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COHOMOLOGICAL RESTRICTIONS ON FINITE GROUP ACTIONS

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Let G be a finite group and X a connected, finite-dimensional G-CW-complex. In this paper we apply methods from cohomology of groups and representation theory to obtain restrictions on the RG-module structure of $H^*(X; R)$, given isotropy subgroups of prescribed size. The two basic techniques which we use are cohomological exponents and complexity. In particular we exhibit a method for generalizing results about free $(\mathbb{Z}/p)^r$-actions to arbitrary ones by using shifted subgroups.

Introduction

Let G be a finite group acting on a space X; then $H_*(X; R)$ is a graded RG-module. Given assumptions about the size of the isotropy subgroups, it is natural to inquire what restrictions this imposes on the structure of this module.

In this paper we attempt to make a systematic analysis of this problem. The main tools which we use are from the cohomology of finite groups. In particular we apply equivariant Tate homology and the notion of complexity for an $\mathbb{F}_pG$-module. Our methods apply particularly well to finite-dimensional G-CW-complexes. As expected, the restrictions obtained are substantial for free actions and they weaken until becoming virtually meaningless for actions where the whole group has a fixed-point.

We start by defining the 'exponent' of a connected, finite-dimensional $\mathbb{Z}G$-chain-complex $C_* \xrightarrow{\varepsilon} \mathbb{Z}$.

Definition 1.4. The exponent of C_*, denoted $e_G(C_*)$, is defined as $e_G(C_*) = |G|/\exp \operatorname{im} \varepsilon_*$, where $\varepsilon_* : \hat{H}_{-1}(G; C_*) \to \hat{H}_{-1}(G; \mathbb{Z})$ is induced by the augmentation ε in Tate homology.

This integer has very natural divisibility properties with respect to subgroups and chain maps. If $H \subset G$ is a subgroup, then $e_H(C_*) \mid e_G(C_*)$, and if $\phi : C_* \to D_*$ is an augmented G-chain map, then $e_G(D_*) \mid e_G(C_*)$.

The following result shows that in a certain way, $e_G(C_*)$ estimates the exponents occurring in the G-cohomology of $H_*(C)$:

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Theorem 2.2.

$$e_G(C_*)\Big|\prod_{i=1}^{\infty} \exp H^{i+1}(G, H_i(C)).$$

This is the natural generalization of the result in the free case, which is due to Browder [4].

The exponent has been defined algebraically, but when applied to connected G-CW-complexes, it acquires interesting geometric properties. We define $e_G(X)=e_G(C_*(X))$, for X a connected, finite-dimensional G-CW-complex with cellular chain complex $C_*(X)$.

The following is a summary of some results about $e_G(X)$:

(1) $e_G(X)=|G| \Leftrightarrow X$ is free.

(2) $e_G(X)$ is determined on the singular set.

(3) $e_G(X)\,|\,[G:G_x]$ for all isotropy subgroups in G. In particular $e_G(X)=1$ if G has a fixed point.

(4) If X is admissible, $e_G(X)\,|\,\chi(X)$.

(5) If X satisfies Poincaré duality and the orientation class in $H_n(X)$ is preserved by G, then $e_G(X)=|G|/\exp \operatorname{im} j^*$, where $j^*: \hat{H}^n_G(X)\to \hat{H}^0(G; H^n(X))$.

(6) If X and $G\cong(\mathbb{Z}/p)^r$ satisfy the conditions in (5), and X is a homology manifold, then $e_G(X)=[G:G_x]$ where G_x is an isotropy subgroup of maximal rank in G.

The results above are similar to those obtained by Gottlieb [11] for G-manifolds. However, they differ in that many of them hold for connected finite-dimensional G-CW-complexes and in the fact that our invariant clearly distinguishes algebraic properties from the geometric ones. Result (6) is an immediate consequence of a theorem due to Browder [5] which was the starting point to this approach.

We now move in a different direction by considering the (co)homology modules of an action with coefficients in $\mathbb{F}_p$. The next inequality is the equivalent of Theorem 2.2 in this setting:

Lemma 4.1. *Let C^* be a connected, finite-dimensional $\mathbb{F}_pG$-cochain complex; then*

$$\dim_{\mathbb{F}_p} \hat{H}^{K+1}(G;\mathbb{F}_p)\leq \sum_{r\geq 1} \dim_{\mathbb{F}_p} \hat{H}^{K-r}(G; H^r(C))+\dim_{\mathbb{F}_p} \hat{H}^{K+1}(G; C^*)$$

for all values of $K\in\mathbb{Z}$.

This inequality was first proved by Heller [12] in the free case, and it yields restrictions different to those from $e_G(C_*)$. It provides a global bound for the free rank of symmetry and it implies that $(\mathbb{Z}/p)^3$ cannot act freely on $S^n\times S^m$, $n\neq m$. Our goal is to analyze this result asymptotically.

If M is a finitely generated $\mathbb{F}_pG$-module, then the growth rate of its minimal projective resolution is well defined, and known as the complexity $\operatorname{cx}_G(M)$ of M (Definition 4.3).

Applying this invariant and the limit form of Lemma 4.1 to a connected G-CW-complex yields

Corollary 4.5. *Let X be a connected, finite-dimensional G-CW-complex such that $\mathrm{cx}_G(H^i(X; \mathbb{F}_p))$ is less than the p-rank of G for all $i>0$. Then there exists a p-elementary abelian subgroup $E \subset G$ with maximum rank and such that $X^E \neq \emptyset$.*

In particular, we obtain a condition for realizing $\mathbb{F}_pG$-modules in the sense of Steenrod, given isotropy subgroups of rank lower than that of G.

Corollary 4.6. *If an $\mathbb{F}_pG$-module M can be realized* (mod p) *on X with isotropy subgroups of p-rank less than that of G, then*

$$\mathrm{cx}_G(M) = p\text{-rank of } G.$$

For elementary abelian groups, Kroll [13] has characterized complexity in terms of certain subgroups of units in the group algebra, provided the ground field is algebraically closed. These are called 'shifted subgroups', (see Definition 5.1) and we use them to extend results about free $(\mathbb{Z}/p)^n$-CW-complexes to arbitrary ones.

Theorem 5.4. *Let $G=(\mathbb{Z}/p)^n$ and X a G-CW-complex such that $d=\max\{\mathrm{rk}\, G_\sigma \mid G_\sigma$ isotropy subgroup$\}$. Then there exists a shifted subgroup $S \subset KG$ of rank $n-d$ such that $C_*(X; K)$ is a free KS-chain complex. S has maximal rank for shifted subgroups with this property.*

The following are some applications of this result.

We denote by X a finite-dimensional G-CW-complex, $G \cong (\mathbb{Z}/p)^n$ with an isotropy subgroup of maximal rank d.

(1) If $X \simeq (S^q)^r$ and the action is trivial in homology, then $n-d \leq r$.

(2) If $X \sim S^q \times S^r$, then $n-d \leq 2$.

(3) There exists a shifted subgroup $S \subset KE$, where K is an algebraically closed field of characteristic p, of rank $n-d$ and such that

$$\dim_K \hat{H}^{k+1}(S; K) \leq \sum_{r=1} \dim_K \hat{H}^{k-r}(S; H^r(X; K))$$

for all $k \in \mathbb{Z}$.

(4) If $p=2$, $n-d \leq 4$ and X is finite, then

$$2^{n-d} \leq \sum_{i \geq 0} \dim_{\mathbb{F}_2} H_i(X; \mathbb{F}_2).$$

(5) If M is realized on $X \bmod p$ in dimension m, then

$$\hat{H}^*(S, M \otimes K) \cong \hat{H}^{*+m+1}(S, M \otimes K)$$

for S a shifted subgroup of rank $n-d$ in KG, K an algebraically closed field of characteristic p.

The proof of these results is based on choosing a shifted subgroup of maximal rank which acts freely at the chain complex level using coefficients in K. This involves looking at the 'isotropy variety' of X, which we define in a way analogous to the rank variety of a module (see [7]). The crucial point is that although X may not be finite, it does have a finite number of distinct isotropy subgroups.

The paper is organized as follows: in Sections 1–3 we define and develop the properties of the exponent; in Sections 4 and 5 we consider actions using field coefficients, applying complexity and shifted subgroups; and finally in Section 6 we conclude by comparing the two approaches previously described.

1. Definition and properties of the exponent

Let G be a finite group.

Definition 1.1. A *complete resolution* is an acyclic complex $F_* = (F_i)_{i \in \mathbb{Z}}$ of projective $\mathbb{Z}G$-modules, together with a map $\varepsilon : F_0 \to \mathbb{Z}$ such that $\varepsilon : F_*^+ \to \mathbb{Z}$ is a resolution in the usual sense, $F_*^+ = (F_i)_{i \geq 0}$.

Now let C_*, be a finite-dimensional G-chain complex (graded through any *finite* range of dimensions). As a natural extension of Tate homology with coefficients in a module, we give the following definition:

Definition 1.2.

$$\hat{H}_k(G; C_*) = H_k\left(F_* \underset{G}{\otimes} C_*\right). \tag{1}$$

This is called the *Tate homology of G with coefficients in C_**.

These groups are well defined up to canonical isomorphism and the usual properties of Tate homology extend to them (e.g. long exact coefficient sequences, restriction and corestriction maps, cup products, etc.). We refer to [6] for more details.

Definition 1.3. A G-chain complex C_i is *connected* if $C_i = 0$, $i < 0$ and $H_0(C_*) \cong \mathbb{Z}$, the trivial G-module.

Throughout this section, we will only consider connected, *finite-dimensional* G-chain complexes.

Definition 1.4. The *G-exponent of C_**, denoted $e_G(C_*)$ is defined as $e_G(C_*) =$

$|G|/\exp \operatorname{im} \varepsilon_*$, where $\varepsilon_* : \hat{H}_{-1}(G; C_*) \to \hat{H}_{-1}(G; \mathbb{Z})$ is induced by the augmentation $C_* \xrightarrow{\varepsilon} \mathbb{Z}$ in Tate homology.

Recall that $\hat{H}_{-1}(G; \mathbb{Z}) \cong \mathbb{Z}/|G|$, hence $e_G(C_*) = \exp(\operatorname{coker} \varepsilon_*)$.

Definition 1.5. Let $C_* \xrightarrow{\varepsilon_C} \mathbb{Z}$, $D_* \xrightarrow{\varepsilon_D} \mathbb{Z}$ be connected G-chain complexes. We call $\eta : C_* \to D_*$ a *map of connected G-chain complexes* if it is a G-map which preserves the augmentation, i.e.

$$\varepsilon_D \circ \eta = \varepsilon_C .$$

We will now describe some properties of the exponent with respect to chain maps and subgroups.

Proposition 1.6. *Let $H \subset G$ be a subgroup, C_* a connected G-chain complex, D_* a connected H-chain complex and $\phi : C_* \to D_*$ a map of connected H-chain complexes. Then*

$$e_H(D_*) \mid e_G(C_*).$$

Proof. We have a commutative diagram

$$\begin{array}{ccc} \hat{H}_{-1}(H; D_*) & \xleftarrow{\phi_* \circ \operatorname{tr}} & \hat{H}_{-1}(G; C_*) \\ \Big\downarrow{\scriptstyle (\varepsilon_D)_*} & & \Big\downarrow{\scriptstyle (\varepsilon_C)_*} \\ \hat{H}_{-1}(H; \mathbb{Z}) & \xleftarrow[\operatorname{tr}]{} & \hat{H}_{-1}(G; \mathbb{Z}) \end{array}$$

This is induced by the naturality of the transfer. The bottom map is just the epimorphism

$$\mathbb{Z}/|G| \twoheadrightarrow \mathbb{Z}/|H|.$$

Hence

$$\exp \operatorname{im}(\varepsilon_C)_* / [G : H] \mid \exp \operatorname{im}(\varepsilon_D)_* .$$

This implies

$$e_H(D_*) \mid |H|[G : H]/\exp \operatorname{im}(\varepsilon_C)_* = e_G(C_*). \quad \square$$

Corollary 1.7. *Let $H \subset G$ be a subgroup and $\phi_* : C_* \to D_*$ a map of connected G-chain complexes. Then*

$$e_H(C_*) \mid e_G(C_*) \quad \textit{and} \quad e_G(D_*) \mid e_G(C_*).$$

Proof. Follows from Proposition 1.6 with $\phi = \operatorname{id}$ and $H = G$ respectively. $\square$

Proposition 1.8. *Let C_*, D_* be connected G-chain complexes and suppose there exist G-equivariant maps of augmented chain complexes*

$$\eta_1 : C_* \to D_*, \qquad \eta_2 : D_* \to C_*.$$

Then

$$\exp_G(C_*) = \exp_G(D_*).$$

Proof. By the hypothesis, Corollary 1.7 implies

$$e_G(C_*) \mid e_G(D_*) \mid e_G(C_*).$$

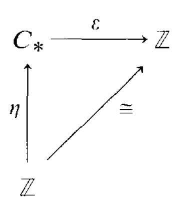
It follows that

$$e_G(C_*) = e_G(D_*). \qquad \square$$

In particular this implies that the exponent is invariant under equivariant retraction.

Corollary 1.9. *If there is an augmented G-chain map $\mathbb{Z} \xrightarrow{\eta} C_*$, then $e_G(C_*) = 1$.*

Proof. We have a commutative diagram



Hence ε_* is an epimorphism, so we obtain

$$e_G(C_*) = 1. \qquad \square$$

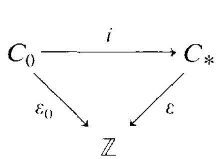
Finally to conclude this section we prove the following relationship between $e_G(C_*)$ and the module map $C_0 \to \mathbb{Z}$:

Proposition 1.10. *Let C_* be a connected G-chain complex. Then*

$$e_G(C_*) \mid |G| / \exp \operatorname{im}(\varepsilon_0)_*$$

where $\varepsilon_0 : C_0 \to \mathbb{Z}$.

Proof. We have a commutative diagram of G-chain complexes



This induces

$$\begin{array}{ccc} \hat{H}_{-1}(G; C_0) & \xrightarrow{\quad i_* \quad} & \hat{H}_{-1}(G; C_*) \\ & {\scriptstyle (\varepsilon_0)_*}\searrow \quad \swarrow{\scriptstyle \varepsilon_*} & \\ & \hat{H}_{-1}(G; \mathbb{Z}) & \end{array}$$

Hence $\exp \operatorname{im}(\varepsilon_0)_* \mid \exp \operatorname{im} \varepsilon_* \Rightarrow e_G(C_*) \mid |G| / \exp \operatorname{im}(\varepsilon_0)_*$. □

2. Exponent and spectral sequences

Let F_* be a complete resolution of the trivial G-module $\mathbb{Z}$. Then if C_* is a finite-dimensional G-chain complex, $F_* \otimes_G C_*$ is the total complex of the double complex of abelian groups $(F_p \otimes_G C_q)$. Hence we have two spectral sequences associated to it (see [6, VII, §3]):

$$E^2_{p,q} = \hat{H}_p(G; H_q(C_*)) \quad \Rightarrow \quad \hat{H}_{p+q}(G; C_*), \tag{1(A)}$$

$$E^1_{p,q} = \hat{H}_q(G; C_p) \quad \Rightarrow \quad \hat{H}_{p+q}(G; C_*). \tag{1(B)}$$

Remark. As before we will assume all our chain complexes are finite-dimensional.

The spectral sequences can be used to estimate the exponent by the following lemma:

Lemma 2.1. *Let $C_* \xrightarrow{\varepsilon} \mathbb{Z}$ be a connected G-chain complex. Then, in the spectral sequence* 1(A),

$$E^{\infty}_{-1,0} = \operatorname{im} \varepsilon_*.$$

Proof. This group is just the image of the edge homomorphisms

$$\hat{H}_{-1}(G; C_*) \twoheadrightarrow E^{\infty}_{-1,0} \subset \hat{H}_{-1}(G; H_0(C_*))$$

which can be verified to be the map induced by the augmentation. □

Hence we have that

$$e_G(C_*) = |G| / \exp E^{\infty}_{-1,0}.$$

This characterization leads to the following theorem relating the exponent with the homology groups considered as G-modules:

Theorem 2.2. *If $C_* \to \mathbb{Z}$ is a connected G-chain complex, then*

$$e_G(C_*) \,\Big|\, \prod_{i=1}^{\infty} \exp \hat{H}^{i+1}(G; H_i(C_*)).$$

Proof. We estimate $\exp E^{\infty}_{-1,0}$ in 1(A). This group is computed by taking successive kernels

$$E^r_{-1,0} \to E^r_{-1-r,r-1},$$

i.e. the sequences

$$E^{r+1}_{-1,0} \to E^r_{-1,0} \xrightarrow{d_r} E^r_{-1-r,r-1}$$

are exact. It follows that

$$(\exp E^r_{-1,0}/\exp E^{r+1}_{-1,0}) \mid \exp E^r_{-1-r,r-1} \quad \text{for all } r \geq 2.$$

Taking the product,

$$\prod_{r=2}^{\infty} (\exp E^r_{-1,0}/\exp E^{r+1}_{-1,0}) \Big| \prod_{r=2}^{\infty} \exp E^r_{-1-r,r-1}.$$

The term on the left is just $\exp E^2_{-1,0}/\exp E^{\infty}_{-1,0}$ which by the lemma is

$$|G|/\exp \operatorname{im} \varepsilon_* = e_G(C_*).$$

The groups $E^r_{-1-r,r-1}$ are subquotients of $E^2_{-1-r,r-1}$, hence their exponents divide those of the latter, which are just $\exp \hat{H}_{-1-r}(G; H_{r-1}(C_*))$. It follows that

$$e_G(C_*) \Big| \prod_{r=2}^{\infty} \exp \hat{H}_{-1-r}(G; H_{r-1}(C)).$$

Adjusting indices and using the isomorphism $\hat{H}_i \cong \hat{H}^{-1-i}$, we obtain (1). □

Corollary 2.3 (Browder [4]). *If in addition to the hypotheses of Theorem 2.2 we add that the C_i are G-cohomologically trivial for all i, then*

$$|G| \Big| \prod_{i=1}^{\infty} \exp \hat{H}^{i+1}(G; H_i(C_*)).$$

This is a consequence of the following lemma:

Lemma 2.4. *If C_* is a G-chain complex such that each chain group C_i is G-cohomologically trivial, then*

$$\hat{H}_*(G; C_*) \equiv 0.$$

Proof. We have $E^2_{p,q} \equiv 0 \equiv E^{\infty}_{p,q}$, hence the abutment must be 0 too. □

As another application of the spectral sequences, we have the following result:

Theorem 2.5. *If C_* and D_* are two connected G-chain complexes and $C_* \xrightarrow{\phi} D_*$ is a G-chain map such that $\hat{H}_*(G; H_k(C)) \xrightarrow{\phi_*} \hat{H}_*(G; H_k(D))$ is an isomorphism for all k, then*

$$e_G(C_*) = e_G(D_*).$$

Proof. ϕ induces a map of spectral sequences which is an isomorphism at the E^2 level

$$E^2_{p,q}(C_*) = \hat{H}_p(G; H_q(C_*)) \xrightarrow[\phi_*]{\cong} \hat{H}_p(G; H_q(D_*)) = E^2_{p,q}(D_*).$$

Hence $\hat{H}_*(G; C_*) \xrightarrow{\phi_*} \hat{H}_*(G; D_*)$ must also be an isomorphism, as these are the respective abutment terms.

There is a commutative diagram

$$\begin{array}{ccc} \hat{H}_{-1}(G; C_*) & \xrightarrow{\cong} & \hat{H}_{-1}(G; D_*) \\ \downarrow \varepsilon_C & & \downarrow \varepsilon_D \\ \hat{H}_{-1}(G; \mathbb{Z}) & \xrightarrow[\cong]{} & \hat{H}_{-1}(G; \mathbb{Z}) \end{array}$$

It follows that

$$e_G(C_*) = e_G(D_*). \qquad \square$$

Corollary 2.6. *If C_*, D_* are connected, weakly equivalent G-chain complexes, then $e_G(C_*) = e_G(D_*)$.* $\square$

3. *G*-CW-complexes and exponents

Consider a finite-dimensional connected G-CW-complex X. Then its cellular chain-complex $C_*(X)$ is finite-dimensional and connected; hence the following definition makes sense:

Definition 3.1.

$$e_G(X) = e_G(C_*(X)).$$

We can apply the algebraic results of Section 2 to G-CW-complexes. In this way the divisibility properties of the exponent give certain restrictions on the type of action. For example Theorem 2.2 becomes

Proposition 3.2. *Let G act cellularly on a finite-dimensional connected* CW-*complex X. Then*

$$e_G(X) \,\Big|\, \prod_{i=1}^{\infty} \exp \hat{H}^{i+1}(G; H_i(X)). \qquad \square$$

The strongest form of this proposition is when the action is free. In [4] this was applied to obtain many restrictions on G-actions; similar results follow from the above.

The fact that $C_*(X)$ is the cellular chain complex of a G-space implies that the chain groups are all direct sums of signed permutation modules

$$\mathbb{Z}G \underset{\mathbb{Z}G_\sigma}{\otimes} \mathbb{Z}_\sigma,$$

G_σ the isotropy subgroup of a cell σ, $\mathbb{Z}_\sigma$ a copy of $\mathbb{Z}$ on which G_σ acts by the 'orientation character' $G_\sigma \to \mathbb{Z}/2$. (See [6, p. 68].)

For the 0-cells, the orientation characters are all trivial. Therefore

$$C_0(X) \cong \mathbb{Z}G \underset{\mathbb{Z}G_\sigma}{\otimes} \mathbb{Z}$$

and $\varepsilon_0 : C_0(X) \to \mathbb{Z}$ is induced by the unique map to a point. Algebraically, it is the usual augmentation map.

Denote $(\varepsilon_0)_* : \hat{H}_{-1}(G; C_0) \to \hat{H}_{-1}(G; \mathbb{Z})$.

Lemma 3.3. *Let X be a finite-dimensional connected G-CW-complex. Then*

$$\exp \operatorname{im}(\varepsilon_0)_* = \text{l.c.m.}\{|G_\sigma| \mid G_\sigma \textit{ isotropy subgroup of a } 0\textit{-cell}\}.$$

Proof.

$$\varepsilon_0 : \bigoplus_\sigma \left(\mathbb{Z}G \underset{\mathbb{Z}G_\sigma}{\otimes} \mathbb{Z}\right) \to \mathbb{Z}.$$

By Shapiro's lemma

$$\hat{H}_{-1}(G; C_0) = \hat{H}_{-1}\left(G; \bigoplus \left(\mathbb{Z}G \underset{G_\sigma}{\otimes} \mathbb{Z}\right)\right) \cong \bigoplus_\sigma \hat{H}_{-1}(G_\sigma; \mathbb{Z}).$$

The map $(\varepsilon_0)_*$ is just $\oplus_\sigma (\operatorname{res}^G_{G_\sigma})_{-1}$. For every σ, $\operatorname{im}(\operatorname{res}^G_{G_\sigma})_{-1}$ is the subgroup generated by $[G : G_\sigma] \cdot x$ in $\mathbb{Z}/|G| = \langle x \rangle$. Therefore $|G_\sigma| \mid \exp \operatorname{im}(\varepsilon_0)_*$ for all σ, so

$$\text{l.c.m.}\{|G_\sigma|\} \mid \exp \operatorname{im}(\varepsilon_0)_*$$

and the converse divisibility is obvious. □

Given a G-CW-complex X, we can G-subdivide to obtain a different cellular structure. There is clearly a chain map between these complexes which is a weak equivalence. Hence by Corollary 2.6, G-subdivision does not alter $e_G(X)$.

Suppose now that G is a p-group, p prime. Then each G_σ maps σ to itself; hence by a theorem of Smith, $\sigma^{G_\sigma} \neq \emptyset$. From this we can subdivide X so that all the G_σ appear as isotropy subgroups of vertices (0-cells). Therefore, combining the preceding lemma with the above remark and Proposition 1.10 gives

Proposition 3.4. *If G is a p-group, X a connected G-CW-complex, then*

$$e_G(X) \mid [G:G_\sigma] \quad \forall G_\sigma \textit{ isotropy subgroup.} \qquad \square$$

This can be applied to arbitrary groups when the action is free, to obtain the next result.

Proposition 3.5. *X is a free G-CW-complex (connected) if and only if $e_G(X)=|G|$.*

Proof. If X is G-free, then $\hat{H}_*(G;C_*(X))=0$ (Lemma 2.4) and so $e_G(X)=|G|$.
Now suppose $e_G(X)=|G|$; for every $p \mid |G|$, let P be a p-subgroup of G.

$$\begin{array}{ccc} \hat{H}_{-1}(G;C_*) & \xleftarrow{\text{res}} & \hat{H}_{-1}(P;C_*) \\ \downarrow \varepsilon_*^G & & \downarrow \varepsilon_*^P \\ \hat{H}_{-1}(G;\mathbb{Z}) & \xleftarrow{\text{res}} & \hat{H}_{-1}(P;\mathbb{Z}) \end{array}$$

commutes, and by hypothesis $\operatorname{im} \varepsilon_*^G=0$. Therefore, $\text{res}\circ\varepsilon_*^P=0$. However, the bottom map is the monomorphism $\mathbb{Z}/|P| \to \mathbb{Z}/|G|$, so $\varepsilon_*^P=0$. It follows that $e_P(X)=|P|$. By Proposition 3.4, P acts freely on X for all P, hence G acts freely on X. $\square$

It the action is free above dimension 0, then $C_0(X)$ determines the exponent:

Proposition 3.6. *Let X be a connected G-CW-complex with cells freely permuted above dimension* 0; *then*

$$e_G(X)=|G|/\textit{l.c.m.}\{|G_\sigma| \mid G_\sigma \textit{ isotropy subgroup}\}.$$

Proof. $0 \to C_0(X) \to C_*(X) \to C_*^+(X) \to 0$ is exact and $C_*^+(X)$ is G-free. Therefore, $i_*:\hat{H}_{-1}(G;C_0(X)) \to \hat{H}_{-1}(G;C_*(X))$ is iso. This implies $\exp \operatorname{im} \varepsilon_* = \exp \operatorname{im} \varepsilon_{0*}$. Lemma 3.3 then implies the result. $\square$

Corollary 3.7. *If G is a p-group in the conditions of Proposition* 3.6, *then*

$$e_G(X)=|G|/\max\{|H_i| \mid H_i \textit{ isotropy subgroup}\}.$$

Proof. If G is a p-group, then

$$\text{l.c.m.}\{|H_i|\}=\max\{|H_i|\}. \qquad \square$$

Example 3.8. Let G be a p-group acting cellularly on M^2, a Riemann surface, preserving orientation. The singular points are discrete, and we may apply Proposition 3.6:

$$e_G(M^2)=|G|/\max\{|H_\sigma| \mid H_\sigma \text{ isotropy subgroup}\}.$$

Hence

$$|G|/\max\{|H_\sigma|\} \mid \exp H^2(G; H_1(M)) \cdot \exp H^3(G; \mathbb{Z}).$$

It can be shown that the H_σ are cyclic. Hence if $G=(\mathbb{Z}/p)^l$, this reduces to

$$p^{l-2} \mid \exp H^2(G; H_1(M)).$$

Intuitively, if free G-cells are attached to a G-CW-complex, the action is not fundamentally altered. More precisely, we have

Proposition 3.9. *Let X be a finite-dimensional G-CW-complex; Y a connected subcomplex such that $G_\sigma = 1$ for every cell σ of X not in Y. Then*

$$e_G(X) = e_G(Y).$$

Proof. There is a short exact sequence of (cellular) G-chain complexes

$$0 \to C_*(Y) \to C_*(X) \to C_*(X)/C_*(Y) \to 0.$$

By hypothesis $C_*(X)/C_*(Y)$ is G-free, hence an application of Lemma 2.4 proves the result. □

Therefore, $e_G(X)$ is determined on the singular set of the action.

Given a G-CW-complex X, we may ask what the relationship is between $e_G(X)$ and $\chi(X)$. If the action is free and X is finite-dimensional, with $H_*(X)$ finitely generated, then $e_G(X) = |G| \mid \chi(X)$. We shall prove this for non-free actions given a certain restriction on X.

Definition 3.10. *A G-CW-complex X is said to be* admissible *if for any cell $\sigma \subset X$, G_σ fixes σ point-wise.*

Proposition 3.11. *If X is a finite-dimensional connected admissible G-CW-complex with $H_*(X)$ finitely generated, then*

$$e_G(X) \mid \chi(X).$$

Proof. As G is finite, there are only a finite number of permutation modules which constitute the chain groups $C_i(X)$. We can subdivide X so that all the isotropy subgroups appear in $C_0(X)$ without affecting our hypothesis.

As in Proposition 3.4

$$e_G(X) \mid [G:G_\sigma] \quad \text{for all cells } \sigma \text{ in } X.$$

Then $e_G(X)$ is an integer which divides the order of every orbit; with our hypotheses this implies that it must divide the Euler characteristic of X (see [6, p. 259]). □

Remark. This result is obvious for finite complexes because we can obtain the Euler characteristic by counting cells.

Suppose X is a finite-dimensional CW-complex satisfying Poincaré duality with an orientation-preserving cellular G-action.

We shall prove that there is a duality isomorphism between the spectral sequences associated to $C_*(X)$ and $C^*(X)=\mathrm{Hom}_{\mathbb{Z}}(C_*(X),\mathbb{Z})$ with a certain shift in dimensions. This will show an isomorphism of the abutments and provide an alternative description of the exponent using the orientation class.

Theorem 3.12. *Let X be a finite-dimensional* CW*-complex satisfying Poincaré duality, with a cellular G-action which is trivial on a chain representative of the fundamental class. Then there is a natural duality isomorphism*

$$\hat{H}^p(G;C^*(X))\cong\hat{H}_{n-p-1}(G;C_*(X))$$

where the top non-zero homology class of X lies in dimension n.

Proof. There is a spectral sequence

$$E_2^{p,q}=\hat{H}^p(G;H^q(X)) \quad\Rightarrow\quad \hat{H}^{p+q}(G;C^*(X))$$

analogous to the one we used for $C_*(X)$. We show that these two spectral sequences are isomorphic at the E_2-level.

Let P be a finitely generated projective $\mathbb{Z}G$-module, and denote $\bar{P}=\mathrm{Hom}_{\mathbb{Z}G}(P,\mathbb{Z}G)$. Then we have an isomorphism for any $\mathbb{Z}G$-module M:

$$\bar{P}\underset{\mathbb{Z}G}{\otimes}M\cong\mathrm{Hom}_{\mathbb{Z}G}(P,M),$$

given by $\phi(u\otimes m)(x)=u(x)\cdot m$.

Now let F_* be a complete G-resolution of $\mathbb{Z}$, C^* a finite-dimensional G-cochain complex. As F_* can be taken of finite type, we can extend this duality isomorphism to obtain

$$\bar{F}_*\underset{\mathbb{Z}G}{\otimes}C^*\cong\mathrm{Hom}_{\mathbb{Z}G}(F_*,C^*).$$

Hence if we consider $\bar{F}_*,C^*$ as negatively graded G-chain *complexes*, i.e.

$$(\tilde{C})_K=C^{-K},\qquad(\tilde{\bar{F}})_K=(\bar{F})_{-K},$$

we obtain

$$H_{-r}\left(\tilde{\bar{F}}_*\underset{\mathbb{Z}G}{\otimes}\tilde{C}_*\right)\cong\hat{H}^r(G;C^*).$$

Except for indexing, $\tilde{\bar{F}}_*$ is a complete resolution; it is shifted by one dimension:

$$H_{-r}\left(\tilde{\bar{F}}_*\underset{\mathbb{Z}G}{\otimes}\tilde{C}_*\right)\cong\hat{H}_{-r-1}(G;\tilde{C}_*).$$

Therefore, $\hat{H}^r(G; C^*) \cong \hat{H}_{-r-1}(G; \tilde{C}_*)$. This is the chain complex version of Tate duality, and by its definition this isomorphism induces an isomorphism of the associated spectral sequences.

Now let C_* be the chain complex of X, and C^* its dual. By our hypotheses there is an isomorphism

$$H^k(C^*) \to H_{n-k}(C_*)$$

given by $C \to [X] \cap C$, where $[X] \in H_n(X)$ is the fundamental class of the complex. The important fact about this isomorphism is that it is induced in homology by a map of chain complexes

$$(\tilde{C})_k \xrightarrow{v \cap -} C_{n+k}, \quad [v] = [X], \ k \leq 0$$

where, as before $(\tilde{C})_k = C^{-k}$. More explicitly, we have that

$$v \cap z = (\Delta v)/z, \quad \Delta : C \to C \otimes C$$

the diagonal. (We refer to [3, Chapter 1] for details.) It is not hard to show that if G acts trivially on v, then $v \cap -$ is a G-chain map, inducing a weak equivalence. Then, as in the proof of Theorem 2.5, it induces an isomorphism of spectral sequences at the E^2-level

$$\hat{H}_p(G; H_q(\tilde{C}_*(X))) = E^2_{p,q} \cong E^2_{p,n+q} = \hat{H}_p(G; H_{n+q}(C_*(X))), \quad q \leq 0.$$

We have shown that there is an isomorphism of spectral sequences (at the E_2-level)

$$\hat{H}^p(G; H^q(C^*)) = E_2^{p,q} \cong E^2_{-p-1,n-q} \cong \hat{H}_{-p-1}(G; H_{n-q}(C_*)). \qquad \square$$

Corollary 3.13.

$$E_\infty^{0,n} \cong E^\infty_{-1,0}, \qquad E_\infty^{0,0} \cong E^\infty_{-1,n},$$

$$E^\infty_{-1,0} = \operatorname{im} \varepsilon_*, \qquad E_\infty^{0,0} = \operatorname{im} \varepsilon^*,$$

$$E_\infty^{0,n} = \operatorname{im} j^*, \qquad E^\infty_{-1,n} = \operatorname{im} j_*$$

where

$$j_* : \hat{H}_{-1}(G; H_n(X)) \to \hat{H}_{-1-n}(G; C_*(X)),$$

$$j^* : \hat{H}^n(G; C^*(X)) \to \hat{H}^0(G; H^n(X)). \qquad \square$$

Therefore

$$e_G(X) = |G| / \exp \operatorname{im} j^*.$$

Suppose that X has no cells above the dimension of its top homology class. Then we have a map of G-cochain complexes

$$C^*(X) \to H^n(X)$$

where $H^n(X)$ is concentrated in dimension n as a cochain complex. Hence it induces

$$\hat{H}^n(G; C^*(X)) \to \hat{H}^0(G; H^n(X)).$$

This map is the homomorphism j^* (similarly for j_*)

For example, if M^n is a homology n-manifold with a G-CW-complex structure which preserves orientation, then we may compute the exponent using j^* in some cases. For example,

Proposition 3.14. *If the hypotheses of Theorem* 3.12 *hold, and G is elementary abelian, then*

$$e_G(M) = [G : G_\sigma]$$

where G_σ is an isotropy subgroup of maximal rank.

Proof. The proof is based on a theorem due to Browder [5], which asserts that $|G|/\exp \operatorname{im} j^* = [G : G_\sigma]$, G_σ an isotropy subgroup of maximal rank. □

Corollary 3.15. *If $G \cong (\mathbb{Z}/p)^n$ acts on a homology manifold satisfying the hypothesis of Theorem* 3.12, *with $r = \max\{\operatorname{rk} G_x\}$, and trivial action in homology, then*

$n - r \leq$ *number of non-zero, reduced homology groups of M with $\mathbb{Z}_{(p)}$ coefficients.* □

For example, if $M \sim (S^m)^K$ in Corollary 3.15, then $n - r \leq K$ (see [4]).

4. *KG*-chain complexes

Let K be a field of characteristic $p > 0$, G a finite group. Then for KG-chain complexes, the exponent is meaningless. However, over K we can use dimension instead of exponent. In this framework we obtain the following inequality for KG-cochain complexes:

Lemma 4.1. *Let C^* be a finite-dimensional KG-cochain complex, with $C^i = 0$ for $i < 0$, and $i > N$. Then*

$$\dim_K \hat{H}^{k+1}(G; H^0(C)) \leq \sum_{r=1}^{N} \dim_K \hat{H}^{k-r}(G; H^r(C)) + \dim_K \hat{H}^{k+1}(G; C^*)$$

for all $k \in \mathbb{Z}$. □

Proof. Consider the spectral sequence (see Section 3)

$$E_2^{p,q} = \hat{H}^p(G; H^q(C)) \Rightarrow \hat{H}^{p+q}(G; C^*).$$

Look at the $E_r^{k,0}$ term; no non-trivial differentials originate from it and hence

$$E_r^{k-r,r-1} \to E_r^{k,0} \to E_{r+1}^{k,0} \to 0$$

is exact. We obtain

$$\dim_K E_r^{k,0} - \dim_K E_{r+1}^{k,0} \leq \dim_K E_r^{k-r,r-1} \quad \text{for all } k.$$

Adding these for $r=2,\dots,N$ yields

$$\dim_K E_2^{k,0} - \dim_K E_\infty^{k,0} \leq \sum_{r=2}^{N} \dim_K E_r^{k-r,r-1}.$$

This implies

$$\dim_K H^k(G; H^0(C)) \leq \sum_{r=2}^{\infty} \dim_K \hat{H}^{k-r}(G; H^{r-1}(C)) + \dim E_\infty^{k,0}.$$

Now $E^{k,0}$ is the image of the edge homomorphism, and so $\dim_K E_\infty^{k,0} \leq \dim_K \hat{H}^k(G; C^*)$. Applying this and adjusting indices, completes the proof. □

Corollary 4.2 (Heller [12]). *If C^* is a free, connected, finite-dimensional KG-chain complex, then*

$$\dim_K \hat{H}^{k+1}(G; K) \leq \sum_{r=1}^{\infty} \dim_K \hat{H}^{k-r}(G; H^r(C)).$$

Proof. In this case $\hat{H}^*(G; C^*) \equiv 0$. □

Corollary 4.2 has several applications: we mention two which are of some interest.

(i) Let C^* be a connected, finite-dimensional, free $K(\mathbb{Z}/p)^n$-cochain complex, char $K=p$, and suppose that $H^i(C) \cong K$, the trivial module, for three distinct non-zero values of i, and vanishes in all other positive dimensions.

Using the well-known formula for $\dim_K \hat{H}^i((\mathbb{Z}/p)^n; K)$ and Theorem 2.2, Heller showed that under the preceding condition, $n \leq 2$ [12]. This shows, for example, that if $(\mathbb{Z}/p)^n$ acts freely on $S^r \times S^m$, then $n \leq 2$. Although this result looks innocent enough, it cannot be shown using the exponent approach in Sections 1–3; the classical cohomological method of Borel does yield a proof of this, but it involves many long and delicate spectral sequence arguments.

(ii) Let X be a finite, connected CW-complex, and define its free p-rank of symmetry as

$$F_p(X) = \max\{\mathrm{rk}(\mathbb{Z}/p)^n \mid (\mathbb{Z}/p)^n \text{ acts freely on } X\}.$$

Then applying Corollary 4.2 to $C^*(X; \mathbb{F}_p)$, we can obtain a global bound on $F_p(X)$:

$$F_p(X) \leq \left(\sqrt{\sum_{i>0} \dim H_i(X; \mathbb{F}_p)}\right)\left(\dim X + \frac{1}{2}\right) - \dim X + \frac{1}{2}.$$

Carlsson [10] has conjectured that in fact

$$F_p(X) \leq \log_2\left(\sum_{i=0} \dim H_i(X; \mathbb{F}_p) \right);$$

for $p=2$ his results [9] imply $F_2(X) \leq \sum_{i>0} \dim H_i(X; \mathbb{F}_2)$.

From now on we will only be concerned with the cellular cochain complex of a finite G-CW-complex X. Note that each cochain group is a direct sum of permutation modules $KG \otimes_{G_\sigma} K$.

The next result is a limit version of Lemma 4.1, and is expressed naturally in terms of the complexity of $H_*(X; K)$. We recall its definition and basic properties.

Definition 4.3 (see [2]). Let M be a finitely generated KG-module; then the complexity of M is

$$\operatorname{cx}_G(M) = \min\left\{ s \geq 0 \,\middle|\, \lim_{n \to \infty} \frac{\dim_K P_n}{n^s} = 0 \right\}$$

where $P_* \to M$ is a minimal projective resolution of M over KG.

The following is a summary of the basic properties of complexity; we refer to [7] for more details:

Properties of complexity. (1) $0 \leq \operatorname{cx}_G(M) \leq \operatorname{cx}_G(K)$.

(2) $\operatorname{cx}_H(M) \leq \operatorname{cx}_G(M)$, $H \subseteq G$ a subgroup.

(3)

$$\operatorname{cx}_G(M) = \max_{\substack{E \subset G \\ \text{elementary} \\ \text{abelian}}} \{\operatorname{cx}_E(M|_E)\} \quad [2].$$

(4) If G is a p-group, then

$$\operatorname{cx}_G(M) = \min\left\{ s \geq 0 \,\middle|\, \lim_{n \to \infty} \frac{\dim H^n(G; M^*)}{n^s} = 0 \right\}.$$

(5) $\operatorname{cx}_G(M) = \operatorname{cx}_G(M^*)$.

Clearly (3)–(5) imply that

$$\operatorname{cx}_G(M) = \max_{E \subset G} \left[\min\left\{ s \geq 0 \,\middle|\, \lim_{n \to \infty} \frac{\dim H^n(E; M)}{n^s} = 0 \right\} \right].$$

This means that complexity is determined by the cohomological growth of the module on elementary abelian subgroups.

For a group G, let $p(G) = \max\{\operatorname{rk} E \mid E \cong (\mathbb{Z}/p)^n, E \subset G\}$ this is known as the p-rank of G.

Theorem 4.4. *Let X be a finite-dimensional G-CW-complex, and let $n = \mathrm{cx}_G(H^0(X; \mathbb{F}_p))$. Then either $\max_{G_\sigma}\{p(G_\sigma)\} \geq n$ or $\max_{i>0}\{\mathrm{cx}_G H^i(X; \mathbb{F}_p)\} \geq n$.*

Proof. Let $E \subset G$ be an elementary abelian p-group, and consider Definition 3.1 for E and $C^*(X; \mathbb{F}_p)$:

$$\dim_{\mathbb{F}_p} \hat{H}^{k+1}(E; H^0(X)) \leq \sum_{r=1}^{\infty} \dim_{\mathbb{F}_p} \hat{H}^{k-r}(E; H^r(X)) + \dim_{\mathbb{F}_p} \hat{H}^{k+1}(E; C^*(X)).$$

Suppose that $\max_{i>0}\{\mathrm{cx}_G H^i(X; \mathbb{F}_p)\} < n$, and choose E so that $\mathrm{cx}_G(H^0(X)) = \mathrm{cx}_E(H^0(X))$. Then the above inequality, which holds for all $k \in \mathbb{Z}$, implies that the growth rate of $\dim_{\mathbb{F}_p} \hat{H}^*(E; C^*(X))$ is at least that of $H^*(E; H^0(X))$, which is n. We now apply a theorem of Quillen [14], which states that the growth of $\dim_{\mathbb{F}_p} H^i(G; C^*(X))$ (as $i \to \infty$) is exactly the maximal p-rank of all the isotropy subgroups. Hence there exists a G_σ of p-rank at least n. □

Corollary 4.5. *Let X be a connected, finite dimensional G-CW-complex such that $\mathrm{cx}_G(H^i(X)) < p$-rank G for all $i > 0$. Then there exists a p-elementary abelian subgroup of maximal rank, E, such that $X^E \neq \emptyset$.*

Proof. In this case $H^0(X) = \mathbb{F}_p$, hence it has the highest possible complexity, the p-rank of G. From this we conclude that there is a p-elementary abelian isotropy subgroup E with this maximal rank. The fact that E is a p-group implies that $X^E \neq \emptyset$. □

The preceding corollary can be applied to a certain version of the Steenrod problem. Given a finite-dimensional G-CW-complex X with $H_n(X; \mathbb{F}_p) \cong M$ as G-modules and $\tilde{H}_i(X; \mathbb{F}_p) = 0$, $i \neq n$, we say that X realizes $M \bmod p$, in dimension n.

Corollary 4.6 (compare with [4, 1.6]). *If an $\mathbb{F}_pG$-module M can be realized* (mod p) *on X with isotropy subgroups of p-rank less than that of G, then*

$$\mathrm{cx}_G(M) = p\text{-rank of } G.$$

Proof. This follows from the fact that if $H_n(X; \mathbb{F}_p) \cong M$, then $H^n(X; \mathbb{F}_p) \cong M^*$, and $\mathrm{cx}_G(M) = \mathrm{cx}_G(M^*)$. □

If an $\mathbb{F}_pG$-module M is realized (mod p) on a free G-CW-complex in dimension n, it is not hard to show that in fact

$$\hat{H}^j(G; M) \cong \hat{H}^{j+n+1}(G; \mathbb{F}_p) \quad \text{for all } j.$$

Consider the spectral sequence $E_2^{p,q} = \hat{H}^p(G; H^q(X))$ converging to $\hat{H}^{p+q}(G; C^*(X))$. At each stage the unique differential must be an isomorphism, because the E_∞ term is zero; this gives the desired isomorphism.

5. Shifted subgroups

In this section we apply shifted subgroups to generalize results about free $(\mathbb{Z}/p)^r$ complexes to arbitrary ones. We refer to [7] for more details.

Let K be an algebraically closed field, $\mathrm{char}(K)=p$ and $E=(\mathbb{Z}/p)^n$. Choose a basis $x_1,\dots,x_n$ for E. Then any $A\in \mathrm{GL}(n,k)$ can be thought of as a transformation

$$\bigoplus_{i=1}^{n} K(x_i-1)\to \bigoplus_{i=1}^{n} K(x_i-1).$$

This extends uniquely to an algebra automorphism $\phi_A : KE\to KE$,

$$\phi_A(x_i)=1+\sum_{j=1}^{n} a_{ji}(x_j-1).$$

Definition 5.1. A *shifted subgroup* $S\subset KE$ *of rank* r, is the image of $\langle x_1,\dots,x_r\rangle$ under ϕ_A for some $A\in \mathrm{GL}(n,k)$.

The notion of rank variety of a module relates complexity to shifted subgroups.

Definition 5.2. Let M be a KE-module, $E=(\mathbb{Z}/p)^n$. Then the *rank variety* $V_E(M)$ *of* M is defined as $V_E(M)=\{(\alpha_1,\dots,\alpha_n)\in K^n \mid u_\alpha=(\sum \alpha_i(x_i-1))+1$ does not act freely on $M\}\cup\{0\}$.

The fundamental result (see [7, Theorem 7.6]) is that $V_E(M)$ is a homogeneous affine variety of dimension $\mathrm{cx}_E(M)$.

Now let X be an E-CW-complex, E as before. We can extend the notion of rank variety to $C_*(X)$:

Definition 5.3. The *isotropy variety of* X is defined as

$$V_E(X)=\{(\alpha_1,\dots,\alpha_n)\in K^n \mid C_*(X;K)|_{u_\alpha} \text{ is not free}\}\cup\{0\}$$

where $u_\alpha=\sum \alpha_i(x_i-1)+1$.

Theorem 5.4. *Let X be an E-CW-complex, $E=(\mathbb{Z}/p)^n$. Then $V_E(X)$ is a homogeneous affine variety such that*

(1) $\dim V_E(X)=\max\{\mathrm{rk}\, E_\sigma \mid E_\sigma\subset E$ *isotropy subgroup*$\}$;

(2) *There exists a shifted subgroup S of rank $n-\dim V_E(X)$, such that $C_*(X;K)|_S$ is free, and S has maximal rank with this property.*

Proof. From the structure of $C_*(X;K)$, it is clear that

$$V_E(X)=\bigcup_{E_\sigma} V_E(K[E/E\sigma])=V_E\Big(\bigoplus_{E_\sigma} K[E/E_\sigma]\Big).$$

Note that there are only a finite number of distinct isotropy subgroups. Hence by the result for modules, $V_E(X)$ is a homogeneous affine variety. Then (1) follows from the fact that $\text{cx}_E(K[E/E_\sigma]) = \text{rk } E_\sigma$.

For (2) we use the second identity above and a result due to Kroll [13] for modules, which in this case is

$$\text{cx}_E(\bigoplus K[E/E_\sigma])$$
$$= \text{rk } E - \max\{\text{rk } S \mid S \text{ shifted subgroup acting freely on } \bigoplus K[E/E_\sigma]\}.$$

We sketch the proof for our situation. As $V_E(X)$ is homogeneous, its image in projective $(n-1)$-space, KP^{n-1}, is a projective variety P, of dimension $q-1$, where $q = \max\{\text{rk } E_\sigma\}$. Then we may choose a projective linear variety $\tilde{W}$ in KP^{n-1} of dimension $n-q-1$ such that $\tilde{W} \cap P = \emptyset$. Now let $W \subset K^n$ be a linear subspace of dimension $n-q$, whose image in KP^{n-1} is $\tilde{W}$. If $\alpha_1, \ldots, \alpha_{n-q}$ is a basis for W, then $u_{\alpha_1}, \ldots, u_{\alpha_{n-q}}$ (notation as before) generate a shifted subgroup S, acting freely on $C_*(X)$. Notice that $\text{rk } S = n-q$, and it clearly has maximal rank with this property. □

Theorem 5.4 allows us to consider arbitrary $(\mathbb{Z}/p)^n$-complexes as free complexes (at the chain level) over a group of rank

$$n - \max_\sigma \{\text{rk } E_\sigma\}.$$

This has many applications, some of which we will now consider.

Proposition 5.5. *Let X be a finite-dimensional connected E-CW-complex, $E = (\mathbb{Z}/p)^n$ and suppose $r = \max\{\text{rk } E_\sigma\}$. Then there exists a shifted subgroup S of rank $n-r$, such that*

$$\dim_K \hat{H}^{k+1}(S; K) \le \sum_{i=1}^{\infty} \dim_K \hat{H}^{k-i}(S; H^i(X; K))$$

for all $k \in \mathbb{Z}$.

Proof. Choose S as in Theorem 5.4, then $C^*(X; K)$ is KS-free, and we apply Corollary 4.2 to obtain Proposition 5.5. □

Just as we applied Corollary 4.2, the same can be done with Proposition 5.5.

(i) If $H^i(X; K) \cong K$ for three distinct positive values of i, and zero elsewhere, then $n-r \le 2$. This implies, for example, that if $(\mathbb{Z}/p)^n$ acts on $S^r \times S^m$, then $n - \max\{\text{rk } E_\sigma\} \le 2$.

(ii) The estimates on the free rank of symmetry can be used to estimate the rank of symmetry with 'small' isotropy subgroups, in the obvious way.

We can generalize a result due to Carlsson [8] in the free case:

Proposition 5.6. *Let* $(\mathbb{Z}/p)^n$ *act on a finite* CW*-complex* X *homotopy equivalent to* $(S^m)^k$, *with trivial action on* $H_*(X;\mathbb{Z})$. *Then, if* $r = \max\{\text{rk } \textit{isotropy subgroups}\}$,

$$n - r \leq k.$$

Proof. Choose S a shifted subgroup of rank $n-r$ such that $C_*(X;K)|_S$ is free. Then we may apply Carlsson's proof. □

This generalization was first proved by Browder, using the exponent approach in Sections 1-3.

Proposition 5.7. *Let* $G = (\mathbb{Z}/p)^n$ *and assume the* $\mathbb{F}_pG$*-module* M *can be realized* (mod p) *on* X *in dimension* m, *with an isotropy subgroup of maximal rank* r. *Then there exists a shifted subgroup* $S \subset KG$ *of rank* $n-r$, *such that*

$$\hat{H}^*(S;K) \cong \hat{H}^{*+m+1}(S; M \otimes K).$$

Proof. Choose S of rank $n-r$ acting freely on $C^*(X;K)$, the rest follows as in the remark after Corollary 4.6. □

Another application of shifted subgroups, is the following conjecture, generalizing the one made by Carlsson [10] for finite free $(\mathbb{Z}/2)^n$-CW-complexes:

Conjecture. Let X be a finite $(\mathbb{Z}/2)^n$-CW-complex with an isotropy subgroup of maximal rank r. Then

$$2^{n-r} \leq \sum_i \dim_k H^i(X;K).$$

In the free case ($r=0$), the conjecture has been proved for $n \leq 4$. Hence using shifted subgroups it follows that the conjecture is true for $n-r \leq 4$.

Carlsson [10] has applied another algebraic invariant to finite-free $\mathbb{F}_2E$-chain complexes, where $E = (\mathbb{Z}/2)^n$: for a KE-module M let $\lambda_E(M)$ be the smallest power of the augmentation ideal which kills M (recall it is nilpotent). This is also known as the Loewy length of M.

Shifted subgroups yield the following generalization of his result:

Proposition 5.8. *Let* X *be a finite* E-CW*-complex, where* $E = (\mathbb{Z}/2)^n$. *Let* K *be an algebraically closed field of characteristic* 2. *Then there exists a shifted subgroup* $S \subset KE$ *of rank* n-$\max\{\text{rk } E_\sigma\} = d$ *such that*

$$\sum_{i>0} \lambda_S(H_i(X,K)) \geq d. \qquad \square$$

In particular this yields another proof of Proposition 5.6 when $p=2$.

6. Conclusions

In the previous sections we have analyzed two different invariants of finite transformation groups. For connected G-chain complexes and orientable G-manifolds, the exponent yields good estimates on the freeness of the action, and a restriction on the G-cohomology of the homology modules. On the other hand, the application of techniques from modular representations yields restrictions in terms of ranks, using coefficients in a field of characteristic dividing $|G|$.

As different as they seem, they nevertheless can be used to prove similar and often complementary results. The inequality in Lemma 4.1 is the analogue of the divisibility relation in Theorem 2.2, and their proofs follow parallel reasoning. However, Theorem 2.2 implies that $(\mathbb{Z}/p)^r$ cannot act freely, with trivial action in homology, on $(S^n)^k$, $k>r$, but not that $(\mathbb{Z}/p)^3$ cannot act freely on $S^n \times S^m$, $n \neq m$. Corollary 4.2 implies the second fact but not the first.

In general, for $(\mathbb{Z}/p)^r$ actions, the exponent approach yields generalizations of results in the free case by estimating $e_G(X)$ in terms of the isotropy subgroups. In particular we have seen that in case X is a homology G-manifold, $e_G(X) = [G:G_x]$, where G_x has maximal rank (Browder's Theorem).

Now taking coefficients in an algebraically closed field of characteristic p, we can recover some of these results by using shifted subgroups. For example Carlsson's Theorem can be generalized by using shifted subgroups and copying the proof for the free case: if $(\mathbb{Z}/p)^r$ acts on $(S^n)^K$, trivially in homology, and $\max\{\mathrm{rk}\, G_x\} = d$, then $r-d \leq K$. This proof does not require Browder's result. For actions on $S^n \times S^m$ we again obtain a stronger result using shifted subgroups: if $(\mathbb{Z}/p)^r$ acts on $S^n \times S^m$, $d = \max\{\mathrm{rk}\, G_x\}$, then $r-d \leq 2$. It seems that away from the free case, the use of shifted subgroups is more powerful, as we can use all the results valid for free $(\mathbb{Z}/p)^q$-chain-complexes.

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