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Cohomological Non-vanishing for Modules over P-Groups

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INTRODUCTION

Let G be a finite p-group, and M a finitely generated $\mathbb{Z}G$-lattice (i.e., M is a $\mathbb{Z}$-torsion free $\mathbb{Z}G$-module). Define the following invariant of M:

$$\gamma_G(M) = |G| \, rk_{\mathbb{Z}} M^G - rk_{\mathbb{Z}} M,$$

where $M^G \subseteq M$ denotes the submodule of invariants. A well-known result due to Nakayama and Rim (see [N1, N2, R]) states that M is projective if and only if $\hat{H}^i(G, M) = 0$ for two consecutive integers (note $\gamma_G(M) = 0$). On the other hand, a result due to Kuo [K] shows that for the trivial module $\mathbb{Z}$, $\hat{H}^{2i}(G, \mathbb{Z}) \neq 0 \ \forall i \in \mathbb{Z}$ (note that $\gamma_G(\mathbb{Z}) = |G| - 1 > 0$).

In this note we will show that the results above are special cases of certain cohomological behaviour determined by the sign of the integer $\gamma_G(M)$. We have

THEOREM 2.4. *If G is a finite p-group and M a $\mathbb{Z}G$-lattice, then one of the following must hold*

(1) $\gamma_G(M) > 0$ *and* $\hat{H}^{2i}(G, M) \neq 0 \ \forall i \in \mathbb{Z}$

(2) $\gamma_G(M) < 0$ *and* $\hat{H}^{2i+1}(G, M) \neq 0 \ \forall i \in \mathbb{Z}$

(3) $\gamma_G(M) = 0$ *and* $\hat{H}^{i}(G, M) \neq 0 \ \forall i \in \mathbb{Z}$ *or*

(4) $\gamma_G(M) = 0$ *and M is projective.*

In particular we obtain a generalization of the projectivity criterion

THEOREM 2.5. *Let G be a finite p-group and M a $\mathbb{Z}G$-lattice. Then M is projective if and only if $\hat{H}^i(G, M) = 0$ for two values of i which are not congruent* mod 2.

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The method consists of analyzing the Euler characteristic of a partial minimal resolution of M over $\mathbb{Z}G$, which enables us to compare $\gamma_G(M)$ with $\dim \hat{H}^i(G, M) \otimes \mathbb{F}_p$.

1. Minimal Resolutions

Let G be a finite p-group, and M a $\mathbb{Z}G$-lattice; from the work in [S], the notion and existence of a minimal projective resolution for M is well-defined.[1] Swan shows that a resolution

$$\cdots \to P_n \xrightarrow{\delta_n} P_{n-1} \to \cdots \to P_0 \to \mathbb{Z} \to 0$$

is minimal, if for each $i = 0, 1, 2, \ldots$

$$rk_{\mathbb{Z}} P_i = |G| \cdot \dim H^i(G, \mathbb{F}_p).$$

Using similar methods one can construct a minimal resolution for a $\mathbb{Z}G$-lattice M. In this case we have:

Lemma 1.0. *A resolution* $\cdots \to P_n \xrightarrow{\delta_n} P_{n-1} \to \cdots \to P_0 \to M \to 0$ *is minimal if and only if* $rk_{\mathbb{Z}} P_n = |G| \dim H^n(G, M_p^*)$, *where* $M_p = M \otimes_{\mathbb{Z}} \mathbb{F}_p$ *and* $*$ *denotes the usual dual.*

Proof. We recall that for an $\mathbb{F}_pG$-module $N \neq 0$, $N_G = N/IN$ (the coinvariants) must be non-zero when G is a p-group. This is merely the dual statement of the fact that $N^G \neq 0$. If $N = M_p$, then in particular $M_G \neq 0$, as M_G maps onto $(M_p)_G$.

Choose $x_1, \ldots, x_k \in M$ liftings of generating classes for M_G and define a G-map $\phi\colon (\mathbb{Z}G)^k \to M$ in the obvious way. By construction $\operatorname{coker} \phi_G = 0$, hence $\operatorname{coker} \phi$ is p'-torsion. Let $P = \phi^{-1}(qM) \subseteq (\mathbb{Z}G)^k$, where we assume that $q(\operatorname{coker} \phi) = 0$. Then P is projective because $(\mathbb{Z}G)^k/P$ is p'-torsion, and ϕ maps P onto $qM \cong M$. From the fact that $M_G \otimes \mathbb{F}_p \cong (M_p)_G$, we conclude that $k = \dim(M_p)_G$, and $k\,|G|$ is the minimal $\mathbb{Z}$-rank for any projective module covering M. Now note that $(M_p)_G \cong (M_p^*)^G$, hence for a minimal projective resolution P_* of M, we have shown that $rkP_0 = |G| \dim H^0(G, M_p^*)$.

Using the above and successive dimension-shifting we deduce that the resolution P_* is minimal if and only if $rkP_n = |G| \dim H^n(G, M_p^*)$. ∎

Now consider a finite stage in this resolution

$$0 \to K_{n+1} \to P_n \xrightarrow{\delta_n} P_{n-1} \to \cdots \to P_0 \to M \to 0.$$

Here $K_{n+1} = \ker \delta_n = \Omega^{n+1}(M)$, the usual minimal dimension-shift of M.

[1] By definition a resolution $\cdots \to P_n \xrightarrow{\delta_n} P_{n-1} \to \cdots \to P_0 \to M \to 0$ is minimal if P_n is a projective of minimal rank mapping onto $\ker \delta_{n-1}$ for all $n \geqslant 0$.

LEMMA 1.1. $rk_{\mathbb{Z}}(K_{n+1}) = |G| \, [\dim H^{n+1}(G, M^*)_p + (-1)^n rk_{\mathbb{Z}}(M^*)^G] + (-1)^{n+1} rk_{\mathbb{Z}} M$.

Proof. Taking Euler characteristics, we find

$$(-1)^n rk_{\mathbb{Z}} K_{n+1} + rk_{\mathbb{Z}} M = \sum_{i=0}^{n} (-1)^i rk_{\mathbb{Z}} P_i$$

$$= \sum_{i=0}^{n} (-1)^i |G| \dim H^i(G, M_p^*).$$

Now from the exact sequence $0 \to M \to M \to M_p \to 0$ one obtains $\dim H^i(G, M_p^*) = \dim H^{i+1}(G, M^*)_p + \dim H^i(G, M^*)_p$. Substituting above yields

$$(-1)^n rk_{\mathbb{Z}} K_{n+1} + rk_{\mathbb{Z}} M = |G| \, ((-1)^n \dim H^{n+1}(G, M^*)_p + rk_{\mathbb{Z}}(M^*)^G)$$

which after rearrangement proves 1.1. ∎

We point out a simple fact about the partial resolutions: if M is not projective, then $K_{n+1} \neq 0$ for all n. Hence 1.1 yields the following corollary, which will be essential for Section 2:

COROLLARY 1.2. *If M is not projective, then for all $n \geqslant 0$*

$$0 < |G| \dim H^{n+1}(G, M)_p + (-1)^n \, [|G| \, rk_{\mathbb{Z}} M^G - rk_{\mathbb{Z}} M].$$

2. A COHOMOLOGICAL NON-VANISHING THEOREM

Motivated by 1.2, we define $\gamma_G(M) = |G| \, rk_{\mathbb{Z}} M^G - rk_{\mathbb{Z}} M$. We list some properties of this invariant

PROPOSITION 2.1. (1) *If $0 \to M'' \to M \to M' \to 0$ is a short exact sequence of $\mathbb{Z}G$-lattices, then $\gamma_G(M) = \gamma_G(M') + \gamma_G(M'')$.*

(2) $\gamma_G(\Omega(M)) = -\gamma_G(M)$.

(3) *If $H \subseteq G$ is a subgroup, then $\forall n \in \mathbb{Z}$*

$$(-1)^n \, [\gamma_H(M) - \gamma_G(M)] \leqslant |G| \dim \hat{H}^{n+1}(G, M)_p - |H| \dim \hat{H}^{n+1}(H, M)_p.$$

Proof. Part (1) follows from the rational splitting $\mathbb{Q}M \cong \mathbb{Q}M'' \oplus \mathbb{Q}M'$. For (2), note that if P is projective, then $\gamma_G(P) = 0$ and use (1).

For (3), we first observe that

$$\Omega^{n+1}(M)|_H \cong \Omega^{n+1}(M|_H) \oplus \text{Projective}$$

and hence

$$rk_{\mathbb{Z}}\Omega^{n+1}(M) \geqslant rk_{\mathbb{Z}}\Omega^{n+1}(M|_H).$$

From 1.1 we deduce

$$|G| \dim H^{n+1}(G, M)_p + (-1)^n \gamma_G(M)$$
$$\geqslant |H| \dim H^{n+1}(H, M)_p + (-1)^n \gamma_H(M)$$

which, after rearrangement, is (3). ∎

Remarks. From (1) it is clear that $\gamma_G(M)$ is an invariant which only depends on the rational representation type of M. Also note that $\gamma_G(M) = \gamma_G(M^*)$.

We list two corollaries for the case $M = \mathbb{Z}$, the trivial $\mathbb{Z}G$-module.

COROLLARY 2.2. *If G is a finite p-group and $H \subseteq G$ a subgroup, then for all $k \in \mathbb{Z}$*

$$|G| - |H| \leqslant |G| \dim \hat{H}^{2k}(G, \mathbb{Z})_p - |H| \dim \hat{H}^{2k}(H, \mathbb{Z})_p.$$

COROLLARY 2.3. *If G is a finite p-group, $H \subseteq G$ a subgroup, and $\hat{H}^r(G, \mathbb{Z}) = 0$ for r odd then* $\dim \hat{H}^r(H, \mathbb{Z})_p \leqslant [G:H] - 1$.

We will now show that the sign of $\gamma_G(M)$ determines cohomological non-vanishing for M. The fact that it is a rational invariant makes this somewhat unexpected. We have

THEOREM 2.4. *If G is a finite p-group and M a $\mathbb{Z}G$-lattice, then one of the following must hold*

(1) $\gamma_G(M) > 0$ *and* $\hat{H}^{2i}(G, M) \neq 0$ $\forall i \in \mathbb{Z}$

(2) $\gamma_G(M) < 0$ *and* $\hat{H}^{2i+1}(G, M) \neq 0$ $\forall i \in \mathbb{Z}$

(3) $\gamma_G(M) = 0$ *and* $\hat{H}^i(G, M) \neq 0$ $\forall i \in \mathbb{Z}$ *or*

(4) $\gamma_G(M) = 0$ *and M is projective.*

Proof. Assume $\gamma_G(M) > 0$; then M is not projective, and from 1.2 we deduce that for n odd, $0 < \gamma_G(M) < \dim H^{n+1}(G, M)_p$. Using M^* instead of M we thus obtain $\hat{H}^{2i}(G, M) \neq 0$ for all $i \in \mathbb{Z}$.

For (2), we simply dimension-shift and use (1) and 2.1(2).

For (3), assuming M not projective, we can apply 1.2 to deduce

$$0 < \dim \hat{H}^{i+1}(G, M)_p \qquad \forall i \in \mathbb{Z},$$

and so $\hat{H}^i(G, M) \neq 0$ for all $i \in \mathbb{Z}$. The remaining situation is when M is projective and hence cohomologically trivial. ∎

A well-known cohomological criterion for the projectivity of a $\mathbb{Z}G$-lattice (G a p-group) is that $\hat{H}^i(G, M) = 0$ for two consecutive values of i. We derive the following generalization of this result:

THEOREM 2.5. *Let G be a finite p-group and M a $\mathbb{Z}G$-lattice. Then the following three statements are equivalent*

(1) *M is projective*

(2) *$\gamma_G(M) = 0$ and $\hat{H}^i(G, M) = 0$ for one value of $i \in \mathbb{Z}$ and*

(3) *$\hat{H}^i(G, M) = 0$ for two values of i which are not congruent* mod 2.

The special case of $M = \mathbb{Z}$ in 2.4(1) was described by Kuo [K].

If M is not $\mathbb{Z}$-torsion free, but is a finite abelian p-group instead, then it fits into a short exact sequence $0 \to Q \to \bigoplus^k \mathbb{Z}G \to M \to 0$ where Q is a $\mathbb{Z}G$-lattice. Clearly $\gamma_G(Q) = 0$, whence we obtain the well-known result that either M is cohomologically trivial, or $\hat{H}^i(G, M) \neq 0$ for all $i \in \mathbb{Z}$.

EXAMPLE 2.6. Let G be a finite 2-group, and $\tilde{\mathbb{Z}}$ a sign twist of $\mathbb{Z}$. Then $\gamma_G(\tilde{\mathbb{Z}}) = -1$ and we obtain $\hat{H}^{2i+1}(G, \tilde{\mathbb{Z}}) \neq 0$ for all $i \in \mathbb{Z}$.

EXAMPLE 2.7. Let $G = (\mathbb{Z}/p)^r$ and M a $\mathbb{Z}G$-lattice. Let $H \subseteq G$ be a subgroup of index p in G, and denote $M(H) = \mathbb{Q}[G/H]/N_{G/H}$. Then $\mathbb{Q}$ together with the $M(H)$ as H ranges over all index p subgroups of G is a complete collection of irreducible $\mathbb{Q}G$-modules. Hence assume $\mathbb{Q}M \cong \mathbb{Q}^t \oplus (\bigoplus_1^s M(H_i))$, with $t \neq 0$.

Then

$$\begin{aligned}\gamma_G(M) &= |G|\,t - (t + (p-1)s) \\ &= (p^r - 1)t - (p-1)s = (p-1)[(p^{r-1} + \cdots + p + 1)t - s].\end{aligned}$$

Thus we obtain

$$\gamma_G(M) > 0 \Leftrightarrow p^{r-1} + \cdots + p + 1 > \frac{s}{t}$$

$$\gamma_G(M) < 0 \Leftrightarrow p^{r-1} + \cdots + p + 1 < \frac{s}{t}$$

$$\gamma_G(M) = 0 \Leftrightarrow p^{r-1} + \cdots + p + 1 = \frac{s}{t}.$$

We can think of γ as a homomorphism $R_{\mathbb{Q}}(G) \to \mathbb{Z}$. Rationalizing, $\gamma_G = 0$ corresponds to a hyperplane in $R_{\mathbb{Q}}(G) \otimes \mathbb{Q} \cong \mathbb{Q}^{(p^r-1)/(p-1)+1}$ (i.e., vectors orthogonal to $(1, ..., 1, -(p^{r-1} + \cdots + p + 1))$ after suitably arranging the basis) and $\gamma_G < 0$, $\gamma_G > 0$ to half-spaces below and above it. Hence for $G = (\mathbb{Z}/p)^r$, $\gamma_G(M)$ is determined by the ratio of the number of non-trivial irreducibles in $\mathbb{Q}M$ to the number of trivial ones. Of course a similar result can be written down for any abelian p-group using $\mathbb{C}M$ instead of $\mathbb{Q}M$ and the one-dimensional representations over $\mathbb{C}$.

3. Final Remarks

For the sake of completeness we would like to point out that our methods can also be used to obtain information about the *asymptotic* cohomological behaviour of finite p-groups. For example, we have

THEOREM 3.1. *If G is a non-periodic finite p-group, then $H^i(G, \mathbb{Z}) \neq 0$ for all sufficiently large i.*

Proof. Let $H \cong \mathbb{Z}/p \times \mathbb{Z}/p$ be a subgroup of G; from the Kunneth formula it is not hard to see that the sequence of integers $\{\dim_{F_p} H^{2n+1}(H, \mathbb{Z})\}$ for $n = 1, 2, ...$ is unbounded and non-decreasing, hence 2.3 and the fact that the even dimensional cohomology is always non-zero imply the result. ∎

We conclude by remarking that these non-vanishing results are not true for arbitrary finite groups, as large gaps can occur in $H^*(G, M)$.

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