Homology Representations of Finite Transformation Groups

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0. Introduction

Let X be a finite dimensional CW complex with a cellular action of a finite group G. Given prescribed isotropy subgroups (possibly trivial) it is natural to expect restrictions on $H_*(X;R)$ (R a ring) as a graded RG-module.

In this paper we will describe efforts to deal with this problem by systematically applying techniques from group cohomology and modular representation theory. The common strategy is the following: if X is a $G-CW$ complex, then let $C_*(X;R)$ be its cellular chain complex. Then the chain groups are direct sums of permutation modules

$$C_i(X;R) \cong \bigoplus_{\sigma}(\mathbf{Z}G \underset{G_\sigma}{\otimes} \mathbf{Z}_\sigma) \otimes R$$

($\mathbf{Z}_\sigma$ denotes $\mathbf{Z}$ twisted by an orientation character.) Hence C_* may be thought of as a functor from $G-CW$ complexes to "permutation chain complexes". Now given one such chain complex, we can apply algebraic functors or invariants to it, and relate them to properties of $H_*(X;R)$.

We will discuss two distinct approaches within this framework:

(1) The Exponent — equivariant Tate Homology can be used to obtain a numerical invariant for a connected G-chain complex C_* (and hence connected $G-CW$ complexes). This invariant provides restrictions on the torsion in $H^*(G, H_*(C))$ and for a space is determined on the singular set and characterizes free actions.

(2) Growth Rate and Shifted Subgroups — using coefficients in a field, we obtain conditions on the G-cohomological growth rates of $H^*(X)$, for X a finite dimensional connected complex. These can be interpreted in terms of complexity, an invariant from modular representation theory. We also describe a method for extending results about free $(\mathbf{Z}/p)^r-CW$ complexes to arbitrary ones by using the notion of "shifted subgroups". These have important applications to group actions.

The purpose of this note is not only to describe recent developments but also to compare existing results. Most important among them are those due to Browder [4,5], Carlsson [7,8,9], Gottlieb [11] and Heller [12]. There is clearly a common thread and formal similarities; at the end of the paper we carry out a brief comparison of some of these results.

Most of the proofs are omitted, as full details will appear elsewhere [1]. The material presented here is a close version of a lecture presented at Arcata during the conference on Algebraic Topology, in August 1986.

I. The Exponent Approach

We need two definitions

Definition 1.1

For a torsion module M over $\mathbf{Z}$,

$$\exp(M) = \min\{n > 0 \mid nx = 0 \quad \text{for all } x \in M\}$$

Definition 1.2

A complete resolution is an acyclic complex $\mathcal{F}_* = (F_i)_{i \in \mathbf{Z}}$ of projective $\mathbf{Z}G$ modules, together with a map $F_0 \longrightarrow \mathbf{Z}$ such that $\mathcal{F}_+ \longrightarrow \mathbf{Z}$ is a resolution in the usual sense. ■

Now let C_* be a finite dimensional $\mathbf{Z}G$-chain complex; its Tate Homology is defined as

$$\widehat{H}_K(G, C_*) = H_K(\mathcal{F}_* \underset{G}{\otimes} C_*)$$

Assume that C_* is connected, with augmentation

$$C_* \xrightarrow{\epsilon} \mathbf{Z}$$

Then ϵ induces a map

$$\widehat{H}_{-1}(G, C_*) \xrightarrow{\epsilon_*} \widehat{H}_{-1}(G, \mathbf{Z})$$

Definition 1.3

The exponent of C_*, $e_G(C_*)$ is defined as

$$e_G(C_*) = |G| / \exp \operatorname{im} \epsilon_* \ ■$$

The following properties follow directly from this definition.

(1) $e_G(C_*)$ is a positive integer dividing $|G|$.

(2) If $H \subset G$ is a subgroup, then

$$e_H(C_*) \mid e_G(C_*)$$

(3) If $\phi : C_* \longrightarrow D_*$ is a map of connected G-chain complexes, then

$$e_G(D_*) \mid e_G(C_*)$$

$\mathcal{F}_* \underset{G}{\otimes} C_*$ is the total complex associated to a double complex, and hence for C_* finite dimensional, we have two convergent spectral sequences

(A) $E^2_{p,q} = \widehat{H}_p(G, H_q(C)) \Rightarrow \widehat{H}_{p+q}(G, C_*)$

(B) $E^1_{p,q} = \widehat{H}_q(G, C_p) \Rightarrow \widehat{H}_{p+q}(G, C_*)$

Using (A) and (B), we can estimate the exponent for C_*, connected finite-dimensional $\mathbf{Z}G$-chain complex.

Proposition 1.4

(1) If C_* is $\mathbf{Z}G$-acyclic, then

$$e_G(C_*) = |G|$$

(2) $e_G(C_*) \mid \prod_1^\infty \exp H^{i+1}(G, H_i(C))$

(3) If $\phi : C_* \longrightarrow D_*$ is a weak equivalence of connected G-chain complexes, then

$$e_G(C_*) = e_G(D_*) \blacksquare$$

We remark that (2) was due to Browder [5] when (1) holds.

Now if X is a connected, finite dimensional $G - CW$ complex, let

$$e_G(X) = e_G(C_*(X))$$

($C_*(X)$ the cellular chain complex of X).

In this situation, the exponent acquires interesting geometric properties. We list without proof the most important ones.

Properties of $e_G(X)$:

(1) $e_G(X) \mid [G : G_\sigma]$ for all G_σ isotropy subgroups

(2) $e_G(X) = |G| <=> X$ is a free $G - CW$ complex

(3) $e_G(X) \mid \chi(X)$ if X is admissible (i.e., isotropy subgroups fix cells pointwise), where $\chi(X)$ is the Euler characteristic.

(4) If X satisfies Poincare Duality and G preserves the n-dimensional orientation class, then

$$e_G(X) = |G|/\exp \text{ im } j^*$$

where $j^* : \widehat{H}^n(G, C^*(X)) \longrightarrow \widehat{H}^\circ(G, H^n(X))$.

(5) In case (4), if $G = (\mathbf{Z}/p)^r$ and X is a manifold, then $e_G(X) =$ co-rank of largest isotropy subgroup.

(6) $e_G(X)$ is determined on the singular set of the action. $\blacksquare$

A couple of remarks: (5) follows from a theorem due to Browder [4]. Some of these properties are similar to those of Gottlieb's trace [11], but their proofs are algebraic and have wider applicability.

We proceed to mention a few examples.

(1) Let M be a Riemann surface, with an orientation preserving action of a p-group

G, with $|G| = p^n$. It can be shown that G has a discrete singular set with cyclic isotropy subgroups (if any).

It is easy to compute the exponent in this case, as it is determined on $C_\circ(M)$, and

$$e_G(M) = p^n / \max\{|G_\sigma|\}$$

Applying 1.4 (2):

$$p^n / \max\{|G_\sigma|\} \mid \exp H^2(G, H_1(M)) \cdot \exp H^3(G, \mathbf{Z})$$

In particular, if $G = (\mathbf{Z}/p)^n$, we have $p^{n-2} \mid \exp H^2(G, H_1(M))$, whether the action is free or not.

This indicates that $H_1(M)$ has an interesting $\mathbf{Z}G$-module structure.

(2) Let $(\mathbf{Z}/p)^r$ act on X, an oriented manifold, trivially in homology. Then

$$\begin{matrix}\text{co-rank of largest} \\ \text{isotropy subgroup}\end{matrix} \leq \begin{matrix}\text{number of non-zero reduced} \\ \text{homology groups of } X \text{ over } \mathbf{Z}_{(p)}\end{matrix}$$

This theorem was first proved by Browder [4] and applies particularly well to $X = (S^n)^k$.

(3) Let D be the dihedral group of order 8; then it acts on S^3, preserving orientation with its element of order four acting freely. Using the join, we may construct actions on any S^{4n+3} with these properties. Can D act on other S^m in this way? We apply the exponent.

Suppose it does:

$$e_{\mathbf{Z}/4}(S^m) \mid e_D(S^m) \mid \exp H^{m+1}(D, \mathbf{Z}) \Rightarrow 4 \mid \exp H^{m+1}(D, \mathbf{Z}) \Rightarrow m \equiv -1 \bmod 4$$

The answer is no.

II Growth Rate and Shifted Subgroups

Let K be a field and C^* a connected, finite dimensional G-cochain complex. In a manner quite analogous to the proof of 1.4 (2), the following inequality can be verified:

Lemma 2.1

$$\dim \widehat{H}^{k+1}(G, K) \leq \sum_{r=1}^{\infty} \dim \widehat{H}^{k-r}(G, H^r(C)) + \dim_k \widehat{H}^{k+1}(G, C^*) \quad \text{for all } k \in \mathbf{Z}. \blacksquare$$

Here we use Tate Cohomology, defined as

$$\widehat{H}^k(G, C^*) = \widehat{H}^k(\mathrm{Hom}_G(\mathcal{F}_*, C^*))$$

for $\mathcal{F}_*$ a complete resolution of K over KG.

The term on the extreme right measures how far the cochain complex is from being free. It vanishes for free actions, and in this case a form of this inequality was first proved by Heller [12].

This can of course be applied to $C^*(X; K)$, for X a connected complex. For the free case we mention two interesting applications.

(1) Let $X = S^n \times S^m \quad n \neq m$, then using well-known formulae in group cohomology, (2.1) can be used to show that $(\mathbf{Z}/p)^3$ cannot act freely on X (see [12]).

(2) Let X be any finite complex; we define its free p-rank of symmetry as

$$F_p(X) = \max\{n \mid (\mathbf{Z}/p)^n \text{ acts freely on } X\}$$

Using 2.1, one can show

$$F_p(X) \leq \left(\sqrt{\sum_{i>0} \dim H^i(X)}\right)(\dim X + 1/2) - \dim X + 1$$

This has the virtue of being a global bound; later on we shall see improvements on this.

For a positively graded K-vector space $X_* = \{X_n\}$, we define its growth rate γ as

$$\gamma(X_*) = \min\left\{s \in \mathbf{N} \mid \lim_{n\to\infty} \frac{\dim X_n}{n^s} = 0\right\}$$

A theorem due to Quillen [14] states that

$$\gamma\left(\widehat{H}^*(G, C^*(X;K))\right) = \begin{array}{l}\text{largest rank of a } p\text{-elementary}\\ \text{abelian isotropy subgroup}\end{array}$$

where $p = \mathrm{char}\,(K)$.

There is an analogue of this for modules, which involves the notion of complexity. If M is a finitely generated KG-module, let $P_* \longrightarrow M$ be a minimal projective resolution of M over KG; then the complexity of M is defined as

$$cx_G(M) = \gamma(P_*)$$

This invariant has been shown to have many interesting properties; the one we require is

$$cx_G(M) = \max_{\substack{E \subset G \\ \text{elementary abelian}}} \{\gamma(H^*(E,M))\} \qquad \text{(see [15])}$$

Combining Quillen's result with this algebraic one, and the fact that (2.1) holds for all values of k, we obtain

Theorem 2.2

Let X be a finite dimensional connected $G-CW$ complex such that

$$cx_G(H^i(X,\mathbf{F}_p)) < p\text{-rank of } G \qquad \text{for } i > 0$$

Then there exists an elementary abelian p-subgroup $E \subset G$ of largest rank, such that $X^E \neq \emptyset$. ∎

This says for example that if the modules $H_i(X,\mathbf{F}_p)$ are projective or periodic for $i > 0$ and p-rank $G \geq 2$, then $X^E \neq \emptyset$ for some subgroup $E \cong \mathbf{Z}/p \times \mathbf{Z}/p$ in G.

Next we specialize to actions of elementary abelian p-groups. Let $E = (\mathbf{Z}/p)^n = \langle x_1,\ldots,x_n\rangle$ and K an algebraically closed field of characteristic p. We will define certain subgroups of units in KE which can be applied very fruitfully to group actions.

Let T be an automorphism of $\overset{n}{\underset{1}{\oplus}} K(x_i - 1)$. Then T extends uniquely to an algebra isomorphism $\varphi_T : KE \longrightarrow KE$.

Definition 2.3 (see [6] or [13])

A shifted subgroup S of order p^r in KE is the image of $\langle x_1,\ldots,x_r\rangle$ under φ_T for some $T \in GL_n(K)$. ∎

The importance of shifted subgroups lies in that there is a much wider choice of them than for real subgroups, making restriction arguments much more powerful. The following theorem illustrates this:

Theorem 2.4

Let X be an $E-CW$ complex, where $E \cong (\mathbf{Z}/p)^n$ with isotropy subgroup of largest rank r. Then there exists a shifted subgroup $S \subset KE$ of rank $n-r$, such that $C_*(X,K)|_S$ is free, and S has maximal rank with this property. ∎

The proof of this requires a remarkable theorem due to Kroll [13] which characterizes complexity in terms of shifted subgroups: under the above conditions, if M is a finitely generated KE-module, then $cx_E(M) = rkE - \max\{rkS \mid S \text{ is a shifted subgroup acting freely on } M\}$.

Clearly 2.4 opens the way to generalizing results about free actions to arbitrary ones at one stroke. The following is a list of some of these; let X be a finite dimensional $E-CW$ complex, where $E \cong (\mathbf{Z}/p)^n$ and r = rank of largest isotropy subgroup:

(1) If $X \simeq (S^m)^k$ and the action is trivial in homology, then $n - r \leq k$ (compare with example (2) in I).

(2) If $X \sim S^m \times S^l$, $\quad m \neq l$, then $n - r \leq 2$

(3) If $\overline{H}^*(X, K) \cong_G \begin{cases} M & i = m \\ 0 & \text{otherwise} \end{cases}$ $\qquad$ (M a KG-module)

then there exists S, a shifted subgroup of rank $n - r$ in KE, such that

$$\widehat{H}^*(S, M) \simeq \widehat{H}^{*+m+1}(S, K)$$

(this is a version of the Steenrod Problem)

(4) If $p = 2$, X is finite and $n - r \leq 4$, then

$$2^{n-r} \leq \sum_{i>0} \dim H_i(X; K)$$

(the proof of this in the free case is due to Carlsson [10])

(5) We can estimate the rank of symmetry of a complex with isotropy subgroups of prescribed rank.

To conclude this section we mention that (1) above can be extended to arbitrary $(\mathbf{Z}/2)^n$ actions on $X \simeq (S^m)^k$, provided $m \neq 1, 3, 7$. As a corollary of this we obtain the first exact value for $F_2((S^m)^k)$:

Corollary 2.5

If $m \neq 1, 3, 7$

$$F_2\left((S^m)^k\right) = k,$$

necessarily achieved by a homologically trivial action. ∎

III Related Results and Conclusions

Let $G = (\mathbf{Z}/2)^n$, and M a finitely generated KG-module (char $K = 2$). Then we define the Loewy length $\lambda_G(M)$ as

$$\lambda_G(M) = \min\left\{\lambda > 0 \mid I^\lambda M = 0\right\}$$

where $I \subset KG$ is the augmentation ideal. This is well-defined because I is nilpotent, and in fact $\lambda_G(M) \leq n + 1$.

Now let C_* be a connected, finite free KG-chain complex. Carlsson [8] proved

Theorem 3.1

Under the above conditions,

$$\sum_{i>0} \lambda_G(H_i(C)) \geq n \blacksquare$$

This result is formally similar to ones mentioned previously but it provides a considerably sharper estimate on the free rank of symmetry of a complex. From the fact that $\lambda_G(M) \leq \dim_K M$, we obtain

$$F_2(X) \leq \sum_{i>0} \dim_K H_i(X; K)$$

Carlsson has in fact conjectured that

$$F_2(X) \leq \log_2(\sum_{i=0} \dim_K H_i(X; K))$$

but this has only been verified for $rk\, G \leq 4$.

It is clear how 3.1 can be painlessly extended to arbitrary finite, connected permutation chain complexes using shifted subgroups. This can be applied to recover example (1) in II.

We have presented an outline of how techniques from group cohomology and representation theory apply successfully to questions about group actions. The methods described here yield a few overlapping results but in general seem to go in different directions. For example, the exponent approach shows that $(\mathbf{Z}/p)^n$ cannot act freely, trivially in homology, on $(S^m)^r$ if $n > r$, but not that $(\mathbf{Z}/p)^3$ cannot act freely on $S^m \times S^l$, $m \neq l$. The approach in II implies the second fact but not the first. Carlsson's result (3.1) provides a sharp estimate on the free 2-rank of symmetry but would not be useful for odd primes (indeed, the Loewy length for $(\mathbf{Z}/p)^n$-modules is bounded by $n(p-1)+1$).

The next stage should be to interpret the algebraic restrictions on the homology of a $G-CW$ complex in terms of more familiar and tractable invariants. However, it first seems necessary to be able to distinguish representations arising from geometric actions on the complex, as the complete family of G-modules (up to isomorphism) is very complicated for almost any group. This has been done only in the simplest of cases (we refer to the work on the Steenrod problem [2], [3], [9]) and a lot remains to be done.

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