Journal of Pure and Applied Algebra 66 (1990) 111-120
North-Holland

DISCRETE GROUPS WITH LARGE EXPONENTS IN COHOMOLOGY

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Communicated by K.W. Gruenberg
Received 31 October 1989

Let Γ be a discrete group of finite virtual cohomological dimension. The Farrell cohomology of $\Gamma, \hat{H}^*(\Gamma, \mathbb{Z})$, depends on the finite subgroups of Γ, in a manner which is not yet well-understood. In this paper we produce examples of groups with nontrivial finite subgroups of order at most p (a prime) but with arbitrarily high p-torsion in their Farrell cohomology. We describe a method for constructing discrete groups with large exponents starting from a finite group acting freely on an aspherical complex.

1. Introduction

A discrete group Γ is said to have finite virtual cohomological dimension if it has a normal subgroup Γ' of finite cohomological dimension such that $\Gamma/\Gamma' = G$ is a finite group. For such a group we can define its Farrel cohomology groups $\hat{H}^*(\Gamma, \mathbb{Z})$ which are torsion groups and an analogue in the discrete case of the Tate cohomology of finite groups. Like the Tate cohomology, the Farrell cohomology coincides with the ordinary cohomology in large enough degrees, i.e. larger than the cohomological dimension of Γ' (see [3]). It has long been known that the groups $\hat{H}^n(\Gamma, \mathbb{Z})$ depend to some extent on the cohomology of the finite subgroups of Γ. However the dependence is poorly understood, largely because so few examples have been constructed. It had been suggested that the least common multiples of the orders of the finite subgroups of Γ might annihilate the Farrell cohomology. The first counterexample appeared in [2] where it was also shown that the cohomology depends on the embeddings of the finite subgroups in Γ or in G.

In this paper we extend the ideas of [2]. Our main offering is the construction of

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a set of examples which reveal some of the complications arising in the cohomology of infinite discrete groups with interesting topological interpretations. Among other things, they show the following:

Theorem A. *Given any prime p and positive integer t, there exists a group Γ, having finite virtual cohomological dimension, whose finite nontrivial subgroups have order p, and such that*

$$\exp \hat{H}^{2m}(\Gamma, \mathbb{Z}) = \exp H^{2m}(\Gamma, \mathbb{Z}) \geq p^t$$

whenever m is sufficiently large. Here 'exp' *indicates the exponent of the group.*

Geometrically the examples can be produced by the following construction. Consider the standard free $(\mathbb{Z}/p)^{2n}$-action on $X=(S^1)^{2n}$ by rotation on each coordinate. The action can be pulled back to an action of E_n on X where E_n is an extra-special group of order p^{2n+1}. Then the groups $\Gamma_n = \pi_1(X \times_{E_n} EE_n)$ demonstrate the theorem provided n is sufficiently large relative to t. However the proof that we give is by purely algebraic means. The examples were originally constructed by looking at infinite-group analogues of large exponential phenomena in finite groups $(G=\Gamma/\Gamma')$. It is hoped that this sort of construction will have other applications.

In Section 3 we show how the construction can be generalized to geometric situations involving group actions on aspherical complexes. In the last section we show how the methods of equivariant cohomology can be applied and point out some unanswered questions. Huebschmann has pointed out that he has analyzed aspects of the cohomology of Γ_2 for different reasons [6].

2. The examples Γ_n

Throughout this section we fix a prime p. Let $U_{2n}=\mathbb{Z}^{2n}$ be a (multiplicative) free abelian group of rank $2n$. Let N be a cyclic group of order p. Now $H^*(U_{2n}, \mathbb{Z}) = \Lambda_{\mathbb{Z}}(x_1, y_1, \ldots, x_n, y_n)$ a $\mathbb{Z}$-exterior algebra. Its reduction modulo p is $H^*(U_{2n}, \mathbb{Z}/p) = \Lambda_{\mathbb{F}_p}(\bar{x}_1, \bar{y}_1, \ldots, \bar{x}_n, \bar{y}_n)$. Define Γ_n to be the central extension

$$1 \longrightarrow N \longrightarrow \Gamma_n \xrightarrow{\sigma} U_{2n} \longrightarrow 1$$

determined by the extension class $\alpha = \bar{x}_1\bar{y}_1 + \cdots + \bar{x}_n\bar{y}_n \in H^2(U_{2n}, \mathbb{Z}/p)$. Then Γ_n is generated by elements $a_1, b_1, \ldots, a_n, b_n$ such that $[a_i, b_j] = [a_i, a_j] = [b_i, b_j] = 1$ if $i \neq j$ and

$$[a_1, b_1] = \cdots = [a_n, b_n]$$

is a central element of order p, a generator for N. Let $\Gamma_n' = \langle a_i^p, b_i^p \mid i = n, \ldots, n \rangle$ and $U_{2n}' = \{x^p \mid x \in U_{2n}\} = \sigma(\Gamma_n')$. Let $V_{2n} = U_{2n}/U_{2n}' \cong (\mathbb{Z}/p)^{2n}$. Then we have the following commutative diagram of groups:

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \Gamma_n' & = & U_{2n}' & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & N & \longrightarrow & \Gamma_n & \xrightarrow{\sigma} & U_{2n} & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & N & \longrightarrow & E_n & \longrightarrow & V_{2n} & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array} \tag{1}$$

Here E_n is an extra-special group of order p^{2n+1}. It can be checked that if p is odd, then E_n has exponent p. If $p=2$, then E_n is a central product of dihedral groups.

Theorem 2.1. *For any n, Γ_n is a discrete group of virtual cohomology dimension $2n$. Also,*

$$H^*(\Gamma_n,\mathbb{Z})\cong\Lambda_{\mathbb{Z}}(x_1,y_1,\ldots,x_n,y_n)\otimes\mathbb{Z}[\theta]/I,$$

where I is the ideal generated by

$$p(1\otimes\theta)-(x_1y_1+\cdots+x_ny_n)\otimes 1.$$

The exterior algebra $\Lambda_{\mathbb{Z}}(x_1,y_1,\ldots,x_n,y_n)\otimes 1$ is the inflation of $H^*(U_{2n},\mathbb{Z})$, and $\theta\in H^2(\Gamma_n,\mathbb{Z})$ has nonzero restriction to N.

Proof. The first statement follows because Γ_n' is free abelian with cohomological dimension $2n$. Notice that V_{2n} is the abelianization of E_n. So in the diagram

$$\begin{array}{ccc}
H^2(V_{2n},\mathbb{Z}) & \longrightarrow & H^2(E_n,\mathbb{Z}) \\
\downarrow & & \downarrow \\
H^2(U_{2n},\mathbb{Z}) & \longrightarrow & H^2(\Gamma_n,\mathbb{Z})
\end{array}$$

the top map is an isomorphism. This can also be verified from the spectral sequence of the bottom row of (1). Moreover $H^2(V_{2n},\mathbb{Z})$ is finite while $H^2(U_{2n},\mathbb{Z})$ is torsion free. So the vertical maps are zero. Now consider the LHS-spectral sequence of the middle column of (1), whose E_2-term is

$$E_2^{r,s}=H^r(E_n,H^s(\Gamma_n',\mathbb{Z}))\quad\Rightarrow\quad H^{r+s}(\Gamma_n,\mathbb{Z}).$$

Observe that $E_2^{1,1}=0$ since $H^1(\Gamma_n',\mathbb{Z})$ is a sum of copies of $\mathbb{Z}$ with trivial E_n-action. By the above, the inflation map from $H^2(E_n,\mathbb{Z})$ to $H^2(\Gamma_n,\mathbb{Z})$ is zero. So $H^2(\Gamma_n,\mathbb{Z})$ is a subgroup of $E_2^{0,2}\cong H^2(\Gamma',\mathbb{Z})$, and hence it is torsion free.

Now consider the LHS-spectral sequence for the middle row of (1). Its E_2-term is

$$E_2^{r,s} = H^r(U_{2n}, H^s(N,\mathbb{Z}) \quad \Rightarrow \quad H^{r+s}(\Gamma_n,\mathbb{Z}).$$

This time $E_2^{r,s}=0$ whenever s is odd. So $E_3^{r,s}=E_2^{r,s}$. Further $d_3: E_3^{0,2}\to E_3^{3,0}$ is the zero map because its domain is finite while its range is torsion free. So we have an exact sequence

$$0\to H^2(U_{2n},\mathbb{Z})\to H^2(\Gamma_n,\mathbb{Z})\to H^2(N,\mathbb{Z})\to 0.$$

Reduction modulo p gives us the diagram

$$\begin{tikzcd} H^2(U_{2n},\mathbb{Z}) \arrow[r] \arrow[d] & H^2(\Gamma_n,\mathbb{Z}) \arrow[d] \\ H^2(U_{2n},\mathbb{Z}/p) \arrow[r] & H^2(\Gamma_n,\mathbb{Z}/p). \end{tikzcd}$$

But the kernel of the inflation map on the bottom is generated by the extension class $\bar{x}_1\bar{y}_1+\cdots+\bar{x}_n\bar{y}_n$. Therefore $x_1y_1+\cdots+x_ny_n=p\theta$ for some $\theta\in H^2(\Gamma_n,\mathbb{Z})$.

In the last spectral sequence θ is represented by a class $\theta'\in E_2^{0,2}$, at least modulo $E_2^{2,0}$. But we know that the spectral sequence is itself multiplicative and is generated as a ring by $\theta', x_1, y_1, \ldots, x_n, y_n$. Because all higher differentials vanish on the generators, they must be zero. Therefore $E_2^{r,s}=E_\infty^{r,s}$. Hence the proof is complete since the relation obtained above is the only one possible. □

Corollary 2.2. *Let p^r be the highest power of p which divides $n!$. Then in $H^*(\Gamma_n,\mathbb{Z})$, $p^{n-r}\theta^t\neq 0$ for all $t\geq 0$.*

Proof. For $i=1,\ldots,n$, let

$$u_i=\frac{1}{(n-i)!}\sum_{\sigma\in S_n} x_{\sigma(1)}y_{\sigma(1)}\cdots x_{\sigma(i)}y_{\sigma(i)},$$

where S_n is the symmetric group on n letters. It can be checked that $u_i\in H^{2i}(\Gamma_n,\mathbb{Z})$. That is, the coefficients on the monomials in $x_1,\ldots,y_n$ are all 1. Also $u_1=x_1y_1+\cdots+x_ny_n$ and $u_1^i=(i!)u_i$. Now $p^n\cdot\theta^n=u_1^n=(n!)u_n$. We claim that we can find an index i such that

$$p^{n-r}\theta^n = au_i\theta^{n-i}, \tag{2}$$

where p does not divide a. We begin with

$$p^{n-r}\theta^n=(n-r)!\,u_{n-r}\theta^r$$

as a starting point. If, in an expression such as (2), $a=pb$, then

$$p^{n-r}\theta^n = bu_i(p\theta)\theta^{n-(i+1)} = b(i+1)u_{i+1}\theta^{n-(i+1)}.$$

The process can be continued until either we reach an expression as in (2) with p not dividing a, or until we reach $i=n$. In the former case we are done since $p^{n-r}\theta^t=(p^{n-r}\theta^n)\theta^{n-t}\neq 0$ for $n\geq t$ by the spectral sequence. In the latter case we have that

$$p^r(p^{n-r}\theta^n)=p^r a u_n=(n!)u_n$$

and hence p can not divide a. $\square$

Remarks. (1) The proof of Theorem A is now apparent. It is only necessary to notice that the difference $n-r$ can be made arbitrarily large. The tricky case occurs when $p=2$. For if $n=2^m$ then $n-r=1$. Relatively speaking the maximums occur when $n=2^m-1$, so that $n-r=m$.

(2) The exponent of the cohomology of Γ_n is a nondecreasing function of n. For if $m<n$, then we can consider Γ_m as a subgroup of Γ_n by injecting $U_{2m}\to U_{2n}$ onto the first $2n$ factors. From Theorem 2.1 it can be seen that the restriction map $H^*(\Gamma_n,\mathbb{Z})\to H^*(\Gamma_m,\mathbb{Z})$ is surjective.

(3) It was shown in [2] that if Γ has finite virtual cohomological dimension then there is a bound on the exponent of $\hat{H}^*(\Gamma,\mathbb{Z})$ in terms of the cohomology of the finite group $G=\Gamma/\Gamma'$. Specifically,

$$\exp \hat{H}^*(\Gamma,\mathbb{Z}) \quad \text{divides} \quad \prod_{i=1}^{s}\exp(\zeta_i),$$

where s is the maximum of the ranks of all finite subgroups of Γ and $\zeta,\dots,\zeta_s\in H^*(G,\mathbb{Z})$ satisfy a condition on the varieties. The examples show how far this bound is from being sharp. For in the case of Γ_n, $s=1$. If, for example, $p=2$ and $n=2^m$, then $\exp(\zeta_1)=2^{2^m}$ [5] while $\exp\hat{H}^*(\Gamma,\mathbb{Z})=2^{m+1}$.

(4) The Farrell cohomology $\hat{H}^*(\Gamma_n,\mathbb{Z})$ coincides with the ordinary cohomology of Γ_n in degrees larger than $2n$. For these groups the Farrell cohomology is periodic of period 2. Also $\hat{H}^0(\Gamma_n,\mathbb{Z})$ is a ring which can be formally written as

$$\sum_{i=1}^{n}\Lambda_{\mathbb{Z}}^{2i}(x_1,y_1,\dots,x_n,y_n)\theta^{-i}/J,$$

where J is the ideal generated by the relation $p\cdot 1=(x_1y_1+\cdots+x_ny_n)\theta^{-1}$.

(5) The abelian group structure of the Farrell cohomology is fairly complicated. For example if p is odd and $n=2$, then

$$\hat{H}^m(\Gamma_n,\mathbb{Z})=\begin{cases}(\mathbb{Z}/p^3)\oplus(\mathbb{Z}/p)^5, & \text{if } m \text{ even},\\ (\mathbb{Z}/p^2)^4, & \text{if } m \text{ odd}.\end{cases}$$

(6) The examples of this section were constructed as infinite analogues of the extra special p-groups which are known to have cohomology with large exponent [5].

3. Topological interpretations

The description which we have given for the groups Γ_n can be reformulated in terms of topological group actions. Take the free action of V_{2n} on $X=(S^1)^{2n}$ given by rotation on the components. Let E_n act on X through the surjection onto V_{2n}. Then it can be seen that $\Gamma_n=\pi_1(X\times_{E_n}EE_n)$ and that the two bundles

$$\begin{array}{ccc} X \longrightarrow X\times_{E_n}EE_n, & \qquad & B(\mathbb{Z}/p) \longrightarrow X\times_{E_n}EE_n, \\ \downarrow & & \downarrow \\ BE_n & & X/V_{2n} \end{array}$$

give rise to the previously described extensions on the fundamental groups. From this point of view the construction of the last section can be generalized using free actions of finite groups on aspherical complexes.

Theorem 3.1. *Let M be an aspherical complex of finite type, G a finite group, and $\varphi: M\to K(G,1)$. Let $\tilde{M}$ denote the pull-back of the universal G-bundle over $K(G,1)$. Assume that $H^3(M,\mathbb{Z})$ is p-torsion free. Then for each $\bar{\alpha}=\varphi^*(\beta)$ in $H^2(M,\mathbb{Z}/p)$ there exists a discrete group Γ_α with the following properties*:

(i) *Γ_α fits into a diagram of the following form*:

$$\begin{tikzcd} & & 1 \arrow[d] & 1 \arrow[d] & \\ & & \pi_1(\tilde{M}) \arrow[r, equal] \arrow[d] & \pi_1(\tilde{M}) \arrow[d] & \\ 1 \arrow[r] & \mathbb{Z}/p \arrow[r] \arrow[d, equal] & \Gamma_\alpha \arrow[r, "\pi"] \arrow[d] & \pi_1(M) \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \mathbb{Z}/p \arrow[r] & \tilde{G} \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & 1 \\ & & 1 & 1 & \end{tikzcd} \tag{3}$$

where $\tilde{G}$ is the extension corresponding to $\beta\in H^2(G,\mathbb{Z}/p)$.

(ii) *There exists an element $\theta\in H^2(\Gamma_\alpha,\mathbb{Z})$ such that $p\theta=\pi^*(\alpha)$, where $\alpha\in H^2(M,\mathbb{Z})$ is an element whose reduction modulo p is $\bar{\alpha}$.*

(iii) *If $H^*(M,\mathbb{Z})$ is p-torsion free, then $H^*(\Gamma_\alpha,\mathbb{Z})\cong H^*(M,\mathbb{Z})\otimes\mathbb{Z}[\theta]/(p\theta=\alpha)$.*

(iv) *The nontrivial finite subgroups of Γ_α have order p.* □

Notice that the finite virtual cohomological dimension of Γ_α is guaranteed by the condition that $\bar{\alpha}$ is in the image of φ^*. Because $\bar{\alpha}=\varphi^*(\beta)$ for $\beta\in H^2(G,\mathbb{Z}/p)$, the extension $1\to\mathbb{Z}/p\to\Gamma_\alpha\to\pi_1(M/G)\to 1$ maps onto the extension in the bottom row of (3) determined by β. Some information about the torsion in $H^*(\tilde{G},\mathbb{Z})$ can be recovered using the fact that $H^*(\Gamma_\alpha,\mathbb{Z})$ is finitely generated as an $H^*(\tilde{G},\mathbb{Z})$-

module. For example, Corollary 2.2 implies the existence of torsion of exponent p^{n+1} in the groups E_n provided $n<p$.

It is relevant to ask under what conditions it is possible to find $\bar{\alpha}\in H^2(M,\mathbb{Z}/p)$ with $0\neq\bar{\alpha}=\varphi^*(\beta)$. We have the following criterion:

Proposition 3.2. *Assume the hypotheses of Theorem* 3.1. *Then* $\mathrm{Im}(\varphi^*)\neq 0$ *in* $H^2(M,\mathbb{Z}/p)$ *if and only if*

$$\dim H^1(\tilde{M},\mathbb{Z}/p)-\dim H^1(M,\mathbb{Z}/p)<\dim \mathrm{Hom}(H_2(G,\mathbb{Z}),\mathbb{Z}/p).$$

Proof. This follows from the five-term exact sequence

$$0 \longrightarrow H^1(G,\mathbb{Z}/p) \longrightarrow H^1(M,\mathbb{Z}/p) \longrightarrow H^1(\tilde{M},\mathbb{Z}/p)^G$$

$$\longrightarrow H^2(G,\mathbb{Z}/p) \xrightarrow{\varphi^*} H^2(M,\mathbb{Z}/p)$$

and the fact that $\dim H^2(G,\mathbb{Z}/p)-\dim H^1(G,\mathbb{Z}/p)=\dim \mathrm{Hom}(H_2(G),\mathbb{F}_p)$. □

Applications. (1) For G cyclic, Proposition 3.2 implies that we must take $\tilde{M}$ so that

$$\dim H^1(\tilde{M},\mathbb{Z}/p)^G<\dim H^1(M,\mathbb{Z}/p).$$

Suppose that $G\cong\mathbb{Z}/p^n$ acts freely on an aspherical homology $2n+1$-sphere X. Then for a generator $\alpha\in H^2(G,\mathbb{Z}/p)$, $\varphi^*(\alpha)$ is not zero in $H^2(X/G,\mathbb{Z}/p)$. So in this case Γ_α has v.c.d. $2n+1$ and

$$H^*(\Gamma_\alpha,\mathbb{Z})\cong\Lambda_{\mathbb{Z}}(x_{2n+1})\otimes(\mathbb{Z}/p^{n+1})[\theta],$$

where the degree of θ is 2 and the only nontrivial finite subgroups of Γ_α have order p.

(2) It is also possible to construct virtual surface group of arbitrary genus and higher torsion. Suppose we have a surface M^{2g} with $H^1(M,\mathbb{Z}/p)$ generated by $\alpha_1,\beta_1,\dots,\alpha_g,\beta_g$. View these as maps $M\to K(\mathbb{Z}/p,1)$ and take their product $\varphi: M\to \prod^{2g} K(\mathbb{Z}/p,1)$. Then clearly $g\mu_M=\alpha_1\beta_1+\cdots+\alpha_g\beta_g$ is in the image of φ^* in $H^2(M,\mathbb{Z})$ where μ_M is the orientation class in $H^2(M,\mathbb{Z})$. Pulling back to the universal G-bundle ($G=(\mathbb{Z}/p)^{2g}$) we get a finite G-covering $\tilde{M}$ with $\tilde{M}/G=M$. By Theorem 3.1, we have a group Γ_α, where $\alpha=g\cdot\mu_M$ and

$$H^*(\Gamma_\alpha,\mathbb{Z})\cong H^*(M,\mathbb{Z})\otimes\mathbb{Z}[\theta]/(p\theta=g\mu_m),$$

where $p^2\theta^2=g^2\mu_M^2=0$ because $\mu_M^2=0$ as M is a surface. Notice that if p divides g, then we are in the degenerate situation where $H^*(\Gamma_\alpha,\mathbb{Z})\cong H^*(M,\mathbb{Z})\otimes\mathbb{F}_p[\theta]$. Again Γ_α maps onto an extra special p-group and the kernel is a surface group.

Remark. The example obtained in [2] is similar to the construction in Theorem 3.1 applied to an aspherical homology 3-sphere with a free $\mathbb{Z}/p$-action.

4. Estimates of torsion and other remarks

Suppose that Γ has finite virtual cohomological dimension. As before, let Γ' be a normal subgroup of finite c.d. such that $G=\Gamma/\Gamma'$ is finite. We can estimate the torsion in $\hat{H}^*(\Gamma,\mathbb{Z})$ using methods from equivariant cohomology. First we should notice that $\hat{H}^0(\Gamma,\mathbb{Z})$ has the highest exponent because it has a unit element. Moreover $\hat{H}^0(G,\mathbb{Z})\cong\mathbb{Z}/|G|$ surjects onto the corresponding summand. Hence we want to estimate the kernel of this map. Let X be a finite dimensional, contractible Γ-complex such that $X^H\neq 0$ if and only if $H\subseteq\Gamma$ is finite. In [2] it was shown that $\hat{H}^*(\Gamma,\mathbb{Z})\cong\hat{H}^*_G(X/\Gamma',\mathbb{Z})$. There is a spectral sequence associated with this construction whose E_2-term is

$$E_2^{r,s}=\hat{H}^r(G,H^s(X/\Gamma',\mathbb{Z})) \quad\Rightarrow\quad \hat{H}^{r+s}(\Gamma,\mathbb{Z}).$$

The point is that X is a contractible complex with free Γ'-action. So $H^*(X/\Gamma',\mathbb{Z})=H^*(\Gamma',\mathbb{Z})$. On the bottom edge of the spectral sequence we have exact sequences for $r=2,3,4,\ldots$

$$E_r^{-r,r-1}\xrightarrow{d_r}E_r^{0,0}\longrightarrow E_{r+1}^{0,0}.$$

Because $\exp E_\infty^{0,0}=\exp\hat{H}^0(\Gamma,\mathbb{Z})$, the above differentials determine the cohomological exponent of Γ. In general, the differentials are very hard to compute but we can easily deduce the following bound:

Proposition 4.1. *With the given notation*

$$|G|/\exp\hat{H}^*(\Gamma)\quad \textit{divides}\quad \prod_{r=2}^{\infty}\exp\hat{H}^{-r}(G,H^{r-1}(\Gamma',\mathbb{Z})). \qquad \square$$

Using the methods of [4] we can also show that $|G|$ divides $\chi(\Gamma')\exp\hat{H}^*(\Gamma,\mathbb{Z})$.

Some weak statements about the structure of Γ can be inferred from the cohomological exponent. As an example we offer the following:

Proposition 4.2. *Let Γ be a discrete group of finite virtual cohomological dimension. Suppose that n is a square-free positive integer such that $n\cdot\hat{H}^m(\Gamma,\mathbb{Z})=0$ for m sufficiently large. Then the finite p-subgroups of Γ are all elementary abelian.*

Proof. Choose Γ' as before with $\Gamma/\Gamma'=G$ finite. Again let X be a finite dimensional, contractible Γ-complex such that $X^H\neq 0$ if and only if $H\subset\Gamma$ is finite. For any $H\subseteq G$, if $\Gamma_H=\pi_1(X/\Gamma'\times_H EH)$, then $\hat{H}^*(\Gamma_H,\mathbb{Z})$ is a finitely generated $\hat{H}^*(\Gamma,\mathbb{Z})$-module. So $n\cdot\hat{H}^m(\Gamma_H,\mathbb{Z})=0$ for m sufficiently large. Now suppose that H is an isotropy subgroup of the G-action on X/Γ'. Then $\hat{H}^*(H,\mathbb{Z})$ injects into $\hat{H}^*(\Gamma_H,\mathbb{Z})$, and $\hat{H}^m(H,\mathbb{Z})$ is annihilated by n for m large enough. Consequently if H is a finite p-subgroup of Γ then H is elementary abelian by [1]. $\square$

As a final result we point out that the local methods of Webb [8] can also be applied to groups Γ of finite virtual cohomological dimension. Again assume that Γ is an extension $1 \to \Gamma' \to \Gamma \xrightarrow{\pi} G \to 1$ with Γ' torsion free and G finite. Let X be an admissible Γ-complex and let $\mathscr{A}_p(G)$ denote the poset of elementary abelian p-subgroups of G with a G-action induced by conjugation. Then from the Brown complex we get the following:

Theorem 4.3. *There is a split exact sequence*

$$0 \to \hat{H}^*(\Gamma, \mathbb{Z}_{(p)}) \to \bigoplus \hat{H}^*(\Gamma_{\sigma(0)}, \mathbb{Z}_{(p)}) \to \cdots \to \bigoplus \hat{H}^*(\Gamma_{\sigma(n)}, \mathbb{Z}_{(p)}),$$

where the ith sum is over a set of representatives $\sigma^{(i)}$ of the G-orbits of the ith simplices in $\mathscr{A}_p(G)/G$, $\Gamma_\sigma = \pi^{-1}(G_\sigma)$ and where G_σ is the isotropy subgroup of σ. □

The proof is obtained by simply applying the methods in [8] to the G-space X/Γ', X as before, instead of to a point. We may note that each Γ_σ fits into an extension

$$1 \to \Gamma' \to \Gamma_\sigma \to G_\sigma \to 1.$$

The fact that $\hat{H}^*_\Gamma(\mathscr{A}_p(G), \mathbb{Z}_{(p)}) \cong \hat{H}^*_G(X/\Gamma' \times \mathscr{A}_p(G), \mathbb{Z}_{(p)})$ implies that $\hat{H}^*_\Gamma(\mathscr{A}_p(G), \mathbb{Z}_{(p)}) \cong \hat{H}^*(\Gamma, \mathbb{Z}_{(p)})$. Of course, a more satisfactory result would be one such as Webb's for the poset $\mathscr{A}_p(\Gamma)$ of elementary abelian p-subgroups of Γ. But this seems far more difficult due to the absence of a suitable transfer map.

We end this paper with some questions.

(1) What does $\exp \hat{H}^*(\Gamma, \mathbb{Z})$ mean in terms of the representation theory? For $\Gamma = G$ finite, $\exp \hat{H}^*(G, \mathbb{Z})$ is the order of G, indicating that $n = |G|$ is the least positive integer such that $n \cdot \mathrm{Id}_{\mathbb{Z}}$ factors through a projective.

(2) It was previously known that $\hat{H}^0(\Gamma, \mathbb{Z})$ is not necessarily cyclic. For example [7]:

$$\hat{H}^0(\mathrm{SL}_3(\mathbb{Z}), \mathbb{Z}) \cong \mathbb{Z}/24 \oplus \mathbb{Z}/12.$$

What do the extra summands represent?

(3) Suppose that $\Gamma/\Gamma' = P$, a finite p-group. Let $e = \exp(\sum_{m>0} H^m(P, \mathbb{Z}))$. By finite generation it can be seen $e \cdot H^m(\Gamma, \mathbb{Z}) = 0$ for m sufficiently large. But is this true for m larger than the virtual cohomological dimension of Γ?

(4) Suppose that $\Gamma = \Gamma_1 \times \Gamma_2$. Is it possible to compute $\hat{H}^0(\Gamma, \mathbb{Z})$ by knowing the same for each factor?

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