JOURNAL OF ALGEBRA 149, 183–196 (1992)

# Euler Characteristics and Cohomology of $p$-Local Discrete Groups

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*Communicated by Richard G. Swan*

Received March 26, 1990

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## 0. INTRODUCTION

Let $\Gamma$ be a discrete group of finite virtual cohomological dimension. Calculating the cohomology of such a group is in general a very difficult task. Indeed, there are very few general theorems about the cohomological structure of such groups. This is in contrast to the situation for a finite group $G$, where it is well known that a local approach is required, i.e., $\mathrm{Syl}_p(G)$ must be considered for each prime $p$ dividing $|G|$.

In this paper we formulate a local approach for the cohomological analysis of the groups $\Gamma$. An essential aspect of this is the use of Farrell Cohomology. Briefly, it can be described as follows. Fix an extension

$$1 \to \Gamma' \to \Gamma \to G \to 1, \tag{0.1}$$

where $\Gamma' \subset \Gamma$ is torsion-free. Now let $P = \mathrm{Syl}_p(G)$, for some prime $p$ dividing $|G|$. Then clearly the extension

$$1 \to \Gamma' \to \Gamma^p \to P \to 1 \tag{0.2}$$

defines a subgroup of index $[G : \mathrm{Syl}_p(G)]$ in $\Gamma$, and it is easy to see that $\mathrm{res}^{\Gamma}_{\Gamma^p}$ defines an embedding $\hat{H}(\Gamma, \mathbb{Z})_{(p)} \subsetneq \hat{H}^*(\Gamma^p, \mathbb{Z})$, whose image consists of stable elements defined in a manner analogous to the finite setting.

* Supported by NSF Grants DMS-8610730 and DMS-8901414.



0021-8693/92 $5.00

It is now evident that in any systematic treatment of the cohomology of discrete groups, one must first understand these local subgroups, which are the analogue of $p$-groups in finite group theory. The main result in this paper is a cohomological non-vanishing theorem for these groups which is determined (rather surprisingly) by their rational cohomology. Before stating the result we recall the definitions of the Euler Characteristic $\chi(\Gamma)$ and the so-called "naive" Euler Characteristic $\tilde{\chi}(\Gamma)$. We assume of course that these are well defined, i.e., that $\Gamma$ is of type VFP or homologically finite. Then $\chi(\Gamma)=\chi(\Gamma')/[\Gamma:\Gamma']$, where $\Gamma'\subseteq\Gamma$ is any torsion-free subgroup of finite index in $\Gamma$ and $\tilde{\chi}(\Gamma)=\sum_{i=0}^{\infty}(-1)^i \dim H^i(\Gamma, \mathbb{Q})$. Note that $\tilde{\chi}(\Gamma')=\chi(\Gamma')$. We have

THEOREM 4.1. *Let $\Gamma$ be a discrete group of finite virtual cohomological dimension with a normal torsion-free subgroup $\Gamma'$ of index a prime power $p^n$, and assume $\chi(\Gamma,)$, $\tilde{\chi}(\Gamma)$ are well defined. Then, if $\Gamma$ is not torsion-free*

$$\dim_{\mathbb{F}_p} \hat{H}^j(\Gamma, \mathbb{Z})\otimes\mathbb{F}_p+(-1)^{j-1}[\tilde{\chi}(\Gamma)-\chi(\Gamma)]>0$$

*for all $j\in\mathbb{Z}$.*

More explicitly, we have

COROLLARY 4.2. *Under the hypotheses of* 4.1, *one of the following must occur*:

(1) *$\Gamma$ is torsion-free and $\tilde{\chi}(\Gamma)=\chi(\Gamma)$;*
(2) *$\hat{H}^i(\Gamma, \mathbb{Z})\neq 0$ $\forall i\in\mathbb{Z}$, and $\tilde{\chi}(\Gamma)=\chi(\Gamma)$;*
(3) $\dim_{\mathbb{F}_p}\hat{H}^{2i}(\Gamma, \mathbb{Z})\otimes\mathbb{F}_p>\tilde{\chi}(\Gamma)-\chi(\Gamma)>0$ $\forall i\in\mathbb{Z}$; *or*
(4) $\dim_{\mathbb{F}_p}\hat{H}^{2i+1}(\Gamma, \mathbb{Z})\otimes\mathbb{F}_p>\chi(\Gamma)-\tilde{\chi}(\Gamma)>0$ $\forall i\in\mathbb{Z}$.

In [B1], K. Brown analyzed the discrepancy between $\chi(\Gamma)$ and $\tilde{\chi}(\Gamma)$, obtaining formulae for the difference in terms of normalizers of subgroups of finite order in $\Gamma$. For example, if we assume every non-trivial finite subgroup of $\Gamma$ is contained in a unique maximal finite subgroup, then

$$\tilde{\chi}(\Gamma)-\chi(\Gamma)=\sum_{H\in\Phi'}\left(1-\frac{1}{|H|}\right)\chi(N(H)/H),$$

where $\Phi'$ is a set of representatives for the conjugacy classes of maximal finite subgroups of $\Gamma$. Using this we obtain the formula in (Thm. 4.1):

$$\dim_{\mathbb{F}_p}\hat{H}^j(\Gamma, \mathbb{Z})\otimes\mathbb{F}_p+(-1)^{j-1}\left(\sum_{H\in\Phi'}\left(1-\frac{1}{|H|}\right)\chi(N(H)/H)\right)>0. \tag{0.3}$$

What is perhaps most interesting about this is that rational cohomology information about the Weyl groups of finite subgroups of $\Gamma$ determines a lower bound for the size of the torsion module $\hat{H}^*(\Gamma, \mathbb{Z})$. Note the particular case $\Gamma = P$, a finite $p$-group. We simply recover the well-known fact that the even-dimensional cohomology of a finite $p$-group is non-trivial in every dimension [Ku]. This fact was the starting point to our analysis, after noticing that $\tilde{\chi}(P) - \chi(P) = 1 - 1/|P|$, and that this number appears in a representation-theoretic version of cohomological non-vanishing described in [A2].

There are two key ideas in our proof. First we reinterpret $\hat{H}^*(\Gamma)$ using equivariant cohomology and restrict to a $p$-subgroup. More precisely there is a finite dimensional model for $B\Gamma'$ ($\Gamma'$ as in 0.1) with a $G$-action. Then, using [A1] we see that $\hat{H}^*(\Gamma) \cong \hat{H}^*_G(B\Gamma')$, and $\hat{H}^*(\Gamma^p) \cong \hat{H}^*_P(B\Gamma')$, where $P = \mathrm{Syl}_p(G)$. Next we construct a representation over $\mathbb{Z}P$, $M(\Gamma')$ such that $\hat{H}^*(\Gamma^p) \cong \hat{H}^*(P, M(\Gamma'))$ and use a minimal projective resolution for this module over $\mathbb{Z}P$ to infer 4.1. We include a section (Sect. 5) with applications of 4.1 to various discrete groups, in particular to virtually free groups and subgroups of $SL_3(\mathbb{Z})$.

One final comment is perhaps in order. To study the cohomology of a group $\Gamma$ at a prime $p$ it is natural to use the finite $p$-subgroups of $\Gamma$, and indeed up to nilpotence they do detect the mod $p$ cohomology. However, it is by now clear from the existence of exotic torsion in $\hat{H}^*(\Gamma)$ first described in [A1] that the embeddings of the finite subgroups of $\Gamma$ in a finite quotient group $G = \Gamma/\Gamma'$ also play an important role in the Farrell Cohomology of $\Gamma$. In contrast the groups $\Gamma^p \subseteq \Gamma$ contain *all* the $p$-local information about $\hat{H}^*(\Gamma)$, and it seems plausible to expect that their cohomology will be closely related to the structure of their $p$-adic and modular representations.

## 1. A Local Approach to Farrell Cohomology

In this section we will review the basic facts which we need about Farrell Cohomology. Throughout this paper we will be dealing with groups $\Gamma$ of finite virtual cohomological dimension.

Definition 1.1. (1) A complete $\Gamma$-resolution is an acyclic complex $F_*$ of projective $\mathbb{Z}\Gamma$-modules together with an ordinary projective resolution $\varepsilon: P_* \to \mathbb{Z}$ such that $F_*$ and $P_*$ coincide in sufficiently high dimensions.

(2) The Farrell Cohomology of $\Gamma$ is defined as $\hat{H}^*(\Gamma, \mathbb{Z}) \cong H^*(\mathrm{Hom}_{\mathbb{Z}\Gamma}(F_*, \mathbb{Z}))$.

Let $\Gamma' \subseteq \Gamma$ be a torsion-free subgroup of finite index. Then we can state the following two basic properties of Farrell Cohomology:

$$[\Gamma : \Gamma'] \cdot \hat{H}^*(\Gamma) \equiv 0 \tag{F1}$$

$$\hat{H}^*(\Gamma) \cong H^*(\Gamma, \mathbb{Z}) \qquad \text{for} \quad * > \text{v.c.d. } \Gamma. \tag{F2}$$

In [A1] an alternative definition was given for $\hat{H}^*(\Gamma)$. We recall how this goes. By work of Serre [Se1] we can choose an admissible, finite dimensional $\Gamma$-complex $X$ (i.e., $X$ is contractible, and $X^H \neq \emptyset$ if and only if $H \subseteq \Gamma$ is finite). Then $\Gamma'$ acts freely on $X$, and $X/\Gamma' \simeq B\Gamma'$. Now assume that $\Gamma' \lhd \Gamma$, $\Gamma/\Gamma' \cong G$, a finite group. Then $G$ acts on $X/\Gamma'$, and under the projection $\pi\colon \Gamma \to G$, the finite subgroups in $\Gamma$ correspond in a 1–1 fashion to the isotropy subgroups of the $G$-action on this complex. Noting that $X/\Gamma'$ is finite dimensional, its equivariant Tate cohomology is well defined, and indeed we have

$$\hat{H}^*_G(X/\Gamma') \cong \hat{H}^*(\Gamma, \mathbb{Z}) \tag{F3}$$

(Also note that from above we have $B\Gamma \simeq X/\Gamma' \times_G EG$).

Now suppose we have an extension as before

$$1 \to \Gamma' \to \Gamma \to G \to 1.$$

Let $p$ be a prime, $p \mid |G|$, and denote $P = \mathrm{Syl}_p(G)$. Then we may form the extension

$$1 \to \Gamma' \to \Gamma^p \to P \to 1.$$

Clearly $\Gamma^p \subseteq \Gamma$, and $[\Gamma : \Gamma^p] = [G : \mathrm{Syl}_p(G)]$. Furthermore, it is clear that $\hat{H}^*(\Gamma^p) \cong \hat{H}^*_P(X/\Gamma')$. Using the usual transfer-restriction arguments in equivariant cohomology, we obtain

LEMMA 1.2.

$$(\mathrm{res}^{\Gamma}_{\Gamma_p})^*\colon \hat{H}^*(\Gamma, \mathbb{Z}) \to \hat{H}(\Gamma^p, \mathbb{Z})$$

*induces an inclusion*

$$\hat{H}^*(\Gamma, \mathbb{Z})_{(p)} \subsetneq \hat{H}^*(\Gamma^p, \mathbb{Z}).$$

The point of this lemma is that the groups $\Gamma^p$ play the role of $p$-Sylow subgroups in finite group cohomology. Therefore a necessary first step in analyzing $\hat{H}^*(\Gamma)$ is analyzing $\hat{H}^*(\Gamma^p)$ for all $p \mid [\Gamma : \Gamma']$, i.e., a local analysis. Indeed the main result we shall prove is an analogue for these groups of non-vanishing results for $p$-groups. To conclude this section we remark that $\mathrm{im}(\mathrm{res}^{\Gamma}_{\Gamma_p})^*$ can be identified as "stable elements" in a manner totally analogous to the finite group case (see [C-E]) by using (F3).

## 2. EULER CHARACTERISTICS

We start out by recalling the definitions of the Euler Characteristic and "naive" Euler Characteristic for discrete groups of finite v.c.d. Of course we require certain homological finiteness for them to be well defined, e.g., $\Gamma$ of type VFP or $H^*(\Gamma, \mathbb{Z}) \otimes \mathbb{F}_p$ totally finite for all primes $p$. We use the same notation as before.

DEFINITION 2.1. (1) The Euler Characteristic of $\Gamma$ is defined as

$$\chi(\Gamma) = \frac{\chi(\Gamma')}{[\Gamma : \Gamma']}.$$

(2) The "naive" Euler Characteristic of $\Gamma$ is defined as

$$\tilde{\chi}(\Gamma) = \sum_{i=0}^{\infty} (-1)^i \dim H^i(\Gamma, \mathbb{Q}).$$

The invariant $\chi(\Gamma)$ carries interesting information about the finite subgroups of $\Gamma$. K. Brown [B1] showed that, if $m$ = least common multiple of the orders of the finite subgroups in $\Gamma$, then $m \cdot \chi(\Gamma) \in \mathbb{Z}$. We will be interested in how these two invariant differ. Their discrepancy can be measured in terms of the torsion in $\Gamma$. Specifically, Brown proved [B1] that if every non-trivial subgroups of $\Gamma$ is contained in a unique maximal finite subgroup, then

$$\tilde{\chi}(\Gamma) - \chi(\Gamma) = \sum_{H \in \Phi'} \left(1 - \frac{1}{|H|}\right) \chi(N(H)/H), \tag{B1}$$

where $\Phi'$ is a set of representatives for the conjugacy classes of maximal finite subgroups of $\Gamma$. Assuming that $C(\gamma)$ is of finite homological type $\forall \gamma \in \Gamma$ of finite order (including 1), then [B2] Brown shows in fact that

$$\tilde{\chi}(\Gamma) - \chi(\Gamma) = \sum_{\gamma \in C - \{1\}} \chi(C(\gamma)), \tag{B2}$$

where $C$ is a set of representatives for conjugacy classes of elements of finite order.

To put this in perspective, we introduce an invariant for $\mathbb{Z}G$-cochain complexes.

DEFINITION 2.2. Let $C^*$ be a $\mathbb{Z}G$-cochain complex of finite homological type. Then we define

$$\gamma_G(C^*) = |G| \; \chi((C^* \otimes \mathbb{Q})^G) - \chi(C^* \otimes \mathbb{Q}).$$

The main property of this invariant is

LEMMA 2.3. *If*

$$0 \to C'^* \to C^* \to C''^* \to 0$$

*is a short exact sequence of $\mathbb{Z}G$-cochain complexes of finite type, then* $\gamma_G(C) = \gamma_G(C') + \gamma_G(C'')$.

*Proof.* Clear from the rational splitting of the sequence. ∎

For later we point out that if in the above $C^*$ is $\mathbb{Z}G$-free or projective, $\gamma_G(C^*) = 0$, and hence $\gamma_G(C'^*) = -\gamma_G(C''^*)$. Now note the following: from the discussion in Section 1, if $C^*(X/\Gamma', \mathbb{Z})$ is homologically finite, we can consider the invariant

$$\begin{aligned}\gamma_G(C^*(X/\Gamma')) &= |G|\ \chi(C^*(X/\Gamma', \mathbb{Q})^G) - \chi(\Gamma') \\ &= |G|\ \tilde{\chi}(\Gamma) - \chi(\Gamma')\end{aligned}$$

and so

$$\gamma_G(C^*(X/\Gamma')) = [\Gamma : \Gamma'](\tilde{\chi}(\Gamma) - \chi(\Gamma)). \tag{2.1}$$

## 3. MINIMAL RESOLUTIONS

In this section the finite groups $G$ which appear will be $p$-groups. Let $M$ be a finitely generated torsion free $\mathbb{Z}G$-module; then by definition a minimal resolution $(P_*, \partial_*) \to M$ is a resolution such that $P_n$ is a projective of minimal rank mapping onto $\ker \partial_{n-1}$ for all $n \geqslant 0$. We recall

PROPOSITION 3.1 (see [A2, Sw]). *A resolution $P_* \to M$ is minimal if and only if*

$$\mathrm{rk}_{\mathbb{Z}} P_n = |G|\ \dim_{\mathbb{F}_p} H^n(G, M^* \otimes \mathbb{F}_p).$$

Now assume that we have a $\mathbb{Z}G$-cochain complex $C^*$. Can one define a "minimal resolution" for $C^*$? As we are interested only in the homological properties, it suffices to find a representation $M(C^*)$ which is $G$-cohomologous to $C^*$, (abbreviated from here on as $M(C^*) \sim_G C^*$), i.e.,

$$\hat{H}^*(G, M(C^*)) \cong \hat{H}^*(G, C^*)$$

the term on the right being the usual Tate Hypercohomology. To achieve this we make use of the following elementary lemma:

LEMMA 3.2. *Let $C^*$ be an n-dimensional G-cochain complex of finite type. Then there exists a short exact sequence*

$$0 \to M^* \to F^* \to C^* \to 0$$

*of $\mathbb{Z}G$-cochain complexes such that*

(1) $F^*$ *is free,* $F^* = 0$ *for* $* < 0$, $* > n$;

(2) $H^i(M) = 0$ *for* $i \neq n$.

Applying $\hat{H}^*(G,-)$ to this, we conclude that $\hat{H}^*(G, M^*) \cong \hat{H}^{*-1}(G, C^*)$. However, from the spectral sequence

$$E_2^{p,q} = \hat{H}^p(G, H^q(M)) \Rightarrow \hat{H}^{p+q}(G, M^*)$$

we see that $\hat{H}^*(G, H^n(M)) \cong \hat{H}^{*+n}(G, M^*)$; hence

$$\hat{H}^{*-n+1}(G, H^n(M)) \cong \hat{H}^*(G, C^*). \tag{3.1}$$

In case $G$ is a $p$-group, we have a uniquely defined Heller operator, and so we define

DEFINITION 3.3. $M(C^*) = \Omega^{n-1}(H^n(M))$.

We have constructed a $\mathbb{Z}G$-lattice $M(C^*)$ such that $M(C^*) \sim_G C^*$. Assume now that $C^* = C^*(Y)$, the cellular or simplicial cochain complex of a connected $G$-complex $Y$ satisfying suitable finiteness conditions. Set $M(Y) = M(C^*(Y))$. Then $M(Y)$ inherits interesting algebraic properties from the geometry of $Y$. For example, $M(Y)$ is projective if and only if $Y$ is a $G$-free space. Similarly, we have

PROPOSITION 3.4. *Let $Y$ be a connected G-CW complex of finite type; then the highest torsion in $\hat{H}^*(G, M(Y))$ occurs in $\hat{H}^0(G, M(Y))$, and if $G = (\mathbb{Z}/p)^r$*

$$\exp \hat{H}^0(G, M(Y)) = \max\{|G_x|\} = p^{cx_G(M(Y))}.$$

*Proof.* Here $cx_G(M(Y))$ denotes the complexity of the module $M(Y)$. The first part follows from the fact that $\hat{H}^0_G(Y)$ contains a unit for the multiplicative structure of the Tate equivariant cohomology ring. From Theorem 4.1 in [A3] we have that for elementary abelian groups this exponent is in fact the order of the largest isotropy subgroup. On the other hand, for these groups, from the fundamental work of Quillen [Q], the Krull dimension of the equivariant cohomology of $Y$ and hence the complexity of the cohomologous module $M(Y)$ must be the power of $p$ in this exponent. Using this one can in fact deduce that for any finite group $G$ our construction provides a module whose complexity is determined by the 0th Tate cohomology group restricted to its elementary abelian subgroups. ∎

In the next sections we will apply $M(\ )$ to $C^*(X/\Gamma')$, and analyze homological invariants of it. The following is an example of this construction, which we include because it represents the first instance of a group $\Gamma$ of finite v.c.d. with torsion in arbitrarily high dimensions larger than the least common multiple of its finite subgroups. Here $\Gamma = \pi_1(Y \times_G EG)$ (see [A1]).

EXAMPLE 3.5. Let $Y$ be an aspherical integral homology 3-sphere (e.g., an appropriate Brieskorn manifold), with a free $\mathbb{Z}/p$-action. Let $G = \mathbb{Z}/p^2$ act on $Y$ through the projection onto $\mathbb{Z}/p$. Then, up to free summands, $C^*(Y)$ can be written as

$$0 \to \mathbb{Z} \to \mathbb{Z}[G/K] \xrightarrow{1-\bar{t}} \mathbb{Z}[G/K] \xrightarrow{\bar{N}} \mathbb{Z}[G/K] \xrightarrow{1-\bar{t}} \mathbb{Z}[G/K] \to \mathbb{Z} \to 0.$$

Here $K$ denotes the kernel of the projection, $\bar{t}$ is a generator for $G/K$, $\bar{N} = 1 + \bar{t} + \cdots + \bar{t}^{p-1}$.

$$\begin{array}{ccccc}
M_3 & \longrightarrow & (\mathbb{Z}G)^3 & \twoheadrightarrow & \mathbb{Z}[G/K] \\
\uparrow & & \uparrow & & \uparrow {\scriptstyle 1-\bar{t}} \\
M_2 & \longrightarrow & (\mathbb{Z}G)^2 & \twoheadrightarrow & \mathbb{Z}[G/K] \\
\uparrow & & \uparrow & & \uparrow {\scriptstyle \bar{N}} \\
M_1 & \longrightarrow & \mathbb{Z}G & \underset{\pi}{\twoheadrightarrow} & \mathbb{Z}[G/K] \\
\uparrow & & \uparrow & & \uparrow {\scriptstyle 1-\bar{t}} \\
M_0 & \longrightarrow & \mathbb{Z}G & \underset{\pi}{\twoheadrightarrow} & \mathbb{Z}[G/K].
\end{array}$$

We get a short exact sequence

$$0 \to H^3(M) \to \mathbb{Z}G \oplus \mathbb{Z} \xrightarrow{p^2} \mathbb{Z} \to 0$$

from which $\hat{H}^*(G, H^3(M)) \cong \hat{H}^*(G, \mathbb{Z}/p^2)$ and hence $M(Y) \sim_G \mathbb{Z}/p^2 \sim IG \oplus \mathbb{Z}$.

Let us now consider a minimal resolution for $M(Y)$, $P_*(Y)$. We have

PROPOSITION 3.6. *$P_*(Y)$ is minimal if and only if*

$$\mathrm{rk}_{\mathbb{Z}} P_i = \begin{cases} |G| \dim(M(Y)^* \otimes \mathbb{F}_p)^G, & i = 0 \\ |G| \dim \hat{H}_G^{-i-1}(Y, \mathbb{F}_p), & i > 0. \end{cases}$$

*Proof.* Use 3.1 and the isomorphism

$$\hat{H}^n(G, M(Y)^* \otimes \mathbb{F}_p) \cong \hat{H}_G^{-n-1}(Y; \mathbb{F}_p). \quad \blacksquare$$

Take a piece of a minimal resolution for $M(Y)$:

$$0 \to \Omega^{k+1}(M(Y)) \to P_k \to \cdots \to P_0 \to M(Y) \to 0.$$

Calculating Euler characteristic yields:

$$\begin{aligned}(-1)^k \operatorname{rk}_{\mathbb{Z}} \Omega^{k+1}(M(Y)) &+ \operatorname{rk}_{\mathbb{Z}} M(Y) \\ &= |G| \left[ \sum_{i=1}^{k} (-1)^i \dim_{\mathbb{F}_p} \hat{H}_G^{-i-1}(Y, \mathbb{F}_p) + \dim_{\mathbb{F}_p}(M(Y)^* \otimes \mathrm{F}_p)^G \right].\end{aligned}$$

Using the long exact sequence induced by

$$0 \to C^*(Y) \xrightarrow{p} C^*(Y) \to C^*(Y, \mathbb{F}_p) \to 0,$$

we see that

$$\dim_{\mathbb{F}_p} \hat{H}_G^{-i-1}(Y, \mathbb{F}_p) = \dim_{\mathbb{F}_p} \hat{H}_G^{-i-1}(Y) \otimes \mathbb{F}_p + \dim_{\mathbb{F}_p} \hat{H}_G^{-i}(Y) \otimes \mathbb{F}_p$$

$$\begin{aligned}\dim_{\mathbb{F}_p}(M(Y)^* \otimes \mathbb{F}_p)^G &= \dim_{\mathbb{F}_p} H^1(G, M(Y)^*) \otimes \mathbb{F}_p + \operatorname{rk}_{\mathbb{Z}}(M(Y)^*)^G \\ &= \dim_{\mathbb{F}_p} \hat{H}_G^{-1}(Y) \otimes \mathbb{F}_p + \operatorname{rk}_{\mathbb{Z}} M(Y)^G.\end{aligned}$$

Substituting above yields

PROPOSITION 3.7.

$$(-1)^k \operatorname{rk}_{\mathbb{Z}} \Omega^{k+1}(M(Y)) = (-1)^k |G| \dim \hat{H}_G^{-k}(Y) \otimes \mathbb{F}_p + \gamma_G(M(Y)).$$

Assuming that $Y$ is not $G$-free, then $M(Y)$ is non-projective, and hence $\Omega^j(M(Y)) \neq 0$ $\forall j$. Recall that $M(Y)$ is obtained in two steps: first we map a free cochain complex of finite type onto $C^*(Y)$, such that the kernel $M^*$ has cohomology concentrated in the top dimension $n$. Note that $\gamma_G(M^*) = (-1)^n \gamma_G(H^n(M))$. Now $M(Y) = \Omega^{n-1}(H^n(M))$; hence by 2.3 and the remark after it, $\gamma_G(C^*(Y)) = -\gamma_G(M^*) = (-1)^{n+1}\gamma_G(H^n(M))$, and $\gamma_G(M(Y)) = (-1)^{n-1}\gamma_G(H^n(M))$. Hence $\gamma_G(M(Y)) = \gamma_G(C^*(Y))$.

This yields

COROLLARY 3.8. *Let $G$ be a finite $p$-group, and $Y$ a $G$-$CW$ complex which is of finite type and not $G$-free. Then*

$$|G| \cdot \dim \hat{H}_G^j(Y) \otimes \mathbb{F}_p + (-1)^{j-1}\gamma_G(C^*(Y)) > 0 \qquad \forall j \in \mathbb{Z}.$$

(N.B. This is extended to all $j \in \mathbb{Z}$ by applying the above to the dual module $M(Y)^*$, using Tate Duality, and the fact that $\gamma_G(M(Y)) = \gamma_G(M(Y)^*)$.)

## 4. Cohomological Non-vanishing for Discrete Groups

In this section we apply 3.7 to the case $Y = X/\Gamma'$ (as described in Sect. 1). We obtain

THEOREM 4.1. *Let $\Gamma$ be a discrete group of finite virtual cohomological dimension with a torsion-free normal subgroup $\Gamma'$ of index a prime power $p^n$. Then, if $\Gamma$ is not torsion-free*

$$\dim_{\mathbb{F}_p} \hat{H}^j(\Gamma) \otimes \mathbb{F}_p + (-1)^{j-1}[\tilde{\chi}(\Gamma) - \chi(\Gamma)] > 0 \qquad \forall j \in \mathbb{Z}.$$

*Proof.* The proof follows by using 3.7 and identifying

$$\hat{H}^*_G(X/\Gamma') = \hat{H}^*(\Gamma), \qquad \gamma_G(C^*(X/\Gamma')) = [\Gamma : \Gamma'](\tilde{\chi}(\Gamma) - \chi(\Gamma)). \quad \blacksquare$$

This can be rewritten perhaps in a more illuminating way as

$$\dim \hat{H}^{2i + ((1-s)/2)}(\Gamma) \otimes \mathbb{F}_p > |\tilde{\chi}(\Gamma) - \chi(\Gamma)|, \tag{4.1}$$

where $s = (\tilde{\chi}(\Gamma) - \chi(\Gamma))/|\tilde{\chi}(\Gamma) - \chi(\Gamma)|$, provided $\tilde{\chi}(\Gamma) \neq \chi(\Gamma)$. We obtain

COROLLARY 4.2. *Under the hypotheses of* 4.1, *one of the following must occur*:

(1) *$\Gamma$ is torsion-free and $\tilde{\chi}(\Gamma) = \chi(\Gamma)$;*
(2) *$\hat{H}^i(\Gamma, \mathbb{Z}) \neq 0$ $\forall i \in \mathbb{Z}$ and $\tilde{\chi}(\Gamma) = \chi(\Gamma)$;*
(3) $\dim_{\mathbb{F}_p} \hat{H}^{2i}(\Gamma) \otimes \mathbb{F}_p > \tilde{\chi}(\Gamma) - \chi(\Gamma) > 0$ $\forall i \in \mathbb{Z}$; *or*
(4) $\dim_{\mathbb{F}_p} \hat{H}^{2i+1}(\Gamma) \otimes \mathbb{F}_p > \chi(\Gamma) - \tilde{\chi}(\Gamma) > 0$ $\forall i \in \mathbb{Z}$.

Using formulae (B1) and (B2) from Section 2, we can relate the above result to the structure of the lattice of finite subgroups in $\Gamma$. What is rather surprising is that a *rational* invariant $(\tilde{\chi}(\Gamma) - \chi(\Gamma))$ determines a lower bound for the size of the Farrell Cohomology of these groups (and consequently for $H^*(B\Gamma, \mathbb{Z})$, $* \gg 0$).

## 5. Some Examples

EXAMPLE 5.1. Let $\Gamma = G_1 * G_2$ (an amalgamated product), fitting into an extension $1 \to \Gamma' \to \Gamma \to P \to 1$, $\Gamma'$ free, $P$ a $p$-group. Then

$$\tilde{\chi}(\Gamma) - \chi(\Gamma) = (1 - \dim H^1(\Gamma', \mathbb{Q})^P) - \left(\frac{1}{|G_1|} + \frac{1}{|G_2|} - \frac{1}{|G_3|}\right).$$

As $\chi(\Gamma')<0$, $\chi(\Gamma)<0$ and so the term $1/|G_1|+1/|G_2|-1/|G_3|<0$. The sign of $\tilde{\chi}(\Gamma)-\chi(\Gamma')$ is determined as follows:

$$\tilde{\chi}(\Gamma)-\chi(\Gamma)>0 \Leftrightarrow 1-\frac{1}{|G_1|}-\frac{1}{|G_2|}+\frac{1}{|G_3|}>\dim H^1(\Gamma, \mathbb{Q})$$

$$\tilde{\chi}(\Gamma)-\chi(\Gamma)<0 \Leftrightarrow 1-\frac{1}{|G_1|}-\frac{1}{|G_2|}+\frac{1}{|G_3|}<\dim H^1(\Gamma)$$

$$\tilde{\chi}(\Gamma)-\chi(\Gamma)=0 \Leftrightarrow 1+\frac{1}{|G_3|}=\frac{1}{|G_1|}+\frac{1}{|G_2|}+\dim H^1(\Gamma).$$

We obtain that if $H^1(\Gamma)=0$, then

$$\dim_{\mathbb{F}_p} \hat{H}^{2i}(\Gamma, \mathbb{Z})\otimes \mathbb{F}_p>1+\frac{1}{|G_3|}-\frac{1}{|G_2|}-\frac{1}{|G_1|}>0$$

and in general for a virtually free group with $H^1(\Gamma)=0$, we have

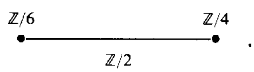
$$\dim_{\mathbb{F}_p} \hat{H}^{2i}(\Gamma, \mathbb{Z})\otimes \mathbb{F}_p>1+\frac{n-1}{[\Gamma:\Gamma']},$$

where $\Gamma'\cong F^{(n)}$ is of finite index in $\Gamma$.

For example, consider the subgroup $\Gamma\subset SL_2(\mathbb{Z})$ given by the extension

$$1\to\Gamma(3)\to\Gamma\to Q_8\to 1.$$

Here $\Gamma(3)=$ level 3 congruence subgroup, $Q_8=\mathrm{Syl}_2(SL_2(\mathbb{F}_3))$. Now as $\chi(SL_2(\mathbb{Z}))=-1/12$ and $|SL_2(\mathbb{F}_3)|=24$, $\Gamma(3)\cong F^{(3)}$. Now recall from work of Serre [Se2] that $SL_2(\mathbb{Z})\cong \mathbb{Z}/4 *_{\mathbb{Z}/2} \mathbb{Z}/6$ and hence acts on a tree with orbit space




It has two types vertices, with isotropy subgroups $\mathbb{Z}/6$ and $\mathbb{Z}/4$, and one edge, with isotropy $\mathbb{Z}/2$. The subgroup $\Gamma(3)$ acts freely on this tree, with quotient a graph $X\simeq \bigvee_1^3 S^1$. This space now has an $SL_2(\mathbb{F}_3)$-action, with isotropy subgroups as before; hence its cellular complex has the form

$$0\to H_1(\Gamma(3))\to \mathbb{Q}[SL_2(\mathbb{F}_3)/\mathbb{Z}/2]\to \mathbb{Q}[SL_2(\mathbb{F}_3)/\mathbb{Z}/4]\oplus \mathbb{Q}[SL_2(\mathbb{F}_3)/\mathbb{Z}/6]$$
$$\to \mathbb{Q}\to 0.$$

Restricting to $Q_8\subseteq SL_2(\mathbb{F}_3)$, this becomes

$$0\to H_1(\Gamma(3))\to (\mathbb{Q}[Q_8/\mathbb{Z}/2])^3\to (\mathbb{Q}[Q_8/\mathbb{Z}/4])^3\oplus \mathbb{Q}[Q_8/\mathbb{Z}/2]\to \mathbb{Q}\to 0.$$

Taking invariants, it is clear that $H_1(\Gamma(3))^{Q_8}=0$; hence $H^1(\Gamma, \mathbb{Q})=0$ and we obtain

$$\dim_{\mathbb{F}_2} \hat{H}^{2i}(\Gamma)\otimes \mathbb{F}_2 \geqslant 2 \qquad \forall i\in \mathbb{Z}.$$

EXAMPLE 5.2. Let $p$ be an odd prime, and $E_n$ the extraspecial $p$-group of order $p^{2n+1}$, all of whose elements have exponent $p$. Then $H_1(E_n)\cong (\mathbb{Z}/p)^{2n}=A$. Let $A$ act freely on $X=(S^1)^{2n}$ through rotation, and pull this back to an $E_n$-action. Now denote

$$\Gamma_n = \pi_1(X\times_{E_n} EE_n);$$

it is a group of v.c.d. $2n$, fitting into an extension

$$1\to \mathbb{Z}^{2n}\to \Gamma_n \to E_n \to 1,$$

$\tilde{\chi}(\Gamma_n)-\chi(\Gamma_n)=0$, and hence as $\Gamma_n$ has $p$-torsion, we conclude that $\hat{H}^i(\Gamma_n, \mathbb{Z})\neq 0$ $\forall i\in\mathbb{Z}$. In particular, for $n=2$ we have

$$\hat{H}^*(\Gamma_2, \mathbb{Z})\cong \begin{cases} \mathbb{Z}/p^3\oplus(\mathbb{Z}/p)^5, & *\text{ even}\\ (\mathbb{Z}/p^2)^4, & *\text{ odd}\end{cases}$$

and if $p(t)=\sum_{i=0}^{\infty}\dim H^i(\Gamma_2, \mathbb{F}_p)\, t^i$ (Poincaré Series), then

$$p(t)=1+4t+5t^2+5t^3+4t^4+t^5/(1-t^2).$$

See [A-C] for more on this type of group. More generally, we have that given an extension

$$1\to \Gamma'\to\Gamma\to P\to 1$$

with $\Gamma'$ torsion-free and a homologically trivial $P$-action on $B\Gamma'$, then $\tilde{\chi}(\Gamma)-\chi(\Gamma)=(1-1/|P|)\,\chi(\Gamma')$, i.e., its sign is completely determined by $\chi(\Gamma')$.

EXAMPLE 5.3. Let $\Gamma(3)$ be the level-3 congruence subgroup in $SL_3(\mathbb{Z})$; then it fits into an extension

$$1\to\Gamma(3)\to SL_3(\mathbb{Z})\to SL_3(\mathbb{F}_3)\to 1.$$

Recall that $|SL_3(\mathbb{F}_3)| = 3^2(3^3-1)(3^3-3) = 2^4\cdot 3^3\cdot 13$. Denote $G = \mathrm{Syl}_2(SL_3(\mathbb{F}_3))$; then $|G|=16$, and it can be described as an extension

$$1\to\mathbb{Z}/8\to G\to\mathbb{Z}/2\to 1.$$

Let $\Gamma^2$ be given by the extension

$$1-\Gamma(3)\to\Gamma^2\to G\to 1.$$

We need to compute $\tilde{\chi}(\Gamma^2)-\chi(\Gamma^2)$. As $\chi(\Gamma(3))=0$, then $\chi(\Gamma^2)=0$; hence we only need to find $\sum_{i=0}^{3}(-1)^i \dim H^i(\Gamma(3))^G$. However, we have that $H^1(\Gamma(3))=0$ [Ka]; hence only $H^2$ and $H^3$ are involved. Lee and Szczarba [L-S] have described these modules over $SL_3(\mathbb{F}_3)$.

Let $P_1$, $P_2$ be the parabolic subgroups in $SL_3(\mathbb{F}_3)$, with Borel subgroup $B$. Denote by $T_1$, $T_2$, respectively, the unique one-dimensional rational representations (non-trivial) of $P_1$, $P_2$ and set

$$V_1 = T_1 \otimes_{P_1} \mathbb{Q}[SL_3(\mathbb{F}_3)], \qquad V_2 = T_2 \otimes_{P_2} \mathbb{Q}[SL_3[\mathbb{F}_3)].$$

Then rationally,

$$H_2(\Gamma(3)) \cong V_1 \oplus V_2 \qquad \text{(26-dimensional)},$$

and so

$$\dim_{\mathbb{Q}} H^2(\Gamma(3))^G = \dim_{\mathbb{Q}}(V_1 \oplus V_2)^G.$$

On the other hand,

$$H^3(\Gamma(3)) \cong \mathrm{St},$$

the 27-dimensional Steinberg module; hence

$$\dim_{\mathbb{Q}} H^3(\Gamma(3))^G = \dim_{\mathrm{Q}}(\mathrm{St})^G.$$

To compute this, note that from the Tits Building associated to $SL_3(\mathbb{F}_3)$ we have a sequence

$$0 \to \mathrm{St} \to \mathbb{Q}[SL_3[\mathbb{F}_3)/B] \to \mathbb{Q}[SL_3(\mathbb{F}_3)/P_1] \oplus \mathbb{Q}[SL_3(\mathbb{F}_3)/P_2] \to \mathbb{Q} \to 0$$

and so

$$\dim_{\mathbb{Q}} \mathrm{St}^G = \dim_{\mathbb{Q}}(\mathbb{Q}[SL_3(\mathbb{F}_3)/B])^G - \dim_{\mathbb{Q}}(\mathbb{Q}[SL_3(\mathbb{F}_3)/P_1])^G$$
$$- \dim_{\mathbb{Q}}(\mathbb{Q}[SL_3(\mathbb{F}_3)/P_2])^G + 1.$$

This can be done more efficiently as follows. Let $\tilde{P}_1$, $\tilde{P}_2$ denote the unique index 2 subgroups in $P_1$, $P_2$, respectively, and consider $\mathbb{Q}[P_i/\tilde{P}_i]$. It is clear that $T_i \oplus \mathbb{Q} \cong_{P_i} \mathbb{Q}[P_i/\tilde{P}_i]$. Hence

$$\dim_{\mathbb{Q}} H^2(\Gamma(3))^G + 1 - \dim_{\mathbb{Q}} H^3(\Gamma(3))^G$$
$$= \dim_{\mathbb{Q}}(\mathbb{Q}[SL_3(\mathbb{F}_3)/\tilde{P}_1])^G + \dim_{\mathbb{Q}}(\mathbb{Q}[SL_3(\mathbb{F}_3)/\tilde{P}_2])^G$$
$$- \dim_{\mathbb{Q}}(\mathbb{Q}[SL_3(\mathbb{F}_3)/B])^G.$$

These invariants can be calculated using double coset decompositions and Mackey's formula. We obtain (choosing $G \subseteq P_1$)

$$\dim_{\mathbb{Q}}(\mathbb{Q}[SL_3(\mathbb{F}_3)/\tilde{P}_1])^G = 13, \qquad \dim_{\mathbb{Q}}(\mathbb{Q}[SL_3(\mathbb{F}_3)/\tilde{P}_2])^G = 5,$$
$$\dim_{\mathbb{Q}}(\mathbb{Q}[SL_3(\mathbb{F}_3)/B])^G = 13$$

and so $\tilde{\chi}(\Gamma^2)=5$. We immediately conclude

$$\dim \hat{H}^{2i}(\Gamma^2)\otimes \mathbb{F}_2 > 5 \qquad \forall i \in \mathbb{Z}.$$

In a similar fashion we can calculate $\chi(\Gamma^3)$, $\chi(\Gamma^{13})$, yielding:

$$0=\tilde{\chi}(\Gamma^3)=\tilde{\chi}(\Gamma^{13})$$

(note that $|SL_3(\mathbb{F}_3)|=2^4\cdot 3^3\cdot 13$). On the one hand, $SL_3(\mathbb{Z})$ has 3-torsion; hence we get

$$\hat{H}^i(\Gamma^3)\neq 0 \qquad \text{for all} \quad i\in\mathbb{Z};$$

as it has no 13-torsion, clearly $\Gamma^{13}$ is torsion-free and $\hat{H}^*(\Gamma^{13})\equiv 0$.

## References


[A1] A. Adem, On the exponent of cohomology of discrete groups, *Bull. London Math. Soc.* **21** (1989), 585–590.

[A2] A. Adem, Cohomological non-vanishing for modules over $p$-groups, *J. Algebra* **141** (1991), 376–381.

[A3] A. Adem, Torsion in equivariant cohomology, *Comment. Math. Helv.* **64** (1989), 401–411.

[A-C] A. Adem and J. F. Carlson, Discrete groups with large exponents in cohomology, *J. Pure Appl. Algebra* **66** (1990), 111–120.

[B] K. Brown, "Cohomology of Groups," Grad. Texts in Mathematics, Vol. 87, Springer-Verlag, Berlin, 1982).

[B1] K. Brown, Euler characteristics of discrete groups and $G$-spaces, *Invent. Math.* **27** (1974), 229–264.

[B2] K. Brown, Complete Euler characteristics and fixed-point theory, *J. Pure Appl. Algebra* **24** (1982), 103–121.

[Ca] J. F. Carlson, Exponents of modules and maps, *Invent. Math.* **95** (1989), 13–24.

[C-E] H. Cartan and S. Eilenberg, "Homological Algebra," Princeton Univ. Press, Princeton, NJ, 1956.

[Ka] D. A. Kajdan, On the connection of the dual space of a group with the structure of its closed subgroups, *Functional Anal. Appl.* **1** (1967), 63–65.

[Ku] T. Kuo, On the exponent of $H^n(G, Z)$, *J. Algebra* **7** (1967), 160–167.

[L-S] R. Lee and R. H. Szczarba, On the homology and cohomology of congruence subgroups, *Invent. Math.* **33** (1976), 15–53.

[Q] D. Quillen, The spectrum of an equivariant cohomology ring, I, *Ann. of Math. (2)* **94**, No. 3 (1971), 549–572.

[Se1] J.-P. Serre, Cohomologie des groupes discrets, *Ann. of Math. Stud.* **70** (1971), 77–169.

[Se2] J.-P. Serre, "Trees," Springer-Verlag, New York/Berlin, 1980.

[Sw] R. G. Swan, Minimal resolutions for finite groups, *Topology* **4** (1965), 193–208.