ERRATUM FOR COMMUTING ELEMENTS AND SPACES OF HOMOMORPHISMS

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Abstract. We correct some cohomology calculations appearing in [1].

In the preprint [2], M. Crabb has pointed out that two of the cohomology calculations in [1], namely Theorem 1.4 and Theorem 4.12 are not consistent with the results in his paper. Here we outline how to correct the results in our paper.

Consider the Mayer-Vietoris spectral sequence in integral cohomology associated to the covering $X[3] = X_{1,2} \cup X_{1,3} \cup X_{2,3}$ where $X[3]$ denotes the non–commuting triples in $SU(2)$ and the $X_{i,j}$ correspond to the non–commutativity of the i–th and j–th coordinates. It will have as its E_1 term

$$E_1^{p,q} \cong \begin{cases} H^q(X_{1,2},\mathbb{Z}) \oplus H^q(X_{1,3},\mathbb{Z}) \oplus H^q(X_{2,3},\mathbb{Z}) & \text{if} \quad p=0 \\ H^q(X_{1,2}\cap X_{1,3},\mathbb{Z}) \oplus H^q(X_{1,2}\cap X_{2,3},\mathbb{Z}) \oplus H^q(X_{1,3}\cap X_{2,3},\mathbb{Z}) & \text{if} \quad p=1 \\ H^q(X_{123},\mathbb{Z}) & \text{if} \quad p=2 \\ 0 & \text{if} \quad p \geq 3. \end{cases}$$

The differentials on the E_1–page are determined as alternating sums of the maps induced by inclusions. The cohomology of the spaces $X_{i,j}$ and their intersections were computed in [1]; there are homotopy equivalences

$$X_{j,k} \simeq SO(3) \times SU(2),\ X_{j,k} \cap X_{i,k} \simeq SO(3) \times \mathbb{S}^1,\ X_{123} \simeq SO(3) \times [\mathbb{S}^1 \bigvee \mathbb{S}^1 \bigvee \mathbb{S}^1].$$

The spectral sequence begins with $d_1^{1,q} : E_1^{i,q} \to E_1^{i+1,q}$ for $i = 0,1,2$. These differentials can be computed explicitly, using our knowledge of these spaces and their intersections. First we must make a digression to consider the fundamental group.

By projecting $X_{1,2} \cap X_{1,3}$ onto its first two coordinates we obtain a split fibration onto the non–commuting pairs $X[2] \simeq SO(3)$ where the fiber is homotopic to $\mathbb{S}^1$. This product can be used to represent the generators of its fundamental group geometrically. Now consider the projection onto the first and third coordinates; again this yields a map onto $X[2]$. However, the previous fiber, homotopic to the circle, is now mapped homotopically

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non–trivially, as it can be identified with the fiber of the projection of $X[2]$ onto $SU(2) - \{\pm 1\}$ (which is simply further projection onto the first coordinate). Thus what we have shown is that if $\pi_1(X_{12} \cap X_{13}) \cong \mathbb{Z}w \oplus \mathbb{Z}/2y$ with these generators corresponding to the geometric decomposition above, then the inclusions into X_{12} and X_{13} induce maps $f_1, f_2 : \mathbb{Z}w \oplus \mathbb{Z}/2y \to \mathbb{Z}/2v$ such that $f_1(w) = 0$, $f_1(y) = v$ and $f_2(w) = v$, $f_2(y) = v$. An immediate consequence of this is that $X[3]$ is simply connected, as can be seen by using the Seifert-Van Kampen Theorem for the same decomposition we are considering; indeed in $\pi_1(X[3])$ the homomorphisms induced by the inclusions of the intersections force the identification of the three generators but they also identify $f_1(w) = 0$ with $f_2(w) = v$ and thus the fundamental group is trivial.

Back to the calculation: the key ingredients are that $E_2^{1,i} = 0$ for all $i \geq 0$ and that

$$E_2^{2,i} \cong \begin{cases} 0 & \text{if} \quad i = 0 \\ \mathbb{Z}/2 & \text{if} \quad i = 1 \\ 0 & \text{if} \quad i = 2 \\ \mathbb{Z}/2 & \text{if} \quad i = 3 \\ \mathbb{Z}/2 & \text{if} \quad i = 4 \\ 0 & \text{if} \quad i \geq 5 \end{cases} \qquad E_2^{0,i} \cong \begin{cases} \mathbb{Z} & \text{if} \quad i = 0 \\ 0 & \text{if} \quad i = 1 \\ \mathbb{Z}/2 & \text{if} \quad i = 2 \\ \mathbb{Z}^4 & \text{if} \quad i = 3 \\ 0 & \text{if} \quad i = 4 \\ (\mathbb{Z}/2)^3 & \text{if} \quad i = 5 \\ \mathbb{Z}^6 & \text{if} \quad i = 6 \\ 0 & \text{if} \quad i \geq 7. \end{cases}$$

It turns out that there is a single non–zero differential on the E_2 page, namely $d_2 : E_2^{0,2} \to E_2^{2,1}$, which is an isomorphism. This differential is forced by the fact that $X[3]$ is simply connected. There are no other non–trivial differentials: the only possible one would be $E_2^{0,5} \to E_2^{2,4}$; however these terms necessarily appear in the cohomology of $X[3]$ as through duality the term $E_2^{0,5}$ represents the top cohomology of the 3 copies of the top pieces of $Hom(\mathbb{Z}^2, SU(2))$ which must split off stably from $Hom(\mathbb{Z}^3, SU(2))$. Using this we can obtain a corrected version of Theorem 5.3 in [1]:

$$H_i(X[3], \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if} \quad i = 0, 1, 2 \\ \mathbb{Z}^4 & \text{if} \quad i = 3 \\ (\mathbb{Z}/2)^4 & \text{if} \quad i = 4 \\ \mathbb{Z}/2 & \text{if} \quad i = 5 \\ \mathbb{Z}^3 & \text{if} \quad i = 6 \\ 0 & \text{if} \quad i \geq 7. \end{cases}$$

Applying Poincaré–Lefschetz duality to the pair $(SU(2)^3, Hom(\mathbb{Z}^3, SU(2)))$ now yields the revised calculation:

Theorem 0.1. *The integral cohomology of the space of (ordered) commuting triples in $SU(2)$ is given by*

$$H^i(Hom(\mathbb{Z}^3, SU(2)), \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \textit{if} \quad i=0 \\ 0 & \textit{if} \quad i=1 \\ \mathbb{Z}^3 & \textit{if} \quad i=2 \\ \mathbb{Z}^3 \oplus \mathbb{Z}/2 & \textit{if} \quad i=3 \\ (\mathbb{Z}/2)^4 & \textit{if} \quad i=4 \\ \mathbb{Z} & \textit{if} \quad i=5 \\ 0 & \textit{if} \quad i \geq 6. \end{cases}$$

This agrees with the results implied in Crabb's paper, and is thus a corrected version of Theorem 1.4 in [1]. As a consequence of these corrections, our identification of $Hom(\mathbb{Z}^3, SU(3))/S_3(SU(2))$ with $SU(2) \wedge (\mathbb{S}^6 - SO(3))$ must be modified; as shown by Crabb this piece is in fact homotopy equivalent to the stunted projective space $\mathbb{R}P^5/\mathbb{R}P^3$. The corresponding modification must be made in Example 6.9 and in Example 6.10.

Similarly there is a mistake in the calculation for the cohomology of $Hom(\mathbb{Z}\times F_2, SU(2))$ (Proposition 4.11) where F_2 is the free group on two generators. The cohomology of the space $SU(2)^3 - Hom(\mathbb{Z} \times F_2, SU(2))$ is actually given by

$$H^i(SU(2)^3 - Hom(\mathbb{Z} \times F_2, SU(2)), \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if} \quad i=0 \\ 0 & \text{if} \quad i=1 \\ \mathbb{Z} & \text{if} \quad i=2 \\ \mathbb{Z}^3 & \text{if} \quad i=3 \\ 0 & \text{if} \quad i=4 \\ \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \text{if} \quad i=5 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if} \quad i=6 \\ 0 & \text{if} \quad i \geq 7. \end{cases}$$

A differential in the spectral sequence used to calculate this was overlooked. Using duality as explained in [1] this leads to the corrected version of Theorem 4.12:

Theorem 0.2. *The integral cohomology of $Hom(\mathbb{Z} \times F_2, SU(2))$ is given by*

$$H^i(Hom(\mathbb{Z} \times F_2, SU(2)), \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if} \quad i = 0 \\ 0 & \text{if} \quad i = 1 \\ \mathbb{Z}^2 & \text{if} \quad i = 2 \\ \mathbb{Z}^4 & \text{if} \quad i = 3 \\ (\mathbb{Z}/2)^2 & \text{if} \quad i = 4 \\ \mathbb{Z} & \text{if} \quad i = 5 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if} \quad i = 6 \\ 0 & \text{if} \quad i \geq 7. \end{cases}$$

From this analysis it also follows that the corresponding claim (Theorem 4.13) made about the cohomology of $Hom(\mathbb{Z} \times F_r, SU(2))$ where F_r is the free group on r generators is incorrect; however the correct calculation can be obtained from [2].

We also take this opportunity to correct a typo: in the proof for Proposition 3.1 the correct value for $H_1(\Gamma_2, \mathbb{Z})$ is $\mathbb{Z}/10$, not $\mathbb{Z}/20$.

We are grateful to M. Crabb for his interest in our work as well as pointing out the above mistakes in our cohomology calculations. The rest of the paper is unaffected.

References

[1] A. Adem, F. Cohen, *Commuting Elements and Spaces of Homomorphisms*, Mathematische Annalen **338**, pp. 587–626 (2007).

[2] M. Crabb, *Spaces of Commuting Elements in $SU(2)$*, preprint 2008.

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