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Characters and K-theory of discrete groups

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0. Introduction

One of the nicest results in algebraic topology is the theorem due to Atiyah [4] that if G is a finite group, then the complex K-theory of its classifying space BG can be computed from the formula

$$K^*(BG) \cong R(G)^\wedge \ , \tag{0.1}$$

where the term on the right is the completion of the complex representation ring of G at its augmentation ideal. The goal of this paper is to prove a similar result for certain infinite discrete groups which include arithmetic groups as well as many other geometrically interesting classes of groups. To be specific, we consider groups Γ of finite virtual cohomological dimension (see §1 for definitions) acting with finite isotropy on a contractible complex X such that X/Γ is *compact* (by the work of Borel and Serre [6], all arithmetic groups of finite *vcd* will satisfy this condition).

The most obvious difficulty in such a project is working with the representation ring of a non-compact group. To deal with this, we construct a modified version of $R(\Gamma)$ using families of subgroups in Γ. We have

Definition. Let $\mathscr{F}$ be a family of finite subgroups in Γ, then

$$R_{\mathscr{F}}(\Gamma) = \lim_{H \in \mathscr{F}} R(H) \ .$$

Our assumptions on Γ imply that it has only finitely many finite subgroups up to conjugacy and in fact we have

Theorem. *If $\mathscr{F}$ is the family of all finite subgroups in Γ, then $R_{\mathscr{F}}(\Gamma)$ is a commutative, unitary ring which is free abelian of finite rank $n(\Gamma)$, the number of conjugacy classes of elements of finite order in Γ. In particular $R_{\mathscr{F}}(\Gamma) \cong \mathbb{Z}$ if and only if Γ is torsion-free.*

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Similarly if we use $\mathcal{F}(p)$, the family of all finite p-subgroups of Γ, we obtain a ring $R_{\mathcal{F}(p)}(\Gamma)$ of rank $n_p(\Gamma)$, the number of conjugacy classes of elements of order a finite power of p in Γ. Analogues of Artin's Theorem of Brauer's Theorem can be proved for these rings.

Next we relate $K^*(B\Gamma)$ with $R_{\mathcal{F}}(\Gamma)$ by using equivariant K-theory. Namely, if we take X as before, then we construct a ring homomorphism

$$K_\Gamma^*(X) \to R_{\mathcal{F}}(\Gamma)\,, \tag{0.2}$$

which is a rational surjection. Fix an extension

$$1 \to \Gamma' \to \Gamma \to G \to 1\,,$$

where Γ' is torsion-free, and G is finite. Then, as Γ' acts freely on X, $K_\Gamma^*(X) \cong K_G^*(X/\Gamma')$. On the other hand, it is not hard to see that $B\Gamma \simeq X/\Gamma' \times_G EG$, hence

$$K_G^*(X/\Gamma')^\wedge \cong K^*(B\Gamma) \tag{0.3}$$

by the Atiyah–Segal Completion Theorem [5] (here we use IG-adic completion, where as before $IG \subseteq R(G)$ is the augmentation ideal, and $K_G^*(X/\Gamma')$ is an $R(G)$-module). Hence we have a "completed map"

$$K^*(B\Gamma) \to R_{\mathcal{F}}(\Gamma)^\wedge\,. \tag{0.4}$$

As one would suspect, this map is often far from being injective (of course if $\Gamma = G$, this is simply Atiyah's isomorphism). Consequently our following step is to analyze it in detail. For this we first remark that unlike the ordinary cohomology of $B\Gamma$, $K^*(B\Gamma)$ will contain substantial information from the finite subgroups of Γ, which is *torsion-free* (again, for $\Gamma = G$, all of it is torsion-free). Hence a calculation of (0.4) modulo torsion is a worthwhile approach to our problem.

For technical reasons it is convenient to work p-locally. Let $K_p(\,)$ denote K-theory with coefficients in the p-adic integers $\mathbb{Z}_p$, and let $\mathbb{Q}_p$ denote the field of p-adic numbers. In the following statements and in the rest of the paper tensor products will be taken over $\mathbb{Z}$ or $\mathbb{Z}_p$; we omit them from our notation, as it will be clear from the context which ring is involved. Our main result is

Theorem 6.3. *There is a surjection of rings*

$$K_p^*(B\Gamma) \otimes \mathbb{Q}_p \overset{\varphi_p}{\twoheadrightarrow} R_{\mathcal{F}(p)}(\Gamma) \otimes \mathbb{Q}_p$$

and its kernel is additively described as

$$I_{\mathcal{F}(p)}^* \otimes \mathbb{Q}_p \cong \bigoplus_{\substack{(\gamma) \\ \gamma \in \mathrm{Tors}_p(\Gamma)}} \tilde{K}^*(B(C(\gamma) \cap \Gamma'))^{H_\gamma} \otimes \mathbb{Q}_p\,,$$

where $C(\gamma)$ is the centralizer of γ in Γ, γ ranges over Γ-conjugacy classes of elements in $\mathrm{Tors}_p(\Gamma) = \{\gamma \in \Gamma \mid |\langle\gamma\rangle| = p^n, \text{for } n \geqq 0\}$, *and H_γ is the finite group $C(\gamma)/C(\gamma) \cap \Gamma'$.*

As $B(C(\gamma) \cap \Gamma')$ is of finite type, we in fact have

$$K^*(B(C(\gamma) \cap \Gamma'))^{H_\gamma} \otimes \mathbb{Q} \cong \begin{cases} \bigoplus_{j \text{ even}} H^j(BC(\gamma), \mathbb{Q}) & * \text{ even} \\ \bigoplus_{j \text{ odd}} H^j(BC(\gamma), \mathbb{Q}) & * \text{ odd.} \end{cases}$$

Consequently, the kernel of φ_p is determined by the rational cohomology of $BC(\gamma)$, $\gamma \in \mathrm{Tors}_p(\Gamma)$ and we have

Corollary. *$K_p^*(B\Gamma) \otimes \mathbb{Q}_p \cong R_{\mathscr{F}(p)}(\Gamma) \otimes \mathbb{Q}_p$ if and only if $\tilde{H}^*(BC(\gamma), \mathbb{Q}) \equiv 0$ for every $\gamma \in \mathrm{Tors}_p(\Gamma)$.*

An interesting example of the above occurs when $p = 2$, $\Gamma = \mathrm{SL}_3(\mathbb{Z})$ (example 3.2):

$$K_2^*(BSL_3(\mathbb{Z})) \otimes \mathbb{Q}_2$$

$$\cong \mathbb{Q}_2[\alpha_1, \alpha_2, \beta_1, \beta_2] \Bigg/ \begin{array}{c} \alpha_1^2 - 1, \alpha_2^2 - 1 \\ \beta_1^2 - 2 - 2\alpha_1, \alpha_1\beta_1 - \beta_1, \\ \beta_2^2 - 2 - 2\alpha_2, \alpha_2\beta_2 - \beta_2, \\ \alpha_1\alpha_2 - \alpha_1 - \alpha_2 + 1, \alpha_1\beta_2 - 2\alpha_1 - \beta_2 + 2, \\ \beta_1\beta_2 - 2\beta_1 - 2\beta_2 + 4, \alpha_2\beta_1 - 2\alpha_2 - \beta_1 + 2. \end{array}$$

If $\Gamma = G_1 *_H G_2$, an amalgamated product of finite groups, then

$$K_p^*(B\Gamma) \otimes \mathbb{Q}_p \cong \begin{cases} R_{\mathscr{F}(p)}(\Gamma) \otimes \mathbb{Q}_p & * \text{ even}, \\ \tilde{K}^*(\bigvee_1^{v_p(\Gamma)} \mathbb{S}^1) \otimes \mathbb{Q}_p & * \text{ odd}, \end{cases}$$

where

$$v_p(\Gamma) = \sum_{\substack{(\gamma) \\ \gamma \in \mathrm{Tors}_p(\Gamma)}} \dim_{\mathbb{Q}} H^1(BC(\gamma), \mathbb{Q}).$$

In particular we obtain the rather curious formula (6.5):

$$n_p(\Gamma) = \mathrm{n}_p(G_1) + n_p(G_2) - n_p(H) + v_p(\Gamma).$$

Another consequence of our results is

Corollary. *If $\Gamma_1 \xrightarrow{f} \Gamma_2$ is a homomorphism between two groups satisfying our hypotheses which induces an integral homology isomorphism, then, for all primes p*

$$R_{\mathscr{F}(p)}(\Gamma_1) \otimes \mathbb{Q}_p \cong R_{\mathscr{F}(p)}(\Gamma_2) \otimes \mathbb{Q}_p$$

(in particular $n_p(\Gamma_1) = n_p(\Gamma_2)$) and

$$\bigoplus_{\substack{(\gamma) \\ \gamma \in \mathrm{Tors}_p(\Gamma_1)}} \left(\bigoplus_{j \text{ even}} \tilde{H}^j(BC(\gamma), \mathbb{Q}) \right) \cong \bigoplus_{\substack{(\mu) \\ \mu \in \mathrm{Tors}_p(\Gamma_2)}} \left(\bigoplus_{j \text{ even}} \tilde{H}^j(BC(\mu), \mathbb{Q}) \right)$$

$$\bigoplus_{\substack{(\gamma) \\ \gamma \in \mathrm{Tors}_p(\Gamma_1)}} \left(\bigoplus_{j \text{ odd}} H^j(BC(\gamma), \mathbb{Q}) \right) \cong \bigoplus_{\substack{(\mu) \\ \mu \in \mathrm{Tors}_p(\Gamma_2)}} \left(\bigoplus_{j \text{ odd}} H^j(BC(\mu), \mathbb{Q}) \right).$$

If, for example, Γ_2 is finite in the preceding corollary, then automatically $\tilde{H}^*(BC(\gamma), \mathbb{Q}) \equiv 0$ for all $\gamma \in \mathrm{Tors}_p(\Gamma_1)$, for all primes p.

Burghelea has shown [9] that there is an additive decomposition for the periodic cyclic homology of $\mathbb{Q}\Gamma$. Comparing this to our results we obtain (see §7 for notation).

Theorem. *For the groups Γ satisfying our hypotheses,*

$$PHC_*(\mathbb{Q}\Gamma) \cong K_\Gamma^*(X) \otimes \mathbb{Q} \oplus \left(\bigoplus_{\substack{(\gamma) \\ \gamma \notin \mathrm{Tors}(\Gamma)}} T_*(\gamma, \mathbb{Q}) \right).$$

(*Here* Tors(Γ) *denotes all elements of finite order in* Γ.)

We remark that under our assumptions, one can express the terms $T_*(\gamma, \mathbb{Q})$ as the stable value of $H_*(B(C(\gamma)/\langle\gamma\rangle), \mathbb{Q})$, which can easily be verified as being periodic from the Gysin sequences of the fibrations $\mathbb{S}^1 \to BC(\gamma) \to B(C(\gamma)/\langle\gamma\rangle)$. What this formula means is not clear at this point; but it indicates that the "elliptic" part of cyclic homology can be expressed using equivariant bundles.

This paper can be divided into two parts. Sections 1–3 deal with preliminaries and the main algebraic properties of the representation rings $R_{\mathscr{F}}(\Gamma)$, where $\mathscr{F}$ is a family of finite subgroups in Γ. Sections 4–7 describe the K-theory of $B\Gamma$.

The complete results in this paper were first announced in [3]. The author would like to thank Thomas for generously sharing some unpublished work [24] and Karoubi and Mislin for very helpful correspondence and conversations.

1. Preliminaries

In this section we provide a brief description of the class of groups which will be considered in this paper, including the main algebraic and geometric properties which we will use in later sections. A good reference for this is [8].

Definition 1.1. A discrete group Γ is of finite cohomological dimension (cd $\Gamma < \infty$) if there exists a finite dimensional $K(\Gamma, 1)$.

Definition 1.2. A discrete group Γ is of finite virtual cohomological dimension (vcd $\Gamma < \infty$) if there exists a subgroup $\Gamma' \subseteq \Gamma$ of finite cohomological dimension such that the index $[\Gamma : \Gamma']$ is finite.

The following is a key example of this kind of group.

Example 1.3. Let $\Gamma = \mathrm{SL}_n(\mathbb{Z})$, p an odd prime.

Then the level p congruence subgroup $\Gamma(p)$ is known to be of finite cohomological dimension, and we have an extension

$$1 \to \Gamma(p) \to \Gamma \to \mathrm{SL}_n(\mathbb{F}_p) \to 1 ,$$

hence verifying that vcd $\Gamma < \infty$.

The above is an example of an *arithmetic group*, which are an important class of groups to which our results will apply. However there are other interesting examples such as mapping class groups and the outer automorphism group of a free group which can be analyzed using our methods.

Note that if $\operatorname{cd}\Gamma < \infty$, then Γ must be torsion-free. Also observe that if $\operatorname{vcd}\Gamma < \infty$, then we may choose $\Gamma' \subseteq \Gamma$ (as in 1.2) to be *normal* in Γ. From now on we assume given a fixed extension

$$1 \to \Gamma' \to \Gamma \to G \to 1, \tag{1.4}$$

with $\operatorname{cd}\Gamma' < \infty$, $|G| < \infty$.

Next we recall a construction due to Serre [19].

Theorem 1.5. *For any discrete group Γ of finite vcd there exists a finite dimensional Γ–CW complex X with the following properties*

(1) $X^H \neq \emptyset \Leftrightarrow H \subseteq \Gamma$ *is a finite subgroup,*

(2) X^H *is contractible for all* $H \subseteq \Gamma$ *finite.*

In later sections we will make extensive use of this complex. Note that $\Gamma' \subseteq \Gamma$ acts freely on X, hence $B\Gamma' \simeq X/\Gamma'$. This inherits a $G = \Gamma/\Gamma'$ action where the isotropy subgroups are exactly those which are images of the finite subgroups in Γ under the projection $\Gamma \overset{\pi}{\twoheadrightarrow} G$. Using this projection, the universal space EG becomes a Γ-space and hence Γ acts diagonally (and freely) on the contractible space $X \times EG$, from which we deduce

$$B\Gamma \simeq X/\Gamma' \underset{G}{\times} EG. \tag{1.6}$$

The fact that X/Γ' is finite dimensional makes this a rather useful formula, but we will need additional finiteness hypotheses.

Definition 1.7. Γ is said to be homologically finite if $H^i(\Gamma, M)$ *is finitely generated* for all $i \geqq 0$ for any $\mathbb{Z}\Gamma$-module M which is finitely generated as an abelian group.

From now on we will assume that Γ is homologically finite. Note in particular that this implies

$$H^*(X/\Gamma', \mathbb{Z}) \cong H^*(\Gamma', \mathbb{Z}) \cong H^*(\Gamma, \mathbb{Z}[\Gamma/\Gamma'])$$

is finitely generated as an abelian group. Next we examine the G-action on X/Γ'.

If $Q \subseteq G$ is a subgroup, it is not hard to see [8] that

$$(X/\Gamma')^Q = \coprod_{H \in C(Q)} X^H/\Gamma' \cap N_\Gamma(H),$$

where $C(Q)$ is a set of representatives for the Γ'-conjugacy classes of finite subgroups in Γ whose image in G is Q. From this we infer

$$rk_{\mathbb{Z}} H_0((X/\Gamma')^Q, \mathbb{Z}) = \text{cardinality of } C(Q)$$

from which it is clear that the number of Γ-conjugacy classes of subgroups mapping onto Q is finite if and only if $(X/\Gamma')^Q$ has a finite number of components. We will make this assumption for every $Q \subseteq G$. Note that if Q is a p-group and $H^*(\Gamma', \mathbb{F}_p)$ is finite, then standard Smith Theory arguments imply this fact. Hence, if Γ is homologically finite, then Γ has only finitely many p-subgroups up to conjugacy.

Borel and Serre [6] showed that if Γ is an arithmetic group, then X/Γ' can be taken to be a finite complex, ensuring that the above finiteness conditions are automatically satisfied. In particular there are only *finitely* many elements of finite order in an arithmetic group, up to conjugacy.

Given $\gamma \in \Gamma$ of finite order let $C(\gamma)$ be its centralizer in Γ. Then this group can be described as an extension

$$1 \to C(\gamma) \cap \Gamma' \to C(\gamma) \to H_\gamma \to 1 , \tag{1.8}$$

where $|H_\gamma| < \infty$. We will assume that $C(\gamma)$ is of finite homological type over $\mathbb{Q}$ for each $\gamma \in \Gamma$ of *finite order.* Note that Brown has shown [7] that if X/Γ' is compact, then this is automatically satisfied, from which it holds for arithmetic groups by [6].

For the rest of this paper we will assume that X/Γ' is a finite complex, which automatically ensures the finiteness properties we need. It will be apparent however, that some of the results can be proved with the weaker assumptions previously described.

2. A reduced representation ring

In this section we will define a variant of the usual complex representation ring $R(\Gamma)$; to begin we recall how it is defined. Let V, W be two finite-dimensional $\mathbb{C}$-vector spaces endowed with a Γ-action by automorphisms ($\mathbb{C}\Gamma$-modules). We can define $V \oplus W$ and with respect to this operation the isomorphism classes of Γ-modules form an abelian semigroup. The associated abelian group is denoted by $R(\Gamma)$; it consists of formal differences $[V] - [W]$ of isomorphism classes of $\mathbb{C}\Gamma$-modules with an equivalence relation generated by

$$[V] - [W] \sim [V \oplus Z] - [W \oplus Z].$$

Using the tensor product, $R(\Gamma)$ becomes a commutative ring with unit (the class of the trivial representation).

Given that Γ in general is not compact means that $R(\Gamma)$ is generally not of finite type. For example if $\Gamma = \mathbb{Z} \cong \langle t \rangle$, then $t \mapsto e^{2\pi i/n}$ $n = 1, 2, \ldots$ is an infinite collection of irreducible, non-equivalent Γ-modules. Our goal will be to construct a suitable modification of this ring which is of finite rank and depends on a family of finite subgroups in Γ. We first recall

Definition 2.1. A family $\mathscr{F}$ of subgroups of Γ is a non-empty collection of subgroups which is closed under (1) conjugation by elements in Γ and (2) taking subgroups.

More formally $\mathscr{F}$ can be considered as a category whose objects are the elements in $\mathscr{F}$ and whose morphisms are generated by inclusions $L \subseteq K$ and conjugation by elements $\gamma \in \Gamma$, $c_\gamma : K \to \gamma K \gamma^{-1}$.

The following are examples of families in Γ which we will need.

Example 2.2. Let $\mathscr{F}(\Gamma)$ denote the family of all finite subgroups in Γ. Up to conjugacy, there are finitely many objects.

Example 2.3. Let $\mathscr{F}_p(\Gamma)$ denote the family of all finite p-subgroups in Γ, for p a fixed prime; it also has a finite number of conjugacy representatives.

Example 2.4. Let $\mathscr{F}_{p'}(\Gamma)$ denote the family of all finite p'-subgroups in Γ (i.e. of order *not* divisible by a fixed prime p); again up to conjugacy there are only finitely many.

Example 2.5. Let Y be a Γ–CW complex and let $\mathscr{F}(Y, \Gamma) = \{K \subseteq \Gamma \mid Kx = x$ for some $x \in X\}$. One easily checks that this is a family of subgroups in Γ.

We now introduce

Definition 2.6. If $\mathscr{F}$ is a family of subgroups in Γ, then

$$R_{\mathscr{F}}(\Gamma) = \lim_{H \in \mathscr{F}} R(H).$$

Note. If $\Gamma = G$, a finite group, then we have (see [20, 14])

(1) $R(G) \cong \lim_{H \in \mathscr{B}} R(H)$, $\mathscr{B}$ = family of elementary subgroups of G. (Brauer's theorem)

(2) $R(G) \otimes \mathbb{Z}[1/|G|] \cong \lim_{H \in \mathscr{C}} R(H) \otimes \mathbb{Z}[1/|G|]$, $\mathscr{C}$ = family of cyclic subgroups of G. (Artin's theorem)

In general, given a family $\mathscr{F}$ of subgroups in Γ, we have a map induced by the restrictions

$$R(\Gamma) \xrightarrow{\varphi_{\mathscr{F}}} \lim_{H \in \mathscr{F}} R(H).$$

The image of this map can be described as follows. Call two modules V, W $\mathscr{F}$-isomorphic if $V|_S \cong W|_S$ for all $S \in \mathscr{F}$. Let $\bar{R}_{\mathscr{F}}(\Gamma)$ be obtained by taking the abelian group associated to $\mathscr{F}$-isomorphism classes; hence it is a quotient

$$R(\Gamma)/I_{\mathscr{F}} = \bar{R}_{\mathscr{F}}(\Gamma) = \operatorname{im} \varphi_{\mathscr{F}} ,$$

where $I_{\mathscr{F}}$ is the ideal generated by the differences $[V] - [W]$, where V and W are $\mathscr{F}$-isomorphic. Equivalently we can express $\operatorname{im} \varphi_{\mathscr{F}}$ as the image

$$R(\Gamma) \xrightarrow{\Pi \text{ res}} \prod_{H \in \mathscr{F}} R(H).$$

In particular if there are finitely many objects in $\mathscr{F}$ up to conjugacy, we have a commutative diagram

$$\begin{array}{ccc} R(\Gamma) & \xrightarrow{\varphi_{\mathscr{F}}} & \bigoplus_{\substack{(H) \\ H \in \mathscr{F}}} R(H) \\ & \searrow \qquad \nearrow J_{\mathscr{F}} & \\ & R_{\mathscr{F}}(\Gamma) & \end{array}$$

from which we deduce $\bar{R}_{\mathscr{F}}(\Gamma) \subseteq R_{\mathscr{F}}(\Gamma)$.

In some cases these two rings will coincide. More generally, we have

Proposition 2.7. *Let $\mathscr{F}$ be a family of finite subgroups in Γ with finitely many objects up to conjugacy. Then $R_{\mathscr{F}}(\Gamma)/\bar{R}_{\mathscr{F}}(\Gamma)$ is a finite group.*

Proof. If $H \in \mathscr{F}$, let $\Gamma_H = \pi^{-1}(\pi(H)) \subseteq \Gamma$, described as an extension

$$1 \to \Gamma' \to \Gamma_H \to \pi(H) \to 1,$$

where $\pi : \Gamma \twoheadrightarrow G$ is the projection (1.4). Then $H \subseteq \Gamma_H$ and it is mapped isomorphically onto $\pi(H)$; hence $R(\Gamma_H)$ restricts onto $R(H)$ (it is a split surjection). Furthermore note that for $\gamma \in \Gamma$, $\gamma \Gamma_H \gamma^{-1} = \Gamma_{\gamma H \gamma^{-1}}$ and that if $H \subset K$, then $\Gamma_H \subset \Gamma_K$. Using the reduced representation ring for each Γ_H, we infer that the restrictions induce an epimorphism

$$\lim_{H \in \mathscr{F}} \bar{R}_{\mathscr{F}(\Gamma_H)}(\Gamma_H) \to \lim_{H \in \mathscr{F}} R(H) = R_{\mathscr{F}}(\Gamma).$$

Note that $[\Gamma : \Gamma_H] < \infty$, hence we have a well-defined induction map $R(\Gamma_H) \to R(\Gamma)$. From the induction-restriction formula it follows that two representations of Γ_H which are isomorphic on finite subgroups will evidently induce up to representations of Γ with this same property. Hence there is a well-defined induction on the reduced rings $\bar{R}_{\mathscr{F}(\Gamma_H)}(\Gamma_H) \to \bar{R}_{\mathscr{F}}(\Gamma)$. As in the case of the proof of Artin's theorem [20], it will now suffice to show that

$$\mathrm{Res} : \bar{R}_{\mathscr{F}}(\Gamma) \otimes \mathbb{C} \to \bigoplus_{(H)} \bar{R}_{\mathscr{F}(\Gamma_H)}(\Gamma_H) \otimes \mathbb{C}$$

is injective, but this follows from the factorization

$$\begin{array}{ccc} \bar{R}_{\mathscr{F}}(\Gamma) & \longrightarrow & \bigoplus_{(H)} R(H) \\ \searrow & & \nearrow \\ & \bigoplus_{(H)} \bar{R}_{\mathscr{F}(\Gamma_H)}(\Gamma_H) & \end{array}$$

□

Note that if $\mathscr{C}$ is the category of all cyclic subgroups in $\mathscr{F}$, then $\bar{R}_{\mathscr{F}}(\Gamma) \cong \bar{R}_{\mathscr{C}}(\Gamma)$; in particular if $\Gamma = G$ (a finite group), then $R(G) \cong \bar{R}_{\mathscr{C}}(G)$.

In later applications we will be concerned with the rationalization of these rings, in which case we can indistinctly use $R_{\mathscr{F}}(\Gamma)$ or $\bar{R}_{\mathscr{F}}(\Gamma)$.

Let $\Gamma_0 \subseteq \Gamma$ denote a subset of elements closed under conjugation; then we have a decomposition (as Γ-sets under conjugation)

$$\Gamma = \Gamma_0 \coprod (\Gamma - \Gamma_0) .$$

Recall the definition of the ring of complex class functions on Γ:

$$C(\Gamma) = \mathrm{Map}_{\Gamma}(\mathrm{Hom}(\mathbb{Z}, \Gamma), \mathbb{C}) .$$

The preceding decomposition yields a direct sum of algebras

$$C(\Gamma) \cong C(\Gamma_0) \oplus C(\Gamma - \Gamma_0) .$$

Now suppose $\mathscr{F}$ is a family of subgroups in Γ, then we may take

$$\Gamma_0 = \Gamma_0(\mathscr{F}) = \{\gamma \in \Gamma \mid \langle \gamma \rangle \in \mathscr{F}\} .$$

Denote by $n(\mathscr{F})$ the number of distinct conjugacy classes of elements in $\Gamma_0(\mathscr{F})$; then $C(\Gamma_0(\mathscr{F}))$ is an algebra of dimension $n(\mathscr{F})$. Given $H \in \mathscr{F}$, there is a character map

$$\chi_H : R(H) \to C(H) .$$

These can be assembled to yield a map

$$\chi_F : R_{\mathscr{F}}(\Gamma) \to \lim_{H \in \mathscr{F}} C(H) .$$

However, using the bijection between $\varinjlim_{H \in \mathscr{F}} \mathrm{Hom}(\mathbb{Z}, H)$ and conjugacy classes of homomorphisms $\psi : \mathbb{Z} \to \Gamma$ with values in Γ_0, we can identify the term on the right with $C(\Gamma_0(\mathscr{F}))$. There is of course also a standard character map

$$\chi_{\mathscr{F}} : \bar{R}_{\mathscr{F}}(\Gamma) \to C(\Gamma_0(\mathscr{F})) ,$$

which relates to the other by a commutative diagram

$$\begin{array}{ccc} \bar{R}_{\mathscr{F}}(\Gamma) & \longrightarrow & C(\Gamma_0(\mathscr{F})) \\ \downarrow & & \downarrow \cong \\ R_{\mathscr{F}}(\Gamma) & \longrightarrow & \lim\limits_{H \in \mathscr{F}} C(H) \end{array} .$$

To simplify matters, we use the notation $C(\Gamma_0(\mathscr{F})) = C_{\mathscr{F}}(\Gamma)$. As in the case of finite groups, it is of interest to examine this character map more closely.

Theorem 2.8. *If $\mathscr{F}$ is a family of finite subgroups in Γ with $n(\mathscr{F}) < \infty$, then the character map*

$$\chi_{\mathscr{F}} : R_{\mathscr{F}}(\Gamma) \to C_{\mathscr{F}}(\Gamma)$$

is an isomorphism after complexification of the domain.

Proof. For any finite group, the character map $R(H) \to C(H)$ induces an isomorphism after tensoring the domain with $\mathbb{C}$. Hence we obtain that $\chi_{\mathscr{F}}$ induces an isomorphism

$$R_{\mathscr{F}}(\Gamma) \otimes \mathbb{C} \cong \left(\lim_{H \in \mathscr{F}} R(H) \right) \otimes \mathbb{C}$$

$$\cong \lim_{H \in \mathscr{F}} (R(H) \otimes \mathbb{C}) \cong C_{\mathscr{F}}(\Gamma). \qquad \square$$

Corollary 2.9. *Assume that $n(\mathscr{F}) < \infty$; then $R_{\mathscr{F}}(\Gamma)$ is a free abelian group of this rank.*

Let $n(\mathscr{F}(\Gamma)) = n(\Gamma)$, $n(\mathscr{F}_p(\Gamma)) = n_p(\Gamma)$ and $n(\mathscr{F}_{p'}(\Gamma)) = n_{p'}(\Gamma)$. Then for the groups we consider all these numbers are finite, and represent the $\mathbb{Z}$-rank of the free abelian groups $R_{\mathscr{F}}(\Gamma)$, $R_{\mathscr{F}(p)}(\Gamma)$ and $R_{\mathscr{F}(p')}(\Gamma)$ respectively.

For computational purposes it is convenient to provide an explicit description of $R_{\mathscr{F}}(\Gamma)$, in terms of its natural embedding in $\prod_{H \in \mathscr{F}} R(H)$. It can be described in terms of stable elements, similar to the finite case (see [20]). More precisely we can identify $R_{\mathscr{F}}(\Gamma)$ with the sequences $\{x_H\}_{H \in \mathscr{F}}$ such that

(1) if $H \subset H'$, then $\mathrm{res}_H^{H'}(x_{H'}) = x_H$

(2) if $H' = \gamma H \gamma^{-1}$, then $c_\gamma^*(x_H) = x_{H'}$.

3. Explicit calculations

The first example we consider is an amalgamated product of finite groups G_1, G_2 along a common subgroup H, denoted $\Gamma = G_1 *_H G_2$.

Example 3.1. $\Gamma = G_1 *_H G_2$.

Every element of finite order in Γ is conjugate to one either in G_1 or G_2; these are maximal finite subgroups in Γ. Hence if $\mathscr{F}$ is a family of finite subgroups in Γ we have an exact sequence

$$0 \to R_{\mathscr{F}}(\Gamma) \to R_{\mathscr{F}}(G_1) \oplus R_{\mathscr{F}}(G_2) \xrightarrow{\varphi = \mathrm{res}_1 - \mathrm{res}_2} R_{\mathscr{F}}(H) .$$

(a) $G_1 = \mathbb{Z}/n$, $G_2 = \mathbb{Z}/m$, $H = \{1\}$ and $(m, n) = 1$, $\mathscr{F} = \mathscr{F}(\Gamma)$; then we have $\Gamma = \mathbb{Z}/n * \mathbb{Z}/m$ and

$$0 \to R_{\mathscr{F}}(\Gamma) \to R(\mathbb{Z}/n) \oplus R(\mathbb{Z}/m) \to \mathbb{Z} \to 0 .$$

In fact if ρ_n, ρ_m are generators for $R(\mathbb{Z}/n)$ and $R(\mathbb{Z}/m)$ respectively, then $R_{\mathscr{F}}(\Gamma)$ is generated by $x = (\rho_n, \rho_m)$ and we have

$$R_{\mathscr{F}}(\mathbb{Z}/n * \mathbb{Z}/m) \cong \mathbb{Z}[x]/(x^n - 1)(x^{m-1} + x^{m-2} + \cdots + x + 1) .$$

Note that this has rank $n + m - 1$, as expected from counting conjugacy classes of elements of finite order in Γ.

(b) $G_1 = \mathbb{Z}/2$, $G_2 = \mathbb{Z}/2$, $H = \{1\}$, $\mathscr{F} = \mathscr{F}(\Gamma)$.

In this case $\Gamma = \mathbb{Z}/2 * \mathbb{Z}/2 = D_\infty$, the infinite dihedral group, we have an exact sequence

$$0 \to R_{\mathscr{F}}(\Gamma) \to R(\mathbb{Z}/2) \oplus R(\mathbb{Z}/2) \to \mathbb{Z} \to 0 ,$$

and

$$R_{\mathscr{F}}(D_\infty) \cong \mathbb{Z}[x, y] \Big/ \begin{matrix} x^2 - 1,\ y^2 - 1 \\ xy - x - y + 1 \end{matrix} ,$$

which is free abelian of rank 3.

(c) $G_1 = \mathbb{Z}/6$, $G_2 = \mathbb{Z}/4$, $H = \mathbb{Z}/2$, $\mathscr{F} = \mathscr{F}(\Gamma)$.

In this case $\Gamma = \mathbb{Z}/6 *_{\mathbb{Z}/2} \mathbb{Z}/4 \cong \mathrm{SL}_2(\mathbb{Z})$ and we have an exact sequence

$$0 \to R_{\mathscr{F}}(\Gamma) \to R(\mathbb{Z}/6) \oplus R(\mathbb{Z}/4) \to R(\mathbb{Z}/2) \to 0 .$$

A simple calculation shows that

$$R_{\mathscr{F}}(\Gamma) \cong \mathbb{Z}[w]/w^8 + w^6 - w^2 - 1 ;$$

note that this has rank 8, the number of conjugacy classes of elements of finite order in $\mathrm{SL}_2(\mathbb{Z})$.

(d) $G_1 = S_3$, $G_2 = S_3$, $H = \mathbb{Z}/3$, $\mathscr{F} = \mathscr{F}_3(\Gamma) = \mathscr{F}(3)$.

We have an exact sequence

$$0 \to R_{\mathscr{F}(3)}(\Gamma) \to R_{\mathscr{F}(3)}(S_3) \oplus R_{\mathscr{F}(3)}(S_3) \xrightarrow{\varphi} R_{\mathscr{F}(3)}(\mathbb{Z}/3) .$$

Now

$$R_{\mathscr{F}(3)}(S_3) \cong R(\mathbb{Z}/3)^{\mathbb{Z}/2} \cong \langle 1, w_3 + w_3^2 \rangle$$

$$R_{\mathscr{F}(3)}(\mathbb{Z}/3) = R(\mathbb{Z}/3)$$

and $R_{\mathscr{F}(3)}(\Gamma) \cong \mathbb{Z}[x]/x^2 - x - 2$, included diagonally in $R_{\mathscr{F}(3)}(S_3) \oplus R_{\mathscr{F}(3)}(S_3)$. Note that $n_3(\Gamma) = 2$ and that coker $\varphi = R(\mathbb{Z}/3)/R(\mathbb{Z}/3)^{\mathbb{Z}/2}$. Later we will see that this cokernel (modulo torsion) can be determined cohomologically.

Example 3.2.

(a) $\Gamma = \mathrm{SL}_3(\mathbb{Z})$, $\mathscr{F} = \mathscr{F}(3) = \mathscr{F}_3(\Gamma)$.

Up to conjugacy, Γ has two 3-subgroups, generated by

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

with the full automorphism group $\mathbb{Z}/2$ present in their Weyl group. Hence we have an exact sequence

$$0 \to R_{\mathscr{F}(2)}(\Gamma) \xrightarrow{J} R(\mathbb{Z}/3)^{\mathbb{Z}/2} \oplus R(\mathbb{Z}/3)^{\mathbb{Z}/2} \to \mathbb{Z} \to 0 .$$

If $R(\mathbb{Z}/3) = \mathbb{Z}[w]/w^3 - 1$, then $\operatorname{im} J$ is generated by $(1, 1)$ and

$$\alpha = (w + w^2, 2) \quad \text{and} \quad \beta = (2, w + w^2) .$$

Thus we obtain

$$R_{\mathscr{F}(3)}(\mathrm{SL}_3(\mathbb{Z})) \cong \mathbb{Z}[\alpha, \beta] \Big/ \begin{array}{c} \alpha^2 - \alpha - 2, \\ \beta^2 - \beta - 2, \\ \alpha\beta - 2(\alpha + \beta - 2) \end{array},$$

where $n_3(\Gamma) = 3$. Note that this simply shows $R_{\mathscr{F}(3)}(\mathrm{SL}_3(\mathbb{Z})) \cong R_{\mathscr{F}(3)}(S_3 * S_3)$.

(b) $\Gamma = \mathrm{SL}_3(\mathbb{Z})$, $\mathscr{F} = \mathscr{C}(2) = \mathscr{C}_2(\Gamma)$, cyclic subgroups of order a power of two. Up to conjugacy, Γ has four non-trivial elements of order 2^r, where $r = 1$ or 2 ($r \geqq 3$ does not occur). They are represented by

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$D = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $A^2 = B^2 = C^4 = D^4 = 1$. Note that $D^2 = A$, and that

$$C^2 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

is conjugate to B, as

$$C^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ -1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ -1 & 1 & 1 \end{pmatrix} B .$$

Hence the subgroups of order 4 are maximal and their automorphism group $\mathbb{Z}/2$ is present in their normalizers. We obtain an exact sequence

$$0 \to R_{\mathscr{C}(2)}(\Gamma) \to R(\mathbb{Z}/4)^{\mathbb{Z}/2} \oplus R(\mathbb{Z}/4)^{\mathbb{Z}/2} \to \mathbb{Z} \to 0 .$$

If $R(\mathbb{Z}/4) = \mathbb{Z}[y]/y^4 - 1$, then $\mathbb{Z}/2$ acts by $y \mapsto y^3$, and so

$$R(\mathbb{Z}/4)^{\mathbb{Z}/2} = \langle 1, y^2, y + y^3 \rangle .$$

Hence a basis for $R_{\mathscr{C}(2)}(\Gamma)$ is given by $(1, 1)$, $\alpha_1 = (1, y^2)$, $\alpha_2 = (y^2, 1)$, $\beta_1 = (2, y + y^3)$ and $\beta_2 = (y + y^3, 2)$. The ring structure is described by

$$R_{\mathscr{C}(2)}(\mathrm{SL}_3(\mathbb{Z})) \cong \mathbb{Z}[\alpha_1, \alpha_2, \beta_1, \beta_2] \Bigg/ \begin{matrix} \alpha_1^2 - 1,\ \alpha_2^2 - 1 \\ \beta_1^2 - 2 - 2\alpha_1,\ \alpha_1\beta_1 - \beta_1\ , \\ \beta_2^2 - 2 - 2\alpha_2,\ \alpha_2\beta_2 - \beta_2\ , \\ \alpha_1\alpha_2 - \alpha_1 - \alpha_2 + 1,\ \alpha_1\beta_2 - 2\alpha_1 - \beta_2 + 2\ , \\ \beta_1\beta_2 - 2\beta_1 - 2\beta_2 + 4,\ \alpha_2\beta_1 - 2\alpha_2 - \beta_1 + 2\ . \end{matrix}$$

Note that as we saw in §2, $\bar{R}_{\mathscr{F}(p)}(\Gamma)$ is isomorphic to $\bar{R}_{\mathscr{C}(p)}(\Gamma)$ for any prime p. Rationally they are both isomorphic to $R_{\mathscr{F}(p)}(\Gamma)$. Also observe that as $\mathrm{SL}_3(\mathbb{Z})$ only has 2 and 3 torsion,

$$R_{\mathscr{F}(2')}(\mathrm{SL}_3(\mathbb{Z})) \cong R_{\mathscr{F}(3)}(\mathrm{SL}_3(\mathbb{Z})) \quad \text{and}$$

$$R_{\mathscr{F}(3')}(\mathrm{SL}_3(\mathbb{Z})) \cong R_{\mathscr{F}(2)}(\mathrm{SL}_3(\mathbb{Z})) .$$

Example 3.3. $\Gamma = \mathrm{GL}_{p-1}(\mathbb{Z})$, $\mathscr{F} = \mathscr{F}(p) = \mathscr{F}_p(\Gamma)$ p odd prime. Let ζ be a primitive pth root of unity and $R = \mathbb{Z}[\zeta] \subseteq \mathbb{Q}(\zeta)$ the ring of algebraic integers. The number of distinct R-ideal classes in $\mathbb{Q}(\zeta)$ is the class number $Cl(p)$ and by the result due to Diederichsen and Reiner [10] they correspond to a complete set of conjugacy classes of elements of order p in Γ. As Γ has no elements of order p^2, then $rk_{\mathbb{Z}} R_{\mathscr{F}(p)}(\Gamma) = Cl(p) + 1$.

Let $\Delta \cong \mathbb{Z}/p - 1$ denote the Galois group of the extension, which acts on the set of ideal classes, with S_i = stabilizer corresponding to the Δ-equivalence class $[A_i]$, $i = 1, \ldots, t(p)$. In this case we have an exact sequence

$$0 \to R_{\mathscr{F}(p)}(\Gamma) \to \left(\bigoplus_1^{Cl(p)} R(\mathbb{Z}/p) \right)^{\Delta} \to \mathbb{Z}^{t(p)-1} \to 0 .$$

We also have

$$\left(\bigoplus_1^{Cl(p)} R(\mathbb{Z}/p) \right)^{\Delta} = \bigoplus_1^{t(p)} R(\mathbb{Z}/p)^{S_i} .$$

Now as an S_i-module,

$$R(\mathbb{Z}/p) \cong \mathbb{Z} \oplus (\mathbb{Z}[S_i])^{[\Delta : S_i]} ,$$

hence,

$$R(\mathbb{Z}/p)^{S_i} \cong \mathbb{Z}^{([\Delta : S_i] + 1)} .$$

To verify that the ranks are correct, note that

$$\left(\sum_{i=1}^{t(p)} [\Delta : S_i] + 1 \right) - (t(p) - 1) = \sum_{i=1}^{t(p)} [\Delta : S_i] + 1 = Cl(p) + 1 .$$

In fact we have shown that if $G_i = \mathbb{Z}/p \times_T S_i$, then

$$R_{\mathscr{F}(p)}(\mathrm{GL}_{p-1}(\mathbb{Z})) \cong R_{\mathscr{F}(p)}(G_1 * G_2 * \cdots * G_{t(p)}) ,$$

in particular for $Cl(p) = 1$,

$$R_{\mathscr{F}(p)}(\mathrm{GL}_{p-1}(\mathbb{Z})) \cong R(\mathbb{Z}/p)^{\mathbb{Z}/p-1} .$$

4. Equivariant K-theory

Let Y be a Γ–CW complex such that Y/Γ is compact. We recall that the Γ-equivariant K-theory of Y, denoted $K^0_\Gamma(Y)$ is the Grothendieck group constructed from the semi-ring of isomorphism classes of complex Γ-bundles. Assume in addition that the isotropy subgroups of the Γ-action are all *finite*. Then Γ' acts freely on Y, and we have an isomorphism [18]

$$K^0_\Gamma(Y) \cong K^0_G(Y/\Gamma') , \tag{4.1}$$

where as before $G = \Gamma/\Gamma'$ (finite) and Γ' is torsion-free. Similarly we have an isomorphism

$$K^1_\Gamma(Y) \cong K^1_G(Y/\Gamma') \tag{4.2}$$

and $K^*_\Gamma(Y)$ will be a $\mathbb{Z}/2$-graded ring with the usual finiteness properties.

Now let $\mathscr{F}$ be a family of finite subgroups in Γ. We have restriction maps

$$K^*_\Gamma(Y) \to K^*_H(Y)$$

for each $H \in \mathscr{F}$, inducing a map

$$K^*_\Gamma(Y) \xrightarrow{\varphi_{\mathscr{F}}(Y)} \lim_{H \in \mathscr{F}} K^*_H(Y) ,$$

as it is compatible with respect to restriction and conjugation. Using the projection onto a point we obtain a commutative square

$$\begin{array}{ccc} K^*_\Gamma(Y) & \xrightarrow{\varphi_{\mathscr{F}}(Y)} & \lim_{H \in \mathscr{F}} K^*_H(Y) \\ \uparrow & & \uparrow \\ R(\Gamma) & \xrightarrow{\varphi_{\mathscr{F}}} & R_{\mathscr{F}}(\Gamma) \end{array} . \tag{4.3}$$

We denote

$$\mathscr{F}K^*_\Gamma(Y) = \lim_{H \in \mathscr{F}} K^*_H(Y) .$$

To make this meaningful, we shall assume that for each $H \in \mathscr{F}$, Y is H-equivariantly homotopic to a finite H–CW complex. Under this condition $\mathscr{F}K_\Gamma^*(Y)$ is a finitely generated $R_{\mathscr{F}}(\Gamma)$-module. The following lemma is derived from (4.3).

Lemma 4.4. *Let Y be a Γ-CW complex with Y H-homotopic to a point for all $H \in \mathscr{F}$, a family of finite subgroups in Γ. Then $\mathscr{F}K_\Gamma^*(Y) \cong R_{\mathscr{F}}(\Gamma)$ and there is a commutative diagram of rings*

$$\begin{array}{ccc} K_\Gamma^0(Y) & & \\ & \searrow^{\varphi_{\mathscr{F}}(Y)} & \\ \uparrow & & R_{\mathscr{F}}(\Gamma) \;. \\ & \nearrow_{\varphi_{\mathscr{F}}} & \\ R(\Gamma) & & \end{array}$$

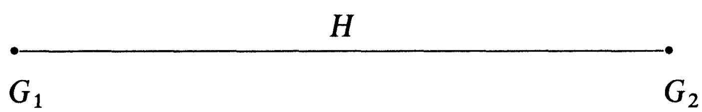
As a consequence of 4.4 and 2.7, we obtain

Corollary 4.5. *Under the hypotheses of* 4.4, *there is a surjection of rings*

$$0 \to I_{\mathscr{F}}(Y) \to K_\Gamma^*(Y) \otimes \mathbb{Q} \xrightarrow{\varphi_{\mathscr{F}}(Y)} R_{\mathscr{F}}(\Gamma) \otimes \mathbb{Q} \to 0 \,.$$

Consider the Γ-complex X described in 1.5: it clearly satisfies the hypotheses of 4.4. In the following section we will analyze the preceding construction for this Γ-space in more detail, showing that $K_\Gamma^*(X)$ determines $K^*(B\Gamma)$ after completion.

Example 4.6. $\Gamma = G_1 *_H G_2$, an amalgamated product of finite groups. We will compute $K_\Gamma^*(X)$. Serre [21] showed that X may be taken to be a tree, with orbit space X/Γ:



Hence Γ' is a free group, X/Γ' a finite graph with a G-action having two orbits of vertices and one orbit of edges. A simple Mayer–Vietoris sequence yields the exactness of

$$0 \to K_\Gamma^0(X) \to R(G_1) \oplus R(G_2) \to R(H) \to K_\Gamma^1(X) \to 0$$

from which we conclude

$$K_\Gamma^*(X) \cong \begin{cases} R_{\mathscr{F}}(\Gamma) & *\,\text{even} \\ \operatorname{coker}(\operatorname{res}_H^{G_1} - \operatorname{res}_H^{G_2}) & *\,\text{odd} \end{cases} ,$$

and hence $I_{\mathscr{F}}(X) = K_\Gamma^1(X) \cong R(H)/\operatorname{im}(\operatorname{res}_H^{G_1} + \operatorname{res}_H^{G_2})$. Applied to 3.1 (a)–(c) we obtain $K_\Gamma^*(X) = R_{\mathscr{F}}(\Gamma)$ in each of these cases. For (d) we see that $K_\Gamma^1(X) \cong \mathbb{Z}$ and so

$$K_\Gamma^*(X) \cong R_{\mathscr{F}}(\Gamma) \oplus \tilde{K}^*(\mathbb{S}^1) \qquad \text{for } \Gamma = S_3 *_{\mathbb{Z}/3} S_3 \,.$$

Given a Γ-CW complex Y satisfying the conditions mentioned at the beginning of this section, we have a spectral sequence [18] converging to $K_\Gamma^*(X)$, with E_1-term

$$E_1^{p,q} = \begin{cases} \bigoplus_{\sigma_q \in (Y/\Gamma)^{(q)}} R(\Gamma_{\sigma_q}) & p \text{ even} , \\ 0 & p \text{ odd} , \end{cases} \tag{4.7}$$

where σ_q is an orbit representative for cells in the q-skeleton of Y. Note that this term is $\mathbb{Z}$-free, of *finite rank*, and the differential d_1 is induced from inclusions $\Gamma_{\sigma_{q+1}} \subsetneq \Gamma_{\sigma_q}$ arising from the Γ-cellular decomposition of Y. As a consequence of (4.7) we obtain

$$K_\Gamma^0(Y) \otimes \mathbb{Q} \oplus \Big(\bigoplus_{\substack{[\sigma_q] \\ q \text{ odd}}} R(\Gamma_{\sigma_q}) \otimes \mathbb{Q} \Big) \cong K_\Gamma^1(Y) \otimes \mathbb{Q} \oplus \Big(\bigoplus_{\substack{[\sigma_q] \\ q \text{ even}}} R(\Gamma_{\sigma_q}) \otimes \mathbb{Q} \Big) .$$

Let $Y_0 = \{ y \in Y \mid \Gamma_y \neq \{1\} \}$, the singular set of the action. Then Γ acts freely on $Y - Y_0$, and we have

$$\chi(K_\Gamma^*(Y) \otimes \mathbb{Q}) = \chi(K^*((Y - Y_0)/\Gamma) \otimes \mathbb{Q}) + \sum_{\substack{[\sigma_q] \\ \Gamma_{\sigma_q} \neq 1}} (-1)^q n(\Gamma_{\sigma_q}) . \tag{4.8}$$

We apply this to the Γ-complex X from (1.5). Now as $(X - X_0)/\Gamma$ is finite,

$$\chi(K^*((X - X_0)/\Gamma) \otimes \mathbb{Q}) = \chi((X - X_0)/\Gamma)$$

(the usual topological Euler characteristic) and hence

$$\chi(K_\Gamma^*(X) \otimes \mathbb{Q}) = \chi((X - X_0)/\Gamma) + \sum_{\substack{[\sigma_q] \\ \Gamma_{\sigma_q} \neq 1}} (-1)^q n(\Gamma_{\sigma_q}) . \tag{4.9}$$

Our goal is to express this in purely group theoretic terms. For this we denote by $S(\Gamma)$ the partially ordered set of non-trivial finite subgroups of Γ (under inclusion); then there is a correspondence

$$H \mapsto X^H .$$

Now $X_0 = \cup X^H$, each X^H is contractible and so are their finite intersections. Let K denote this poset of fixed-points and $|K|$ its nerve; our hypotheses imply that X_0 is homotopy equivalent to $|K|$. The preceding correspondence can be thought of as a map of posets

$$S(\Gamma) \to K$$

which was shown in [8] to be a Γ-equivariant equivalence. Hence we have that $X_0 \simeq |S(\Gamma)|$.

Now for a group Γ as defined in (1.4), consider its group-theoretic Euler characteristic,

$$\chi(\Gamma) = \chi(B\Gamma')/[\Gamma : \Gamma'] . \tag{4.10}$$

Recall that if Y is a Γ-CW complex with Y/Γ finite, its equivariant Euler characteristic is defined (see [8]) as

$$\chi_\Gamma(Y) = \sum_{[\sigma_i]} (-1)^i \chi(\Gamma_{\sigma_i}) . \tag{4.11}$$

We can now state a result due to Brown [8] expressing $\chi((X - X_0)/\Gamma)$ in group theoretic terms:

$$\chi((X - X_0)/\Gamma) = \chi(\Gamma) - \chi_\Gamma(|S(\Gamma)|) \tag{4.12}$$

the proof of which requires the equivalence of posets described above. Combined with (4.9), this yields

$$\chi(K_\Gamma^*(X) \otimes \mathbb{Q}) = \chi(\Gamma) - \chi_\Gamma(|S(\Gamma)|) + \sum_{[\sigma_q]} (-1)^q n(\Gamma_{\sigma_q}) .$$

More elegantly expressed, we have a functor

$$S(\Gamma) \xrightarrow{\mathscr{R}} \text{Rings}$$

$$H \mapsto R(H) \otimes \mathbb{Q}$$

and an associated $H^*(S(\Gamma), \mathscr{R})$ (see [17]). The final term in the expression above is $\chi(H^*(|S(\Gamma)|/\Gamma, \mathscr{R}))$, where we identify Γ-conjugate elements and hence we obtain

Theorem 4.13. *If X is a Γ-complex satisfying the conditions of* (1.5) *and such that X/Γ is compact, then*

$$\chi(K_\Gamma^*(X) \otimes \mathbb{Q}) - \chi(H^*(|S(\Gamma)/\Gamma|, \mathscr{R})) = \chi(\Gamma) - \chi_\Gamma(|S(\Gamma)|) .$$

5. *K*-theory of classifying spaces

In this section we will show how the complex topological K-theory of the classifying space $B\Gamma$ can be calculated from the previous representation-theoretic data.

Let $EG^{(n)} = G * \cdots * G$ (n times) denote the nth stage of the model given by Milnor for the universal G-space EG. Then, using the fact that $B\Gamma \simeq X/\Gamma' \times_G EG$ (1.6), we can define

$$K^*(B\Gamma) = \varprojlim K^*\left(X/\Gamma' \underset{G}{\times} EG^{(n)}\right) . \tag{5.1}$$

Let $R(G)$ denote the representation ring of G, with augmentation ideal $I \subseteq R(G)$. If Y is any G-space, there is a map induced by the projection $Y \times EG^{(n)} \to Y$ which factors as

$$\alpha_n : K_G^*(Y)/I^n K_G^*(Y) \to K^*\left(Y \underset{G}{\times} EG^{(n)}\right)$$

for all $n \geqq 0$. Atiyah and Segal [5] proved

Theorem 5.2 (Completion theorem). *Let Y be a compact G-space such that $K_G^*(Y)$ is finite over $R(G)$. Then the homomorphisms $\{\alpha_n\}$ induce an isomorphism of pro-rings and hence*

$$K_G^*(Y)^\wedge \cong \varprojlim K^*\left(Y \underset{G}{\times} EG^{(n)}\right) = K^*\left(Y \underset{G}{\times} EG\right),$$

the term on the left being the I-adic completion of $K_G^*(Y)$.

From (5.1) and (5.2) we infer that

$$K^*(B\Gamma) \cong K^*_G(X/\Gamma')^\wedge \tag{5.3}$$

provided (as we have assumed) that X/Γ' is compact. Now the map from §4

$$\varphi_{\mathscr{F}}(X): K^*_\Gamma(X) \to R_{\mathscr{F}}(\Gamma)$$

is an $R(G)$-module map, as can be seen from the commutative diagrams, for $H \subseteq \Gamma$ finite:

$$\begin{array}{ccc} \Gamma & \xrightarrow{\pi} & G \\ \uparrow & \nearrow & \\ H & & \end{array}$$

(*that is*, $\Gamma' = \ker \pi$ is torsion-free). Therefore it induces a map

$$K^*(B\Gamma) \to R_{\mathscr{F}}(\Gamma)^\wedge \tag{5.4}$$

which in the case $\Gamma = G$, a finite group, is a case of the isomorphism (5.2), first described by Atiyah [4]; this is not an isomorphism in general.

We note that our description of $K^*(B\Gamma)$ is independent of the choice of the extension

$$1 \to \Gamma' \to \Gamma \to G \to 1 .$$

Any other such extension

$$1 \to \Gamma'' \to \Gamma \to H \to 1$$

yields a third extension

$$1 \to \Gamma' \cap \Gamma'' \to \Gamma \to K \to 1 .$$

Let $Q = \Gamma'/\Gamma' \cap \Gamma''$ (a finite group), then $G = K/Q$, and

$$K^*_G(X/\Gamma') = K^*_{K/Q}((X/\Gamma' \cap \Gamma'')/Q) \cong K^*_K(X/\Gamma' \cap \Gamma'')$$

as Q acts freely on $X/\Gamma' \cap \Gamma''$. This isomorphism induces an isomorphism of the associated completions, using the ring map $R(G) \to R(K)$. Hence by symmetry we obtain an isomorphism

$$K^*_G(X/\Gamma') \cong K^*_H(X/\Gamma'')$$

(and between the completed objects).

Example 5.5. $\Gamma = G_1 *_H G_2$.

We have already computed

$$K^*_\Gamma(X) \cong \begin{cases} R_{\mathscr{F}}(\Gamma) & * \text{ even} \\ R(H)/(\operatorname{im} \operatorname{res}^{G_1}_H + \operatorname{im} \operatorname{res}^{G_2}_H) & * \text{ odd} \end{cases}$$

and hence we obtain

$$K^*(B\Gamma) \cong \begin{cases} R_{\mathscr{F}}(\Gamma)^\wedge & * \text{ even} \\ R(H)^\wedge/(\operatorname{im} \operatorname{res}^{G_1}_H + \operatorname{im} \operatorname{res}^{G_2}_H) & * \text{ odd} . \end{cases}$$

Here we are using the fact that $R(H)$ is a module over $R(G)$, that $I(G)$-adic completion corresponds to $I(H)$-adic completion (see [At]) and that, as $R(G)$ is

Noetherian, IG-adic completion is exact.

$$K^*(B\mathbb{Z}/m * \mathbb{Z}/n) \cong (\mathbb{Z}[x]/(x^n - 1)(x^{m-1} + x^{m-2} + \cdots + x + 1)^\wedge \ (m, n) = 1$$

$$K^*(BD_\infty) \cong \left(\mathbb{Z}[x, y] \Big/ \begin{matrix} x^2 - 1,\ y^2 - 1 \\ xy - x - y + 1 \end{matrix}\right)^\wedge$$

$$K^*(BSL_2(\mathbb{Z})) \cong (\mathbb{Z}[w]/w^8 + w^6 - w^2 - 1)^\wedge$$

$$K^*(B(S_3 *_{\mathbb{Z}/3} S_3))$$

$$\cong \left[\tilde{K}^*(\mathbb{S}^1) \oplus \left(\mathbb{Z}[z_1, z_2, z_3] \Big/ \begin{matrix} z_1^2, z_2^2, z_3^2 - z_3 - 2, \\ z_1 z_2 - z_2 + z_1 - 1, z_1 z_3 - z_3 - 2z_1 + 2 \\ z_2 z_3 - z_3 - 2z_2 + 2 \end{matrix}\right)\right]^\wedge$$

It is interesting to note that the abelianizations

$$\mathbb{Z}/n * \mathbb{Z}/m \to \mathbb{Z}/n \times \mathbb{Z}/m \quad (n, m) = 1$$
$$\mathrm{SL}_2(\mathbb{Z}) \to \mathbb{Z}/12$$

induce cohomology isomorphisms, and hence applying the Atiyah–Hirzebruch spectral sequence their classifying spaces will have isomorphic K-theory. Hence we have isomorphisms

$$R(\mathbb{Z}/n \times \mathbb{Z}/m)^\wedge \cong (\mathbb{Z}[x]/x^{nm} - 1)^\wedge \cong (\mathbb{Z}[x]/(x^n - 1)(x^{m-1} + \cdots + x + 1))^\wedge$$
$$R(\mathbb{Z}/12)^\wedge = (\mathbb{Z}[w]/w^{12} - 1)^\wedge \cong (\mathbb{Z}[w]/(w^6 - 1)(w^2 + 1))^\wedge.$$

These examples exhibit properties of completion using the representation ring of a finite group of composite order. On the other hand, for D_∞ the completion is simply 2-adic completion, as in this case $G = \mathbb{Z}/2 \times \mathbb{Z}/2$, a 2-group.

To make our approach truly effective, we need to have an understanding of the ideal $\ker \varphi_{\mathscr{F}} \subseteq K_\Gamma^*(X)$. To simplify matters, we look at its rationalization

$$0 \to I_{\mathscr{F}} \to K_\Gamma^*(X) \otimes \mathbb{Q} \to R_{\mathscr{F}}(\Gamma) \otimes \mathbb{Q} \to 0 \tag{5.6}$$

Let $\gamma \in \Gamma$ denote an element of finite order and denote its centralizer in Γ by $C(\gamma)$. This group can be expressed as an extension

$$1 \to \Gamma' \cap C(\gamma) \to C(\gamma) \to H_\gamma \to 1 \tag{5.7}$$

where H_γ is *finite*. We have the following additive description of the ideal $I_{\mathscr{F}}$, where (γ) denotes the conjugacy class containing $\gamma \in \Gamma$.

Theorem 5.8. *Let $\mathscr{F}$ be the family of finite subgroups of Γ and $I_{\mathscr{F}}$ the ideal previously defined for X, a Γ-complex satisfying* 1.5. *Then*

$$I_{\mathscr{F}}^* \cong \begin{cases} \bigoplus_{\substack{(\gamma) \\ \gamma \in \mathrm{Tors}(\Gamma)}} \left(\bigoplus_{j \text{ even}} \tilde{H}^j(BC(\gamma), \mathbb{Q})\right) & * \text{ even} \\ \bigoplus_{\substack{(\gamma) \\ \gamma \in \mathrm{Tors}(\Gamma)}} \left(\bigoplus_{j \text{ odd}} H^j(BC(\gamma), \mathbb{Q})\right) & * \text{ odd} \end{cases}$$

where $\mathrm{Tors}(\Gamma)$ *is the set of elements in* Γ *of finite order.*

Proof. In [2], we obtained an *additive* decomposition, as γ ranges over conjugacy classes of elements in $\Gamma_0(\mathscr{F}) = \mathrm{Tors}(\Gamma)$,

$$K_\Gamma^*(X) \otimes \mathbb{Q} \cong \bigoplus_{(\gamma)} K^*(B(C(\gamma) \cap \Gamma'))^{H_\gamma} \otimes \mathbb{Q}$$

which follows from a general decomposition

$$K_G^*(Y) \otimes \mathbb{Q} \cong \bigoplus\nolimits_{(g)} K^*(Y^g/C(g)) \otimes \mathbb{Q}$$

for a finite group G acting on a finite complex Y, applied to $G = \Gamma/\Gamma'$ acting on X/Γ' (this result is due to Kuhn [14]). Note however that $B(C(\gamma) \cap \Gamma')$ is of finite type, hence

$$K^{\mathrm{even}}(B(C(\gamma) \cap \Gamma'))^{H_\gamma} \otimes \mathbb{Q} \cong \bigoplus_{j\ \mathrm{even}} H^j(B(C(\gamma) \cap \Gamma'), \mathbb{Q})^{H_\gamma}$$

$$K^{\mathrm{odd}}(B(C(\gamma) \cap \Gamma'))^{H_\gamma} \otimes \mathbb{Q} \cong \bigoplus_{j\ \mathrm{odd}} H^j(B(C(\gamma) \cap \Gamma'), \mathbb{Q})^{H_\gamma} .$$

On the other hand, as H_γ is finite,

$$H^*(B(C(\gamma) \cap \Gamma'), \mathbb{Q})^{H_\gamma} \cong H^*(BC(\gamma), \mathbb{Q}) .$$

To complete the proof it suffices to observe that

$$\dim K_\Gamma^0(X) \otimes \mathbb{Q} = \dim I_{\mathscr{F}}^0 + n(\Gamma) ,$$

where $n(\Gamma)$ is exactly the number of distinct conjugacy classes of elements in $\mathrm{Tors}(\Gamma)$.

We obtain

Corollary 5.9. *$K_\Gamma^*(X) \otimes \mathbb{Q}$ is isomorphic to $R_{\mathscr{F}}(\Gamma) \otimes \mathbb{Q}$ if and only if $\tilde{H}^*(BC(\gamma), \mathbb{Q}) \equiv 0$ for every element in Γ of finite order.*

Corollary 5.10. *Let $\Gamma = G_1 *_H G_2$, an amalgamated product of finite groups. Then*

(1) $K_\Gamma^0(X) \otimes \mathbb{Q} \cong R_{\mathscr{F}}(\Gamma) \otimes \mathbb{Q}$

(2) $K_\Gamma^1(X) \otimes \mathbb{Q} \cong \bigoplus_{\substack{(\gamma) \\ \gamma \in \mathrm{Tors}(\Gamma)}} H^1(BC(\gamma), \mathbb{Q}) \cong R(H) \otimes \mathbb{Q}/\mathrm{im}(\mathrm{res}_H^{G_1} + \mathrm{res}_H^{G_2})$

(3) $n(\Gamma) = n(G_1) + n(G_2) - n(H) + \sum_{\substack{(\gamma) \\ \gamma \in \mathrm{Tors}(\Gamma)}} \dim H^1(BC(\gamma), \mathbb{Q})$

Proof. We already knew (1); as for (2) this follows from the fact that every subgroup of an amalgamated product of finite groups is virtually free (hence one-dimensional) and 5.5 (3) is just a calculation using the Mayer–Vietoris sequence in (4.6). □

Among our examples of amalgamated products, only $\Gamma = S_3 *_{\mathbb{Z}/3} S_3$ will have an element γ of finite order with $H^1(BC(\gamma), \mathbb{Q}) \neq 0$. We also have

Corollary 5.11.

$$\sum_{\substack{(\gamma) \neq (1) \\ \gamma \in \mathrm{Tors}(\Gamma)}} [\chi(BC(\gamma)) + \chi(C(\gamma))] = \chi(H^*(|S(\Gamma)/\Gamma|, \mathscr{R})) - \chi_\Gamma(|S(\Gamma)|) .$$

Proof. Brown [7] proved the formula

$$\chi(B\Gamma) = \sum_{\substack{(\gamma) \\ \gamma \in \mathrm{Tors}(\Gamma)}} \chi(C(\gamma)) ;$$

applying 4.13 and 5.8 completes the proof. □

In the following section we will prove a local version of our results which allows us to apply 5.8 to obtain a calculation for $B\Gamma$.

6. A local formula

If G is a finite group of composite order, then IG-adic completion can be difficult to describe. In contrast, if G is a finite p-group, this completion coincides with the better understood p-adic completion. To reduce completion to this situation, it is technically useful to work p-locally. Let $\mathbb{Z}_p$ denote the p-adic integers; we denote p-adic K-theory by

$$K_p^*(X) = K^*(X; \mathbb{Z}_p) ,$$

i.e. ordinary K-theory with p-adic coefficients (see [1] for details). This is now a p-local theory.

Using the same notation as before, with $G = \Gamma/\Gamma'$, let $P = p$-Sylow subgroup of G. Then we have a map

$$K_p^*(X/\Gamma' \times_G EG) \xrightarrow{\varphi_*} K_p^*(X/\Gamma' \times_P EP) ,$$

which by transfer arguments in this p-local theory must be injective. On the other hand, we have

$$\tilde{K}_p^*(X/\Gamma' \times_P EP) \cong \tilde{K}_P^*(X/\Gamma') \otimes \mathbb{Z}_p$$

(p-adic completion). The above facts can be combined to yield

$$K_p^*(B\Gamma) \otimes \mathbb{Q}_p \cong (K_P^*(X/\Gamma') \otimes \mathbb{Q}_p)^S , \tag{6.1}$$

where on the right we have the "stable" elements, another expression for the image of

$$\mathrm{res}_P^G : K_G^*(X/\Gamma') \otimes \mathbb{Q}_p \to K_P^*(X/\Gamma') \otimes \mathbb{Q}_p .$$

This map fits into a commutative diagram

$$\begin{array}{ccc} K_G^*(X/\Gamma') \otimes \mathbb{Q}_p & \xrightarrow{\mathrm{res}_P^G} & K_P^*(X/\Gamma') \otimes \mathbb{Q}_p \\ \downarrow \cong & & \downarrow \cong \\ \bigoplus_{\substack{(g) \\ g \in G}} K^*((X/\Gamma')^{\langle g \rangle})^{C_G(g)} \otimes \mathbb{Q}_p & \xrightarrow{\phi} & \bigoplus_{\substack{(g) \\ g \in P}} K^*((X/\Gamma')^{\langle g \rangle})^{C_P(g)} \otimes \mathbb{Q}_p \end{array}$$

the map ϕ is zero for g not conjugate to an element of P. If $hgh^{-1} \in P$ for some $h \in G$, then ϕ on the corresponding component is simply the inclusion

$$K^*((X/\Gamma')^{\langle g \rangle})^{C_G(g)} \otimes \mathbb{Q}_p \subsetneq K^*((X/\Gamma')^{\langle g \rangle})^{C_P(g)} \otimes \mathbb{Q}_p .$$

This local approach is based on a proof appearing in [15]. Let $\mathrm{Tors}_p(\Gamma)$ denote the elements in Γ of order a finite power of p. Identifying this image, we conclude:

Proposition 6.2.

$$K_p^*(B\Gamma)\otimes \mathbb{Q}_p \cong \bigoplus_{\substack{(\gamma)\\ \gamma\in \mathrm{Tors}_p(\Gamma)}} K^*(BC(\gamma)\cap\Gamma'))^{H_\gamma}\otimes \mathbb{Q}_p$$

From the commutative diagram

$$\begin{array}{ccc} K_G^*(X/\Gamma')\otimes \mathbb{Q}_p & \longrightarrow\!\!\!\!\rightarrow & R_{\mathscr{F}}(\Gamma)\otimes \mathbb{Q}_p \\ \downarrow & & \downarrow \\ K_P^*(X/\Gamma')\otimes \mathbb{Q}_p & \longrightarrow\!\!\!\!\rightarrow & R_{\mathscr{F}}(\pi^{-1}(P))\otimes \mathbb{Q}_p \end{array}$$

we obtain a map

$$(K_P^*(X/\Gamma')\otimes \mathbb{Q}_p)^S \longrightarrow\!\!\!\!\rightarrow (R_{\mathscr{F}}(\pi^{-1}(P))\otimes \mathbb{Q}_p)^S$$

and the term on the right is easily recognized as $R_{\mathscr{F}(p)}(\Gamma)\otimes \mathbb{Q}_p$. We therefore obtain a local version of the results in §5, namely

Theorem 6.3. *There is a ring map*

$$0\to I^*_{\mathscr{F}(p)}\otimes \mathbb{Q}_p \to K_p^*(B\Gamma)\otimes \mathbb{Q}_p \xrightarrow{\varphi_p} R_{\mathscr{F}(p)}(\Gamma)\otimes \mathbb{Q}_p \to 0$$

with

$$I^*_{\mathscr{F}(p)} \cong \begin{cases} \displaystyle\bigoplus_{\substack{(\gamma)\\ \gamma\in \mathrm{Tors}_p(\Gamma)}} \left(\bigoplus_{j\ \mathrm{even}} \tilde{H}^j(BC(\gamma),\mathbb{Q})\right) & *\ \mathit{even} \\ \displaystyle\bigoplus_{\substack{(\gamma)\\ \gamma\in \mathrm{Tors}_p(\Gamma)}} \left(\bigoplus_{j\ \mathrm{odd}} H^j(BC(\gamma),\mathbb{Q})\right) & *\ \mathit{odd} \end{cases}$$

We now apply 6.3 to calculate some examples.

Example 6.4. $\Gamma = S_3 *_{\mathbb{Z}/3} S_3$.
Then

$$K_3^*(B\Gamma)\otimes \mathbb{Q}_3 \cong \tilde{K}^*(\mathbb{S}^1)\otimes \mathbb{Q}_3 \oplus \mathbb{Q}_3[x]/x^2-x-2\,,$$

$$K_2^*(B\Gamma)\otimes \mathbb{Q}_2 \cong \mathbb{Q}_2[x,y]\Big/ \begin{matrix} x^2-1,\, y^2-1 \\ xy-x-y+1 \end{matrix}\,.$$

Note that local versions of 5.9 and 5.10 can be proved using 6.3, in particular we have for $\Gamma = G_1 *_H G_2$ (as in 5.9):

$$n_p(\Gamma) = n_p(G_1) + n_p(G_2) - n_p(H) + \sum_{\substack{(\gamma)\\ \gamma\in \mathrm{Tors}_p(\Gamma)}} \dim_{\mathbb{Q}} H^1(BC(\gamma),\mathbb{Q}) \qquad (6.5)$$

from which we conclude that there exists an element $\gamma\in\Gamma$ of order 3 such that $H^1(BC(\gamma),\mathbb{Q}) \neq 0$. Indeed if $\gamma = (123)\in S_3$, then $(12)*(12)$ centralizes (123) and in fact

$$C(\gamma)\cong \mathbb{Z}\times\mathbb{Z}/3\,.$$

Example 6.6. $\Gamma = \mathrm{SL}_3(\mathbb{Z})$
For this group, we know that $C(\gamma)$ is rationally acyclic for $\gamma \in \mathrm{Tors}_p(\Gamma)$, $p = 2, 3$ (the only torsion in this group). Hence $K_p^*(B\Gamma) \otimes \mathbb{Q}_p \cong R_{\mathscr{F}(p)}(\Gamma) \otimes \mathbb{Q}_p$, $p = 2, 3$ and

$$K_3^*(B\Gamma) \otimes \mathbb{Q}_3 \cong \mathbb{Q}_3[\alpha, \beta] \Big/ \begin{matrix} \alpha^2 - \alpha - 2 \,, \\ \beta^2 - \beta - 2 \,, \\ \alpha\beta - 2(\alpha + \beta - 2) \end{matrix}$$

$$K_2^*(B\Gamma) \otimes \mathbb{Q}_2 \cong \mathbb{Q}_2[\alpha_1, \alpha_2, \beta_1, \beta_2] \Big/ \begin{matrix} \alpha_1^2 - 1,\ \alpha_2^2 - 1 \\ \beta_1^2 - 2 - 2\alpha_1,\ \beta_2^2 - 2 - 2\alpha_2 \\ \alpha_1\beta_1 - \beta_1,\ \alpha_2\beta_2 - \beta_2 \\ \alpha_1\alpha_2 - \alpha_1 - \alpha_2 + 1,\ \alpha_1\beta_2 - 2\alpha_1 - \beta_2 + 2 \\ \beta_1\beta_2 - 2\beta_1 - 2\beta_2 + 4,\ \alpha_2\beta_1 - 2\alpha_2 - \beta_1 + 2 \,. \end{matrix}$$

These calculations should be compared to the calculation of $H^*(\mathrm{SL}_3(\mathbb{Z}), \mathbb{Z})$ in [22] and to the recent work of Tezuka and Yagita [23].

Example 6.7. $\Gamma = GL_{p-1}(\mathbb{Z})$, p an odd prime. As we have seen, Γ has no elements of order p^2. Now if $\gamma \in \Gamma$ of order p, then its centralizer in Γ is isomorphic to the group of units $\mathscr{U}$ in $\mathbb{Z}[\zeta]$, where ζ is a primitive p-th root of unity. This group is well-known to be isomorphic to

$$C(\gamma) \cong \mathbb{Z}/p \times (\mathbb{Z})^{\left(\frac{p-3}{2}\right)} \times \mathbb{Z}/2$$

and hence

$$K_p^*(B(C(\gamma) \cap \Gamma'))^{H_\gamma} \cong K_p^*((\mathbb{S}^1)^{\left(\frac{p-3}{2}\right)}) \,.$$

Our exact sequence becomes

$$0 \to I_{\mathscr{F}(p)} \otimes \mathbb{Q}_p \to K_p^*(B\Gamma) \otimes \mathbb{Q}_p \twoheadrightarrow R_{\mathscr{F}(p)}(\Gamma) \otimes \mathbb{Q}_p \to 0 \,,$$

where $R_{\mathscr{F}(p)}(\Gamma)$ is of rank $Cl(p) + 1$ and

$$I_{\mathscr{F}(p)} \otimes \mathbb{Q}_p \cong \left[\bigoplus_1^{Cl(p)} \tilde{K}_p^*((\mathbb{S}^1)^{\left(\frac{p-3}{2}\right)}) \otimes \mathbb{Q}_p \right] \oplus \tilde{K}_p^*(B\Gamma')^G \otimes \mathbb{Q}_p \,.$$

We obtain, for $p \geqq 5$:

$$\dim_{\mathbb{Q}_p} K_p^*(BGL_{p-1}(\mathbb{Z})) \otimes \mathbb{Q}_p = \begin{cases} \sum\limits_{j \text{ even}} \beta_j + (Cl(p))(2^{\left(\frac{p-5}{2}\right)}) & * \text{ even} \\ \sum\limits_{j \text{ odd}} \beta_j + (Cl(p))(2^{\left(\frac{p-5}{2}\right)}) & * \text{ odd} \end{cases}$$

where $\beta_j = \dim_{\mathbb{Q}} H^j(BGL_{p-1}(\mathbb{Z}), \mathbb{Q})$, while

$$K_3^*(BGL_2(\mathbb{Z})) \otimes \mathbb{Q}_3 \cong \mathbb{Q}_3[w]/w^2 - w - 2 \,.$$

Example 6.8. Let Γ_n denote the mapping class group of the n-punctured sphere. Hodgkin [11] has obtained an *additive* description of $K_p^*(B\Gamma_n) \otimes \mathbb{Q}_p$ using the results in [2]. Using representations one can compute the ring structure in some cases. Assume that $n = p$, an odd prime. Then there are no elements of order p^2, $n_p(\Gamma_p) = \frac{1}{2}(p+1)$, and we have an exact sequence

$$0 \to R_{\mathscr{F}(p)}(\Gamma_p) \to \bigoplus_{1}^{\frac{1}{2}(p-1)} R(\mathbb{Z}/p)^{\mathbb{Z}/p-1} \to (\mathbb{Z})^{\frac{1}{2}(p-3)} \to 0 .$$

From Hodgkin's rank calculations we infer that

$$K_p^*(B\Gamma_p) \otimes \mathbb{Q}_p \cong R_{\mathscr{F}(p)}(\Gamma_p) \otimes \mathbb{Q}_p .$$

For example, for $p = 5$, we obtain

$$K_5^*(B\Gamma_5) \otimes \mathbb{Q}_5 \cong \mathbb{Q}_5[\alpha, \beta] \Big/ \begin{matrix} \alpha^2 - 3\alpha - 4 \ , \\ \beta^2 - 3\beta - 4 \ , \\ \alpha\beta - 4\alpha - 4\beta + 16 \ . \end{matrix}$$

We have the following corolaries of 6.3

Corollary 6.9. *$K_p^*(B\Gamma) \otimes \mathbb{Q}_p \cong R_{\mathscr{F}(p)}(\Gamma) \otimes \mathbb{Q}_p$ if and only if $\tilde{H}^*(BC(\gamma), \mathbb{Q}) \equiv 0$ for all $\gamma \in \mathrm{Tors}_p(\Gamma)$.*

Corollary 6.10. *Let $f: \Gamma_1 \to \Gamma_2$ be a homomorphism between two groups of the type considered previously. If f induces an isomorphism $K_p^*(B\Gamma_2) \otimes \mathbb{Q}_p \xrightarrow{f^*} K_p^*(B\Gamma_1) \otimes \mathbb{Q}_p$, then*

(1) $R_{\mathscr{F}(p)}(\Gamma_2) \otimes \mathbb{Q}_p \cong_{f^*} R_{\mathscr{F}(p)}(\Gamma_1) \otimes \mathbb{Q}_p$

(2) $$\bigoplus_{\substack{(\gamma) \\ \gamma \in \mathrm{Tors}_p(\Gamma_1)}} \left(\bigoplus_{j \text{ even}} \tilde{H}^j(BC(\gamma), \mathbb{Q}) \right) \cong \bigoplus_{\substack{(\mu) \\ \mu \in \mathrm{Tors}_p(\Gamma_2)}} \left(\bigoplus_{j \text{ even}} \tilde{H}^j(BC(\mu), \mathbb{Q}) \right)$$

(3) $$\bigoplus_{\substack{(\gamma) \\ \gamma \in \mathrm{Tors}_p(\Gamma_1)}} \left(\bigoplus_{j \text{ odd}} H^j(BC(\gamma), \mathbb{Q}) \right) \cong \bigoplus_{\substack{(\mu) \\ \mu \in \mathrm{Tors}_p(\Gamma_2)}} \left(\bigoplus_{j \text{ odd}} H^j(BC(\mu), \mathbb{Q}) \right) .$$

Using the Atiyah–Hirzebruch spectral sequence, it is apparent that we may replace the hypothesis in 6.10 with the assumption that $f^*: H^*(B\Gamma_2, \mathbb{Z}_p) \to H^*(B\Gamma_1, \mathbb{Z}_p)$ be an isomorphism, obtaining the same conclusions. Of particular interest is

Corollary 6.11. *If $f: \Gamma_1 \to \Gamma_2$ induces a $\mathbb{Z}_p$-cohomology isomorphism, then $R_{\mathscr{F}(p)}(\Gamma_2) \otimes \mathbb{Q}_p \cong_{f^*} R_{\mathscr{F}(p)}(\Gamma_1) \otimes \mathbb{Q}_p$ and in particular $n_p(\Gamma_1) = n_p(\Gamma_2)$.*

Another interesting consequence is

Corollary 6.12. *If $f: \Gamma \to H$, H finite, induces an integral cohomology isomorphism, then for every prime p,*

(1) *$R_{\mathscr{F}(p)}(\Gamma) \otimes \mathbb{Q}_p \cong R_{\mathscr{F}(p)}(H) \otimes \mathbb{Q}_p$ and*

(2) *$\tilde{H}^*(BC(\gamma), \mathbb{Q}) \equiv 0$ for all $\gamma \in \mathrm{Tors}_p(\Gamma)$.*

Note. There are many examples of the type of behaviour described in 6.12, at least p-locally. For example, let H be any finite group of p-rank two. If $A_p(H)$ denotes the poset of p-elementary abelian subgroups of H, then its realization $X = |A_p(H)|$ is a finite graph. H acts on X via conjugation; let $\Gamma = \pi_1(X \times_H EH)$. Note that $B\Gamma \simeq X \times_H EH$ and that $\pi : \Gamma \to H$ induces a p-local cohomology equivalence (see [8]). Hence Γ is a virtually free group such that $R_{\mathscr{F}(p)}(\Gamma) \otimes \mathbb{Q}_p \cong R_{\mathscr{F}(p)}(H) \otimes \mathbb{Q}_p$ and $\tilde{H}^*(BC(\gamma), \mathbb{Q}) \equiv 0$ for all $\gamma \in \mathrm{Tors}_p(\Gamma)$.

7 Final Remarks

Let $\wedge B\Gamma = \mathrm{Map}(\mathbb{S}^1, B\Gamma)$ denote the free loop space on $B\Gamma$. There is a homotopy equivalence

$$\wedge B\Gamma \simeq \coprod_{(\gamma)} BC(\gamma) .$$

Now if $\gamma \in \Gamma$ has infinite order, then there is a fibration

$$\begin{array}{ccc} \mathbb{S}^1 & \to & BC(\gamma) \\ & & \downarrow \\ & & BC(\gamma)/\langle \gamma \rangle \end{array}$$

and hence if $BC(\gamma)$ is of finite type, $\chi(BC(\gamma)) = 0$. We deduce that

$$\chi(\wedge B\Gamma) = \sum_{\substack{(\gamma) \\ \text{finite order}}} \chi(BC(\gamma)) \tag{7.1}$$

and so

$$\chi(\wedge B\Gamma) = \chi(K_\Gamma^*(X) \otimes \mathbb{Q}) . \tag{7.2}$$

This seems related to other results of this nature obtained for classifying spaces of finite groups. These admit generalizations to the so-called Morava K-theories [12] which one would expect to also be true for groups such as those considered in this paper. Lee has recently studied aspects of this question [16].

From a purely algebraic perspective, we have

$$HH_*(\mathbb{Q}\Gamma) \cong H_*(\wedge B\Gamma, \mathbb{Q})$$

(see [9, 13]) where the term on the left is Hochschild homology. Indeed, the ring $K_\Gamma^*(X) \otimes \mathbb{Q}$ is contained (additively) in $HH_*(\mathbb{Q}\Gamma)$. For our purposes (given the periodicity in K-theory) we would prefer the compare $K_\Gamma^*(X)$ to a $\mathbb{Z}/2$-periodic object. Let $HC_*(\mathbb{Q}\Gamma)$ denote the cyclic homology of $\mathbb{Q}\Gamma$; topologically it can be defined as (see [9])

$$HC_*(\mathbb{Q}\Gamma) = H_*^{\mathbb{S}^1}(\wedge B\Gamma; \mathbb{Q}) = H_*(E\mathbb{S}^1 \times_{\mathbb{S}^1} \wedge B\Gamma, \mathbb{Q}) . \tag{7.3}$$

Using the bundle

$$\mathbb{S}^1 \to \wedge B\Gamma \to E\mathbb{S}^1 \times_{\mathbb{S}^1} \wedge B\Gamma ,$$

we obtain a map

$$HC_{*+2n}(\mathbb{Q}\Gamma) \xrightarrow{S} HC_{*+2n-2}(\mathbb{Q}\Gamma)$$

and periodic cyclic homology is defined as

$$PHC_*(\mathbb{Q}\Gamma) = \varprojlim HC_{*+2}(\mathbb{Q}\Gamma) \tag{7.4}$$

which is now a $\mathbb{Z}/2$-periodic theory.

Let $\gamma \in \Gamma$ of infinite order, and consider the fibration

$$\mathbb{S}^1 \to BC(\gamma) \to BC(\gamma)/\langle\gamma\rangle . \tag{7.5}$$

As before we obtain a Gysin sequence

$$\to H_{k+1}(BC(\gamma), \mathbb{Q}) \to H_{k+1}(B(C(\gamma)/\langle\gamma\rangle), \mathbb{Q}) \to H_{k-1}(BC((\gamma)/\langle\gamma\rangle), \mathbb{Q}) \to H_k(BC(\gamma), \mathbb{Q}) \to \cdots .$$

Our hypotheses on Γ imply that $H_*(BC(\gamma), \mathbb{Q}) = 0$ for $* \gg 0$, hence we have $\mathbb{Z}/2$-periodic stable values for $H_*(B(C(\gamma)/\langle\gamma\rangle), \mathbb{Q})$. Following the notation in [9], we denote these by $T_*(\gamma, \mathbb{Q})$.

Theorem 7.1.

$$PHC_*(\mathbb{Q}\Gamma) \cong K_\Gamma^*(X) \otimes \mathbb{Q} \oplus \Bigg(\bigoplus_{\substack{(\gamma) \\ \gamma \notin \mathrm{Tors}(\Gamma)}} T_*(\gamma, \mathbb{Q}) \Bigg) .$$

Proof. Burghelea has shown [9] that

$$PHC_*(\mathbb{Q}\Gamma) \cong \bigoplus_{\substack{(\gamma) \\ \gamma \in \mathrm{Tors}(\Gamma)}} \underline{H}_*(BC(\gamma)/\langle\gamma\rangle, \mathbb{Q}) \oplus \Bigg(\bigoplus_{\substack{(\gamma) \\ \gamma \notin \mathrm{Tors}(\Gamma)}} T_*(\gamma, \mathbb{Q}) \Bigg)$$

where we define

$$\underline{H}_*(X, \mathbb{Q}) = \begin{cases} \bigoplus_{j \text{ even}} H_j(X, \mathbb{Q}) & * = 0 , \\ \bigoplus_{j \text{ odd}} H_j(X, \mathbb{Q}) & * = 1 . \end{cases}$$

However, if $\gamma \in \mathrm{Tors}(\Gamma)$ then $\langle\gamma\rangle$ is finite, hence $B\langle\gamma\rangle$ is rationally acyclic and $H_*(B(C(\gamma)/\langle\gamma\rangle), \mathbb{Q}) \cong H_*(BC(\gamma), \mathbb{Q})$. The result follows from the decomposition

$$K_\Gamma^*(X) \otimes \mathbb{Q} \cong \bigoplus_{\substack{(\gamma) \\ \gamma \in \mathrm{Tors}(\Gamma)}} K^*(B(C(\gamma) \cap \Gamma'))^{H_\gamma} \otimes \mathbb{Q} .$$

Noting (as we did before) that

$$K^*(BC(\gamma) \cap \Gamma')^{H_\gamma} \otimes \mathbb{Q} \cong \underline{H}_*(BC(\gamma), \mathbb{Q}) . \qquad \square$$

What (if any) applications this result may have remains to be seen.

Finally, it is interesting to reformulate (6.3) from a homotopy-theoretic point of view. It indicates a way of constructing classes in $[B\Gamma, BU_p^\wedge]$ using representations of finite subgroups.

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