neq \varnothing$, a contradiction.

So L is simple; now set $H = \mathrm{C}_G(L)$. Then we have that $H \lhd G$ (this follows from the fact that $L \lhd G$) and so $X^H \neq \varnothing$, by Lemma 3.4. However we have assumed that the action is essential, whence $H = 1$. □

The condition $\mathrm{C}_G(L) = 1$ is equivalent to $G \subsetneq \mathrm{Aut}(L)$. Using this proposition we can decide which groups admit essential fixed point free actions on acyclic 2-dimensional complexes by first determining the *simple groups* with such actions and then looking at automorphism groups only for that restricted collection.

As the proof of the main theorem will require explicit knowledge about the finite simple groups, it seems appropriate to briefly recall their classification, we refer to [9] for a detailed explanation. We should point out that it is by now common knowledge that complete details of the proof of the Classification Theorem were not available when it was announced in 1981; crucial work involving the so-called *quasithin groups* was never published and is known to contain gaps. Fortunately this has been resolved thanks to more recent work by Aschbacher and Smith and although a full account has not yet been published, a draft of their manuscript (over 1200 pages long!) is now available on the world wide web (see [3]).

The following theorem encapsulates our understanding of finite simple groups, and its proof requires literally thousands of pages of mathematical arguments by many authors.

THEOREM 3.7. — *Let L denote a non-abelian finite simple group, then it must be isomorphic to one of the following groups:*

- *an alternating group A_n for $n \geqslant 5$*
- *a finite group of Lie type, i.e. a finite Chevalley group or a twisted analogue*[4]
- *one of the 26 sporadic simple groups.*

4. TECHNIQUES FOR CONSTRUCTING ACTIONS

One of the main results in [15] is an explicit listing of conditions which imply the existence of fixed point free actions on acyclic complexes. We first introduce

(4) We should mention that the Tits group ${}^2\mathrm{F}_4(2)'$ is actually of index 2 in the full Lie type group ${}^2\mathrm{F}_4(2)$.

DEFINITION 4.1. — *A non-empty family*[5] $\mathcal{F}$ *of subgroups of a group* G *is said to be* separating *if it has the following three properties: (a)* $G \notin \mathcal{F}$*; (b) any subgroup of an element in* $\mathcal{F}$ *is in* $\mathcal{F}$*; and (c) for any* $H \lhd K \subseteq G$ *with* K/H *solvable,* $K \in \mathcal{F}$ *if* $H \in \mathcal{F}$.

It is not hard to see that any maximal subgroup in a separating family of subgroups of G is self-normalizing. If G is solvable, then it has no separating family of subgroups. For G not solvable we let $\mathcal{SLV}$ denote the family of solvable subgroups, which is the minimal separating family for G.

DEFINITION 4.2. — *Given* G *and a family of subgroups* $\mathcal{F}$*, a* $(G,\mathcal{F})$*-complex is a* G*-complex such that all of its isotropy subgroups lie in* $\mathcal{F}$*. It is said to be* universal *(respectively* H*-universal) if the fixed point set of each* $K \in \mathcal{F}$ *is contractible (respectively acyclic).*

The following proposition relates the two previous concepts in our situation.

PROPOSITION 4.3. — *Let* X *denote a* 2*-dimensional acyclic* G*-complex without fixed points. Let* $\mathcal{F} = \{H \subset G \mid X^H \neq \varnothing\}$*. Then* $\mathcal{F}$ *is a separating family of subgroups of* G*, and* X *is an* H*-universal* $(G,\mathcal{F})$*-complex.*

Given a family of subgroups $\mathcal{F}$, let $N(\mathcal{F})$ denote the nerve of $\mathcal{F}$ (regarded as a poset via inclusion) with a G-action induced by conjugation. Given any set $\mathcal{H}$ of subgroups in G, we let $\mathcal{F}_{\geqslant \mathcal{H}}$ denote the poset of those subgroups in $\mathcal{F}$ which contain some element of $\mathcal{H}$. For a single subgroup H we use the notation $\mathcal{F}_{\geqslant H}$ and $\mathcal{F}_{>H}$ to denote the posets of subgroups containing H or strictly containing H, respectively. We denote $X^{\mathcal{H}} = \cup_{H\in\mathcal{H}} X^H$.

The following are two key technical lemmas which will be required:

LEMMA 4.4. — *If* X *denotes a universal (H-universal)* $(G,\mathcal{F})$*-complex then there exists a* G*-map* $X \to N(\mathcal{F})$ *which induces a homotopy equivalence (homology equivalence) between* $X^{\mathcal{H}}$ *and* $N(\mathcal{F}_{\geqslant \mathcal{H}})$*.*

LEMMA 4.5. — *Let* $\mathcal{F}$ *be any family of subgroups of* G*, and let* $\mathcal{F}_0 \subseteq \mathcal{F}$ *be any subfamily such that* $N(\mathcal{F}_{>H}) \simeq *$ *for all* $H \in \mathcal{F} - \mathcal{F}_0$*. Then any (H-)universal* $(G,\mathcal{F}_0)$*-complex is also an (H-)universal* $(G,\mathcal{F})$*-complex; and* $N((\mathcal{F}_0)_{\geqslant \mathcal{H}}) \simeq N(\mathcal{F}_{\geqslant \mathcal{H}})$ *for any set* $\mathcal{H}$ *of subgroups of* G*.*

A complex Y is said to be homologically m-dimensional if $H_n(X,\mathbb{Z}) = 0$ for all $n > m$ and $H_m(X,\mathbb{Z})$ is $\mathbb{Z}$-free. For later use we observe that, for $m \geqslant 1$, if X is an m-dimensional acyclic complex, then any subcomplex of X is homologically $(m-1)$-dimensional and that the intersection of a finite number of homologically $(m-2)$-dimensional complexes is also homologically $(m-2)$-dimensional.

[5] A family is a collection of subgroups of a group G which is closed under conjugation.

The following is a crucial criterion for the constructions we are seeking.

PROPOSITION 4.6. — *Let G be any finite group and let $\mathcal{F}$ be a separating family for G. Then the following are equivalent:*

- *There is a (finite) 2-dimensional H-universal $(G, \mathcal{F})$-complex.*
- *$N(\mathcal{F}_{>H})$ is homologically 1-dimensional for each subgroup $H \in \mathcal{F}$.*
- *$N(\mathcal{F}_{\geqslant \mathcal{H}})$ is homologically 1-dimensional for every set $\mathcal{H}$ of subgroups of G.*

Given a separating family $\mathcal{F}$ of subgroups of G, we say that $H \in \mathcal{F}$ is a *critical* subgroup if $N(\mathcal{F}_{>H})$ is not contractible. Given the above, we can concentrate our attention on the family $\mathcal{SLV}$ and its subfamily of critical subgroups, denoted $\mathcal{SLV}_c$.

First we record conditions which allow one to show that certain subgroups in a family are not critical.

LEMMA 4.7. — *Let $\mathcal{F}$ be any family if subgroups of G which has the property that $H \subseteq H' \subseteq H''$ and $H, H'' \in \mathcal{F}$ imply that $H' \in \mathcal{F}$. Fix a subgroup $H \in \mathcal{F}$; then $N(\mathcal{F}_{>H})$ is contractible if any of the following holds:*

- *H is not an intersection of maximal subgroups in $\mathcal{F}$.*
- *There is a subgroup $\widehat{H} \in \mathcal{F}$ properly containing H and such that $H \subsetneq K \cap \widehat{H}$ for all $H \subsetneq K \in \mathcal{F}_c$.*

We can now state a simple sufficient condition for the existence of a 2-dimensional H-universal $(G, \mathcal{F})$-complex:

PROPOSITION 4.8. — *Let $\mathcal{F}$ be any separating family of subgroups of G. Assume for every non-maximal critical subgroup $1 \neq H \in \mathcal{F}$, that $N_G(H) \in \mathcal{F}$, and that $H \subsetneq K \cap N_G(H)$ for all non-maximal critical subgroups $K \in \mathcal{F}$ properly containing H. Then there exists a 2-dimensional H-universal $(G, \mathcal{F})$-complex.*

We can in fact give a concrete description of the complex. For this we must introduce an integer associated to $H \in \mathcal{F}$.

DEFINITION 4.9. — *If $H \in \mathcal{F}$, a family of subgroups of G, we define*

$$i_{\mathcal{F}}(H) = \frac{1}{[N_G(H) : H]} \cdot (1 - \chi(N(\mathcal{F}_{>H}))).$$

Now let $M_1, \dots, M_n$ be conjugacy classes representatives for the maximal subgroups of $\mathcal{F}$, and let $H_1, \dots, H_k$ be conjugacy class representatives for all non-maximal critical subgroups of $\mathcal{F}$. Then there is a 2-dimensional H-universal $(G, \mathcal{F})$-universal complex X which consists of one orbit of vertices of type G/M_i for each $1 \leqslant i \leqslant n$, $[-i_{\mathcal{F}}(H_j)]$-orbits of 1-cells of type G/H_j for each $1 \leqslant j \leqslant k$, and free orbits of 1- and 2-cells. If G is simple[(6)] then X can be constructed to contain exactly $i_{\mathcal{F}}(1)$ free orbits of 2-cells, and no free orbits of 1-cells.

[(6)] In fact G must satisfy an additional technical condition which does not affect the results here.

5. EXPLICIT ACTIONS

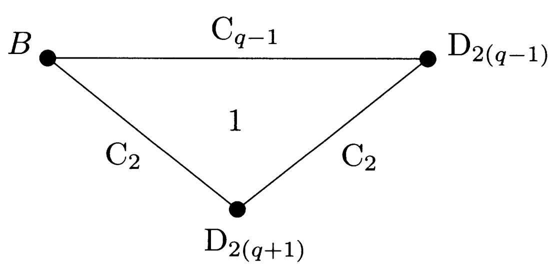
In this section we will outline the construction of fixed point free actions on acyclic 2-dimensional complexes for the simple groups $\mathrm{PSL}_2(2^k)$, for $k \geqslant 2$; $\mathrm{PSL}_2(q)$ for $q \equiv \pm 3 \pmod 8$ and $q \geqslant 5$; and for $\mathrm{Sz}(2^k)$ for odd $k \geqslant 3$.

Example 5.1. — Let $G = \mathrm{PSL}_2(q)$, where $q = 2^k$ and $k \geqslant 2$. Then there is a 2-dimensional acyclic fixed point free G-complex X all of whose isotropy subgroups are solvable. The complex X can be constructed with three orbits of vertices, with isotropy subgroups isomorphic to $B = \mathbb{F}_q \rtimes \mathrm{C}_{q-1}$, $\mathrm{D}_{2(q-1)}$ and $\mathrm{D}_{2(q+1)}$; three orbits of edges with isotropy subgroups isomorphic to C_{q-1}, C_2 and C_2; and one free orbit of 2-cells.

Here B denotes a Borel subgroup, expressed as a semi-direct product isomorphic to $(\mathrm{C}_2)^k \rtimes \mathrm{C}_{q-1}$, identified with the subgroup of projectivized upper triangular matrices. In our notation C_r denotes the cyclic group of order r and D_r denotes the dihedral group of order r. In fact $\mathrm{D}_{2(r-1)}$ can be identified with the subgroup of *monomial matrices.*

This example can be explained from the following analysis. The conjugacy classes of maximal solvable subgroups of G are represented precisely by the groups B, $\mathrm{D}_{2(q-1)}$ and $\mathrm{D}_{2(q+1)}$. The non-maximal critical subgroups must be intersections of maximal subgroups, one can check that up to conjugacy we get C_{q-1}, C_2 and 1. The precise numbers of orbits which appear is determined by calculating the integers $i_{\mathcal{SLV}}(H)$ for the isotropy subgroups.

This example can actually be constructed directly using the 1-skeleton Y_1 of the coset complex Y for the triple of subgroups $(K_1, K_2, K_3) = (B, \mathrm{D}_{2(q-1)}, \mathrm{D}_{2(q+1)})$ in $G = \mathrm{PSL}_2(\mathbb{F}_q)$ given by the maximal solvable subgroups. We can describe Y as the G-complex with vertex set $G/K_1 \sqcup G/K_2 \sqcup G/K_3$, where G acts by left translation, and with a 1-simplex for every pair of cosets with non-empty intersection and a 2-simplex for every triple of cosets with non-empty intersection. The following picture describes the orbit space Y/G:



It is not hard to see that as $G = \langle K_1, K_2, K_3 \rangle$, the complex Y is connected; however (as shown in [2], § 9) it is not acyclic for $k \geqslant 3$, where $q = 2^k$. However, one can show that the module $H_1(Y_1,\mathbb{Z})$ is stably free — this involves a geometric argument based on the fact that Y_1 is a graph such that the fixed point sets Y_1^H are either contractible or empty for all subgroups $1 \neq H \subset G$ and contractible for all p-subgroups in G. The fact that G is a nonabelian simple group implies that the module must in fact be free (for a proof see [15], Prop. C.4.). Now we can simply attach a single free G-cell to Y_1 to kill its homology, yielding the acyclic complex X.

Carrying out this analysis in the classical case $G = \mathrm{A}_5$ yields an acyclic complex X (which in this case is actually identical to the complex Y) whose cellular chains give a complex of the form[(7)]

$$0 \to \mathbb{Z}[\mathrm{A}_5] \to \begin{array}{c}\mathbb{Z}[\mathrm{A}_5\,/\,\mathrm{C}_2]\\ \oplus \\ \mathbb{Z}[\mathrm{A}_5\,/\,\mathrm{C}_2] \\ \oplus \\ \mathbb{Z}[\mathrm{A}_5\,/\,\mathrm{C}_3]\end{array} \to \begin{array}{c}\mathbb{Z}[\mathrm{A}_5\,/\,\mathrm{A}_4]\\ \oplus \\ \mathbb{Z}[\mathrm{A}_5\,/\,\mathrm{D}_6] \\ \oplus \\ \mathbb{Z}[\mathrm{A}_5\,/\,\mathrm{D}_{10}]\end{array} \to \mathbb{Z} \to 0.$$

Example 5.2. — Let $G = \mathrm{PSL}_2(\mathbb{F}_q)$, where $q = p^k$, $q \geqslant 5$ and $q \equiv \pm 3 \pmod 8$. Then there exists a 2-dimensional acyclic fixed point free G-complex X, all of whose isotropy subgroups are solvable. More precisely, X can be constructed to have four orbits of vertices with isotropy subgroups isomorphic to $\mathbb{F}_q \rtimes \mathrm{C}_{(q-1)/2}$, D_{q-1}, D_{q+1} and A_4; four orbits of edges with isotropy subgroups isomorphic to $\mathrm{C}_{(q-1)/2}$, $\mathrm{C}_2 \times \mathrm{C}_2$, C_3 and C_2; and one free orbit of 2-cells.

These examples are slightly more complicated as the structure of the complex will depend on the value of q modulo 8.

Before explaining the final set of examples, we briefly recall the structure of the Suzuki groups $\mathrm{Sz}(q)$ (see [6], [11], [19] for details). Fix $q = 2^{2k+1}$ and let $\theta \in \mathrm{Aut}(\mathbb{F}_q)$ be the automorphism $x^\theta = x^{2^{k+1}} = x^{\sqrt{2q}}$ (note that $(x^\theta)^\theta = x^2$). For $a, b \in \mathbb{F}_q$ and $\lambda \in (\mathbb{F}_q)^*$, define the elements

$$S(a,b) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ b & a^\theta & 1 & 0 \\ a^{2+\theta}+ab+b^\theta & a^{1+\theta}+b & a & 1 \end{pmatrix}$$

and

$$M(\lambda) = \begin{pmatrix} \lambda^{1+2^k} & 0 & 0 & 0 \\ 0 & \lambda^{2^k} & 0 & 0 \\ 0 & 0 & \lambda^{-2^k} & 0 \\ 0 & 0 & 0 & \lambda^{-1-2^k} \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

(7) If we consider the original construction discussed in § 1 of an acyclic A_5-complex, then one can obtain the cellular structure below by subdividing each pentagon into a union of ten triangles.

Let $S(q,\theta) = \langle S(a,b) | a,b \in \mathbb{F}_q \rangle$, $T = \langle M(\lambda) | \lambda \in (\mathbb{F}_q)^* \rangle \cong \mathrm{C}_{q-1}$ and

$$B = M(q,\theta) = S(q,\theta) \rtimes T \text{ and } N = \langle T, \tau \rangle \cong \mathrm{D}_{2(q-1)}.$$

Then $\mathrm{Sz}(q) \cong \langle M(q,\theta), \tau \rangle$, and under this identification the following hold

(1) $S(q,\theta)$ is the 2-Sylow subgroup of $\mathrm{Sz}(q)$.

(2) There are four conjugacy classes of maximal subgroups in $\mathrm{Sz}(q)$ which are solvable: (B), (N), (M_+) and (M_-), where

$$M_+ \cong \mathrm{C}_{q+\sqrt{2q}+1} \rtimes \mathrm{C}_4 \text{ and } M_- \cong \mathrm{C}_{q-\sqrt{2q}+1} \rtimes \mathrm{C}_4.$$

These are all the maximal solvable subgroups in $\mathrm{Sz}(q)$.

(3) $|\mathrm{Sz}(q)| = q^2(q-1)(q^2+1) = q^2(q-1)(q+\sqrt{2q}+1)(q-\sqrt{2q}+1)$; note that the four factors in this expression are relatively prime.

We now describe the third set of examples.

Example 5.3. — Let $q = 2^{2k+1}$, for any $k \geqslant 1$. Then there is a 2-dimensional acyclic fixed point free $\mathrm{Sz}(q)$-complex X, all of whose isotropy subgroups are solvable. X can be constructed to have four orbits of vertices with isotropy subgroups isomorphic to $M(q,\theta)$, $\mathrm{D}_{2(q-1)}$, $\mathrm{C}_{q+\sqrt{2q}+1} \rtimes \mathrm{C}_4$ and $\mathrm{C}_{q-\sqrt{2q}+1} \rtimes \mathrm{C}_4$; four orbits of edges with isotropy subgroups isomorphic to C_{q-1}, C_4, C_4 and C_2; and one free orbit of 2-cells.

6. NON-EXISTENCE OF FIXED POINT FREE ACTIONS

In this section we outline methods for showing that *most* finite simple groups cannot act on an acyclic 2-dimensional complex without fixed points. The first result in this direction is due to Segev [16].

THEOREM 6.1. — *If G is the alternating group A_n, with $n \geqslant 6$, then there is no fixed point free action of G on any acyclic 2-dimensional complex.*

Later this was substantially extended by Aschbacher-Segev [2], who proved:

THEOREM 6.2. — *If G is a finite simple group which acts on an acyclic 2-dimensional complex without fixed points, then G must be isomorphic to either a group of Lie type and Lie rank one, or isomorphic to the sporadic simple group J_1 (the first Janko group).*

We will now sketch the key arguments used to establish these results, which (by the Classification Theorem) rule out most of the finite simple groups. The following lemma will be referred to as the *four subgroup criterion.*

LEMMA 6.3. — *Let G be a finite group and X a 2-dimensional acyclic G-complex. If $H_1, H_2, H_3, H_4 \subset G$ are subgroups such that $X^{\langle H_i, H_j, H_k \rangle} \neq \varnothing$ for any i,j,k then*

$$X^{\langle H_1, H_2, H_3, H_4 \rangle} \neq \varnothing.$$

Proof. — Suppose that in fact $X^{\langle H_1,H_2,H_3,H_4\rangle} = \varnothing$. Let $\mathcal{H} = \{H_1, H_2, H_3, H_4\}$. Now $X^{\mathcal{H}}$ is the union of the acyclic subcomplexes X^{H_i}, which are such that any two or three of them have acyclic intersection, but the four have empty intersection. The homology of $X^{\mathcal{H}}$ is isomorphic (see [5], p. 168) to that of the nerve of the corresponding acyclic covering; yielding $H_2(X^{\mathcal{H}}, \mathbb{Z}) \cong H_2(\mathbb{S}^2, \mathbb{Z}) \cong \mathbb{Z}$. However we know that $X^{\mathcal{H}}$ must be homologically 1-dimensional, which yields a contradiction. □

We now apply this result to multiply transitive groups.

Proposition 6.4. — *Suppose that G acts 4-transitively on a set S with a point stabilizer $H \subset G$. If X is a 2-dimensional acyclic G-complex such that $X^H \neq \varnothing$, then $X^G \neq \varnothing$.*

Proof. — If $|S| = 4$ then G is an extension of the form $1 \to Q \to G \to K \to 1$, where $K \subseteq \Sigma_4$ and $Q \subset H$. By Theorem 3.1, $\varnothing \neq X^Q$ must be acyclic, and as K is solvable its action on X^Q must have a fixed point and we are done. So assume that $|S| \geqslant 5$, and fix four elements $s_1, s_2, s_3, s_4 \in S$. For each $i = 1, 2, 3, 4$, let $H_i \subset G$ be the subgroup of elements which fix s_j for all $j \neq i$. For each $\{i, j, k, r\} = \{1, 2, 3, 4\}$, $\langle H_i, H_j, H_k\rangle$ is the point stabilizer of s_r and therefore fixes a point in X by assumption. Hence by Lemma 6.3 (where $G = \langle H_1, H_2, H_3, H_4\rangle$), $X^G \neq \varnothing$. □

We apply this to show that the alternating groups A_n for $n \geqslant 6$ do not admit fixed point free actions on acyclic 2-complexes. Note that A_n is $(n-2)$-transitive on $\{1, 2, \dots, n\}$, with point stabilizer A_{n-1} and that m-transitivity implies k-transitivity for $k \leqslant m$; hence A_n is 4-transitive on $\{1, \dots, n\}$ for all $n \geqslant 6$ (see [1], page 56). If X is a 2-dimensional acyclic A_n-complex, then by our previous proposition, $X^{\mathrm{A}_n} \neq \varnothing$ if $X^{\mathrm{A}_{n-1}} \neq \varnothing$. Hence by induction we are reduced to considering the case when $G = \mathrm{A}_6$; assume that X is a 2-dimensional acyclic G-complex with $X^G = \varnothing$. Using the subgroups $H_i = \langle (i, 5, 6)\rangle$ for $i = 1, 2, 3, 4$ we can show by contradiction that $X^H = \varnothing$ for $H = \mathrm{Alt}\{1, \dots, 5\}$ (using the covering argument as before). Using an outer automorphism we can thus establish that $X^H = \varnothing$ for any $H \subset G$ with $H \cong \mathrm{A}_5$. Next we consider the collection of subgroups

$$\mathcal{M} = \{\langle (12)(36)\rangle, \langle (12)(45)\rangle, \langle (12)(34)\rangle, \langle (25)(36)\rangle, \langle (26)(35)\rangle\};$$

again applying the covering arguments and comparing with the homology of the nerve of this covering we see that $H_2(X^{\mathcal{M}}, \mathbb{Z}) \neq 0$, a contradiction. We refer to [16], page 39 for details.

This method can also be applied to the Mathieu groups M_n; for $n = 11, 12, 23, 24$ they all act 4-transitively on a set with point stabilizer M_{n-1}. Now M_{10} contains A_6 as a subgroup of index two, hence every action of M_{11} or M_{12} on an acyclic 2-complex must have a fixed point. To obtain the same result for M_{23} and M_{24}, it suffices to establish it for M_{22}, which we will do subsequently.

The case of simple groups of Lie type, and of Lie rank at least equal to 2 can also be handled with these arguments (see [6] for background). We start with a basic lemma about parabolic subgroups.

LEMMA 6.5. — *Let G be a finite simple group of Lie type. Let Σ be the root system associated with G and let Σ_+ and Σ_- be the sets of positive and negative roots. Fix a set J of simple roots which does not contain all of them, and let L_J be the subgroup generated by the diagonal subgroup H together with the root subgroups X_r for all $r \in \langle J\rangle$. Let U_J and V_J be the subgroups generated by all X_r for roots $r \in \Sigma_+$ or $r \in \Sigma_-$, respectively, which are not in $\langle J\rangle$. Then $U_J \lhd P_J = U_J L_J$ and $V_J \lhd P'_J = V_J L_J$, U_J and V_J are nilpotent and $\langle U_J, V_J\rangle = G$.*

In our context we obtain the following fixed point theorem

LEMMA 6.6. — *Let G be a finite simple group of Lie type, and let $P \subsetneq G$ be one of the parabolic subgroups P_J or $P_{J'}$ in the previous lemma. Then for any action of G on an acyclic 2-complex X, $X^P \neq \varnothing$.*

Proof. — Let us assume that $X^G = \varnothing$. Then there are subgroups $U_J \lhd P_J$, $V_J \lhd P'_J$ and $L_J = P_J \cap P'_J$ such that U_J and V_J are nilpotent, $P_J = U_J L_J$, $P'_J = V_J L_J$, and $\langle U_J, V_J\rangle = G$. Note that X^{U_J} and X^{V_J} are acyclic, disjoint and L_J-invariant. Considering the subspaces $A = X^{U_J}$ and $B = X^{V_J}$ and the action of L_J, we see that $X^{L_J} \neq \varnothing$ (Corollary 3.3); similarly we conclude from Lemma 3.4 that X^{P_J} and $X^{P'_J}$ are non-empty. □

Now we can prove

THEOREM 6.7. — *If G is a simple group of Lie type and Lie rank at least two, then every G-action on an acyclic 2-dimensional complex has a fixed point.*

Proof. — Take a root system $\Sigma = \Sigma_+ \sqcup \Sigma_-$ for G and let $J_1 \sqcup J_2$ be a decomposition of the set of simple roots as a disjoint union of non-empty subsets. For each $i = 1, 2$, set $H_i^+ = \langle H, X_s \mid s \in J_i\rangle$, and $H_i^- = \langle H, X_{-s} \mid s \in J_i\rangle$. The subgroup generated by any three of the $H_i^\pm$ is contained in one of the parabolic subgroups P_{J_i} or P'_{J_i} and so has non-empty fixed point set in X. But in fact one can verify that $\langle H_1^\pm, H_2^\pm\rangle = G$, since it contains all subgroups X_s, X_{-s} for simple roots s and hence $X^G \neq \varnothing$ by the four subgroup criterion. □

In [2], Aschbacher and Segev were able to apply the four subgroup criterion to prove that any sporadic simple group other than the Janko group J_1 acting on an acyclic 2-complex has a fixed point. In [15] a different treatment is given, showing that all the sporadics can be handled using a consistent technique which relies on understanding the subgroup structure of these groups in some detail. The essential result is the following.

PROPOSITION 6.8. — *Let $\mathcal{F}$ be a separating family for G and let $K_1, K_2, K_3 \in \mathcal{F}$ be three subgroups such that neither K_2 nor K_3 is conjugate to K_1. Let $K_{ij} = K_i \cap K_j$ and $K = K_1 \cap K_2 \cap K_3$. Let $\mathcal{F}_0 \subseteq \mathcal{F}$ denote the subfamily consisting of $\mathcal{F}_c$ together with all subgroups conjugate to any of the K_i, K_{ij} or K. Assume that the following conditions hold, where $G' = \langle K_1, K_2, K_3 \rangle$:*

- $\dfrac{1}{[K_{12}:K]} + \dfrac{1}{[K_{13}:K]} + \dfrac{1}{[K_{23}:K]} \leqslant 1 + \dfrac{1}{[K_1:K]} + \dfrac{1}{[K_2:K]} + \dfrac{1}{[K_3:K]} - \dfrac{1}{[G':K]}$
- *K_1 is maximal in $\mathcal{F}$.*
- *There is no $H \in \mathcal{F}_0$ such that $K \subsetneq H \subsetneq K_{12}$ or $K_{12} \subsetneq H \subsetneq K_1$.*
- $N_G(K_1) \cap N_G(K_{12}) \cap N_G(K) = K$
- *The triples (K_1, K_{12}, K) and (K_1, K_{13}, K) are not G-conjugate.*

Then $H_2(N(\mathcal{F}_{\geqslant (K)}), \mathbb{Z}) \neq 0$ and so there is no 2-dimensional, H-universal $(G, \mathcal{F})$-complex.

This result can be proved as follows: the coset complex Y for the triple (K_1, K_2, K_3) must have $H_2(Y, \mathbb{Z}) \neq 0$ by the first hypothesis (this follows from a counting argument); the other conditions allow one to push a non-zero class non-trivially into

$$H_2(N((\mathcal{F}_0)_{\geqslant (K)}), \mathbb{Z}) \cong H_2(N(\mathcal{F}_{\geqslant (K)}), \mathbb{Z})$$

via the homomorphism induced by the G-equivariant simplicial map $Y^* \to N((\mathcal{F}_0)_{\geqslant (K)})$ sending each vertex in the barycentric subdivision Y^* of Y to its isotropy subgroup. By Proposition 4.6, this implies the stated result. This proposition can be applied systematically to yield

THEOREM 6.9. — *Let G be any of the sporadic simple groups; then there is no 2-dimensional acyclic G-complex without fixed points.*

We illustrate how this may be applied with two examples. Here we assume that we are given a 2-dimensional acyclic X with a fixed point free G-action, and take $\mathcal{F}$ to be the separating family of $H \subset G$ with $X^H \neq \varnothing$.

Example 6.10. — Let $G = M_{22}$, one of the Mathieu groups. We can take $K_3 \cong 2^4 : \mathrm{A}_6$, the subgroup which leaves invariant some hexad in the Steiner system of order 22, and it has an obvious action on this set of order 6 (see [10], Thm 6.8). Then K_1 can be taken to be the stabilizer of a point z in the hexad, and K_2 the stabilizer of some pair of points in the hexad including z. In this case $K_1 \cong \mathrm{L}_3(4)$, $K_2 \cong 2^4 : S_5$, $K_{12} \cong 2^4 : \mathrm{A}_5$, $K_{13} \cong 2^4 : \mathrm{A}_5$, $K_{23} \cong 2^4 : S_4$ and $K \cong 2^4 : \mathrm{A}_4$. Note that the K_{12} and K_{13} are distinct parabolic subgroups in $\mathrm{L}_3(4)$. The conditions in our previous proposition can be checked to hold (note that from our previous results we can see that the K_i all act with fixed points and hence are in $\mathcal{F}$) and so we have completed the verification that the Mathieu groups have no fixed point free actions on acyclic 2-complexes.

Next we deal with the case J_1 which was not covered by [2].

Example 6.11. — Let $G = J_1$, the first Janko group. Take $K_1 \cong (\mathrm{C}_2)^3 \rtimes \mathrm{G}_{21}$, where G_{21} is the Frobenius group of order 21, i.e. $\mathrm{C}_7 \rtimes \mathrm{C}_3$. K_1 is a maximal subgroup in J_1. Let $K_2 \cong \mathrm{C}_7 \times \mathrm{C}_6$ be the normalizer of a subgroup of order 7 in K_{12}, and let $K_3 \cong \mathrm{C}_3 \times \mathrm{D}_{10}$ be the centralizer in G of a subgroup of order 3 in K_2. Then $K_{12} \cong \mathrm{C}_7 \rtimes \mathrm{C}_3$, $K_{13} \cong \mathrm{C}_6 \cong K_{23}$, and $K = K_1 \cap K_2 \cap K_3 \cong \mathrm{C}_3$. Note that all these subgroups are solvable, and so are in $\mathcal{F}$. We can verify that

$$\sum_{i<j} \frac{1}{[K_{ij} : K]} = \frac{1}{7} + \frac{1}{2} + \frac{1}{2} < 1 + \frac{1}{14} + \frac{1}{10} = 1 + \frac{1}{[K_2 : K]} + \frac{1}{[K_3 : K]}$$

while the other conditions are also easy to check, hence showing that J_1 has no fixed point free action on an acyclic 2-complex.

We now consider the finite groups of Lie type which have Lie rank exactly equal to one. There are four families of such groups: the two dimensional projective special linear groups $\mathrm{L}_2(q)$, the three dimensional projective special unitary groups $\mathrm{U}_3(q)$, the Suzuki groups $\mathrm{Sz}(2^{2k+1})$, and the Ree groups $\mathrm{Ree}(3^{2k+1}) = {}^2\mathrm{G}_2(3^{2k+1})$.

The following propositions are used to handle these groups.

Proposition 6.12. — *Let L be one of the simple groups $\mathrm{L}_2(q)$ or $\mathrm{Sz}(q)$, where $q = p^k$ and p is prime ($p = 2$ in the second case). Let $G \subset \mathrm{Aut}(L)$ be any subgroup containing L and $\mathcal{F}$ a separating family for G. Then there exists a 2-dimensional H-universal $(G, \mathcal{F})$-complex if and only if $G = L$, $\mathcal{F} = \mathcal{SLV}$ and q is a power of 2 or $q \equiv \pm 3 \pmod 8$.*

Proposition 6.13. — *Let $G = \mathrm{U}_3(q)$, or ${}^2\mathrm{G}_2(3^{2k+1})$. Then there is no 2-dimensional acyclic G-complex without fixed points.*

These results are proved by combining our previous non-existence techniques with the following additional notion. For any family of subgroups $\mathcal{F}$ and any maximal element $M \in \mathcal{F}$, we set $Lk_{\mathcal{F}>1}(M) = N(\mathcal{F}_{>1}^{<M}) = N(\{H \in \mathcal{F} \mid 1 \neq H \subsetneq M\})$. Then we have

Lemma 6.14. — *Let $\mathcal{F}$ denote a separating family for G. Let $\mathcal{F}_0 \subset \mathcal{F}$ be any subfamily which contains $\mathcal{F}_c$, and such that each non-maximal subgroup in $\mathcal{F}_0$ is contained in two or more maximal subgroups. Assume that $\mathcal{F}$ satisfies the following two conditions*

(1) *$N(\mathcal{F}_{>1})/G$ is connected and $H_1(N(\mathcal{F}_{>1})/G, \mathbb{Z}) = 0$.*
(2) *There is a maximal subgroup $M \in \mathcal{F}$ such that $Lk_{(\mathcal{F}_0)}(M)$ is not connected.*

Then there is no H-universal 2-dimensional $(G, \mathcal{F})$-complex.

Roughly speaking the proof of this lemma goes as follows: if such a complex did exist, then by (1) the singular set must be acyclic; but the prescribed conditions imply that the links at all vertices must be connected — hence contradicting (2).

In many instances this allows one to prove non-existence of a fixed point free action by contradiction; assuming its existence we can then find a maximal subgroup in the separating family such that the corresponding link is not connected. This of course requires a rather intricate knowledge of the maximal subgroups and more generally the finer structure of the groups under consideration. We refer to [15] § 6 for complete details.

7. PROOF OF THE MAIN THEOREM

We are now prepared to sketch a proof of the main theorem. We recall the statement.

THEOREM 7.1. — *If G is any finite group, then there is an essential fixed point free 2-dimensional finite acyclic G-complex if and only if G is isomorphic to one of the simple groups* $\mathrm{PSL}_2(2^k)$ *for* $k \geqslant 2$; $\mathrm{PSL}_2(q)$ *for* $q \equiv \pm 3 \pmod 8$ *and* $q \geqslant 5$; *or* $\mathrm{Sz}(2^k)$ *for odd* $k \geqslant 3$. *Moreover the isotropy subgroups of any such G-complex are all solvable.*

Proof. — We know that if G has an essential action on an acyclic 2-complex X without fixed points, then there is a non-abelian simple normal subgroup $L \lhd G$ with a fixed point free action and such that $G \subseteq \mathrm{Aut}(L)$. By the Classification Theorem, we know that L must be an alternating group, a group of Lie type, or a sporadic simple group. The results in the previous section rule out all groups on this list[(8)] except possibly the ones in the statement of the theorem. However we have already seen that these groups do in fact act on an acyclic 2-complex without fixed points, and that the isotropy subgroups are all solvable. This completes the proof. □

The work of Oliver and Segev has provided a complete picture for understanding fixed point free group actions on acyclic 2-dimensional complexes. There remains however the problem of considering actions on *contractible* 2-dimensional complexes. In fact Aschbacher and Segev [2] have raised the following

QUESTION 7.2. — *If X is a finite contractible 2-dimensional G-complex, then is* $X^G \neq \varnothing$*?*

This remains open. The results described here are a basic step towards investigating this question but it will probably require a substantially different approach.

[(8)] In fact the Tits group ${}^2\mathrm{F}_4(2)'$ must be handled separately because it is not the full Lie type group ${}^2\mathrm{F}_4(2)$.

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