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INVARIANTS AND COHOMOLOGY OF GROUPS

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This paper is dedicated to the memory of José Adem (1921-1991)

1. Introduction

Let G be a finite group and p a prime number dividing its order. The effective calculation of its cohomology ring $H^*(G;\mathbb{F}_p)$ becomes difficult as soon as the p-rank of G is larger than one. There are, however, well-known local methods ([Q1], [W1]) which allow the use of the cohomology rings of the normalizers of elementary abelian p-subgroups of G to determine the cohomology of G.

These methods are quite effective, reducing the question to studying the cohomology rings and cohomology maps induced from a lattice of proper subgroups of G (usually each much smaller than G). However, the methods run into difficulties as soon as G normalizes one of the elementary p-subgroups. Thus, from a calculational point of view, extensions of the form

$$1 \longrightarrow V \longrightarrow G \longrightarrow K \longrightarrow 1$$

with $V \cong (\mathbb{Z}/p)^n$ are of great importance in group cohomology. In particular we have an induced restriction map

$$H^*(G;\mathbb{F}_p) \longrightarrow H^*(V;\mathbb{F}_p)^K$$

indicating that rings of invariants play an important part in many calculations. Indeed, existing results for the symmetric groups [AMM1] and the general linear groups [Q2] rely heavily on determining rings of invariants.

In this paper we will describe a cohomological decomposition for group extensions where rings of invariants play a significant role. To state it we need to recall a few definitions. For a finite group K let $|A_p(K)|$ denote the geometric realization of the partially ordered set of elementary abelian p-subgroups of K. For any i-simplex σ_i in this K-complex, we denote its stablizer by K_{σ_i} and its orbit representative by $[\sigma_i]$. We then have

THEOREM (2.2). *Let*

$$1 \longrightarrow H \longrightarrow G \longrightarrow K \longrightarrow 1$$

be an extension of finite groups. Then, with $\mathbb{F}_p$ coefficients we have an isomorphism

$$H^*(G) \quad \oplus \quad \left(\bigoplus_{[\sigma_i],\ i\ odd} H^*(\pi^{-1}(K_{\sigma_i})) \right) \oplus \left(\bigoplus_{[\sigma_i],\ i\ even} H^*(H)^{K_{\sigma_i}} \right)$$

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$$\cong \; H^*(H)^K \oplus \left(\bigoplus_{[\sigma_i],\ i\ even} H^*(\pi^{-1}(K_{\sigma_i})) \right) \oplus \left(\bigoplus_{[\sigma_i],\ i\ odd} H^*(H)^{K_{\sigma_i}} \right)$$

where the $[\sigma_i]$ run over the simplexes of $|A_p(K)|/K$. ∎

The point of this expression is that it measures very precisely how far $H^*(G)$ differs from $H^*(H)^K$; note in particular the extreme case $(|K|,p) = 1$ when $H^*(G) \cong H^*(H)^K$. The proof of this theorem relies on a splitting for group cohomology due to P. Webb [W1] together with some techniques from equivariant cohomology.

As particular examples we note that there are a number of extensions of the form

$$(\mathbb{Z}/2)^n \longrightarrow G \longrightarrow \mathcal{A}_m$$

where $n = 3$ or 4 and $\mathcal{A}_m$ is the alternating group on m letters which are critical in studying the cohomology of some of the sporadic groups. More precisely, there are representations

$$\mathcal{A}_n \longrightarrow GL_4(\mathbb{F}_2) \cong \mathcal{A}_8$$

for $n = 5, 6, 7$ (two distinct representations of $\mathcal{A}_5$) which give rise to rings of invariants involved in the cohomology of the sporadic simple groups $O'N$, M_{22}, M_{23}, and M^cL which are critical ingredients in the determination of their cohomology rings ([AM2], [AM3]). Many of these examples are discussed in §4.

After we have proved (2.2) in §2 we give a detailed computer assisted determination of the rings of invariants mentioned above. Some of the invariant subrings of $GL_4(\mathbb{F}_2)$ are determined in [AM1]. Here we determine most of the remaining rings which play a role in the structure of the sporadic groups. The main result along these lines is

THEOREM.

(1) *There are two non-conjugate copies of $\mathcal{A}_5 \subset \mathcal{A}_8$. The ring of invariants for the first is given in [AM1] and for the second its invariant subring in $\mathcal{C} = \mathbb{F}_2[x_1, \ldots, x_4]$ is $\mathbb{F}_2[v_2, v_3, v_4, v_5](1, b_{10})$.*

(2) $\mathcal{C}^{\mathcal{A}_6} \cong \mathbb{F}_2[\gamma_3, \gamma_5, D_8, D_{12}](1, \gamma_9, b_{15}, \gamma_9 b_{15})$. *Here, $\gamma_3 = \mathrm{Sq}^1(\sigma_2)$ where σ_2 is the symmetric monomial in $x_1, \ldots, x_4$, $\gamma_5 = \mathrm{Sq}^2(\gamma_3)$ and $\gamma_9 = \mathrm{Sq}^4(\gamma_5)$.*

(3) $\mathcal{C}^{\mathcal{A}_7} \cong \mathbb{F}_2[D_8, D_{12}, D_{14}, D_{15}](1, d_{18}, d_{21}, d_{22}, d_{23}, d_{24}, d_{27}, d_{45})$.

This is a summary of the main results of §3. In all cases the subscript denotes the dimension of the generator and the D_i are the generators for the Dickson algebra $\mathcal{C}^{GL_4(\mathbb{F}_2)}$.

Remark. The ring in (1) above occurs in studying the Janko groups J_2 and J_3. In particular, in J_2, the normalizer of one of the two conjugacy classes of involutions has the form $2^{1+4} \rtimes \mathcal{A}_5$, and the quotient semi-direct product $(\mathbb{Z}/2)^4 \rtimes \mathcal{A}_5$, is given via this action. The other invariant subrings are needed in studying the cohomology of the Mathieu groups M_{22}, M_{23}, and the group $O'N$.

There is every indication that similar rings of invariants play an essential role in other cohomology calculations, hence it is important to develop systematic methods for computing them. In this regard Peter Kropholler informs us that he has general results on the rings $\mathbb{F}_2[x_1,\ldots x_{2n}]^{SP_{2n}(\mathbb{F}_2)}$ and $\mathbb{F}_2[x_1,\ldots,x_n]^{O_n(\mathbb{F}_2)}$. For $Sp_{2n}(\mathbb{F}_2)$, he shows that the ring of invariants has the form

$$\mathbb{F}_2[\gamma_3,\gamma_5,\gamma_9,\ldots,\gamma_{2^{n-2}+1},D_{2^{2n-1}},\ldots,D_{2^{2n}-2^n}](1,\gamma_{2^{n-1}+1})$$

which explains most of the elements in part (2) of our theorem.

In §4 we apply (2.2) to decompose the $\mathrm{mod}(2)$ cohomology of certain sporadic simple groups explicitly exhibiting the contribution of the invariants discussed above. In particular we give a complete discussion of the Mathieu group M_{11}, replacing the much longer work of [BC]. We also discuss some of the relations between the Mathieu group M_{12} and the exceptional Chevalley groups G_2 and $G_2(q)$. Further details will appear in [AM2], [AM3]. Unless otherwise indicated, $\mathbb{F}_p$ coefficients will be assumed throughout.

This paper is dedicated to the memory of José Adem. He was an elegant mathematician whose understanding of classical questions in algebra led to far-reaching contributions in algebraic topology. As an individual he was an example of that rare combination of style and substance. To one of us he was a valued friend whose support when he was just starting out was crucially appreciated and whose ideas and comments were very very important in his early work. To the other he was a lifelong example and inspiration.

2. Invariants from local methods

We consider a fixed extension

$$1 \longrightarrow H \longrightarrow G \xrightarrow{\pi} K \longrightarrow 1$$

and let p be a prime dividing the order of G. The image of the restriction map

$$(i_H^G)^*\colon H^*(G) \longrightarrow H^*(H)$$

lies in the ring of invariants $H^*(H)^K$, but i^* is neither injective nor surjective in general. However, if $(|K|,p) = 1$ then using the transfer map $tr^*\colon H^*(H) \longrightarrow H^*(G)$ together with the fact that $i^*tr^*\colon H^*(H) \longrightarrow H^*(H)$ is the sum $\sum_{k\in K} k^*$ which is valid when H is normal in G we see that $\operatorname{im}(i^*)$ is exactly $H^*(H)^K$, moreover, using the general fact that $tr^*i^*: H^*(G) \longrightarrow H^*(G)$ is just multiplication by $|G:H|$ we have

$$H^*(G) \cong H^*(H)^K.$$

The goal of this section is to see how this situation changes when $Syl_p(K) \neq 1$ and to develop a systematic method for evaluating this discrepancy.

Consider the partially ordered set $A_p(K)$ of p-elementary abelian subgroups of K, and let $X = |A_p(K)|$ be the geometric realization of the associated category consisting of objects in $A_p(K)$ and maps (proper) inclusions. X is a finite cell complex with a cellular K-action induced by conjugation. An n-simplex $\sigma \in X$ corresponds to a flag

$$\mathcal{F}(\sigma) = \{0\} \neq (\mathbb{Z}/p)^{i_1} \subsetneq \cdots \subsetneq (\mathbb{Z}/p)^{i_{n+1}}.$$

Let K_σ denote the stabilizer of $\mathcal{F}(\sigma)$. Then the action of K on $A_p(K)$ extends to a simplicial action on X and the fixing subgroup of K on the simplex σ is exactly K_σ. We recall a theorem due to P. Webb [W1]

THEOREM (2.1). *There exist projective $\mathbb{F}_pK$-modules P and Q so that*

$$\mathbb{F}_p \oplus \left(\bigoplus_{[\sigma_i],\, i\ odd} \mathbb{F}_p[K/K_{\sigma_i}] \right) \oplus P \cong_K \left(\bigoplus_{[\sigma_i],\, i\ even} \mathbb{F}_p[K/K_{\sigma_i}] \right) \oplus Q$$

where the $[\sigma_i]$ range over orbit representatives in $(X/K)^{(i)}$. ∎

The virtual projective module $P - Q$ is called the Steinberg module of K at p as it generalizes the usual projective module arising from a Tits building [W2].

Returning to our extension let $G_\sigma = \pi^{-1}(K_\sigma) \subseteq G$. It is reasonable to assume that the G_σ should play a part in any cohomological splitting for $H^*(G)$ arising from K; however, the representation theoretic discrepancy in (2.1) (expressed as $P - Q$) gives rise to an "error term" involving invariants. The precise result is

THEOREM (2.2). *Let*

$$1 \longrightarrow H \longrightarrow G \longrightarrow K \longrightarrow 1$$

be an extension of finite groups. Then, with $\mathbb{F}_p$ coefficients we have an isomorphism

$$H^*(G) \;\oplus\; \left(\bigoplus_{[\sigma_i],\, i\ odd} H^*(\pi^{-1}(K_{\sigma_i})) \right) \oplus \left(\bigoplus_{[\sigma_i],\, i\ even} H^*(H)^{K_{\sigma_i}} \right)$$

$$\cong H^*(H)^K \oplus \left(\bigoplus_{[\sigma_i],\, i\ even} H^*(\pi^{-1}(K_{\sigma_i})) \right) \oplus \left(\bigoplus_{[\sigma_i],\, i\ odd} H^*(H)^{K_{\sigma_i}} \right).$$

Proof. Let EG denote a free, contractible G-CW complex with mod(p) cellular cochain complex $C^*(EG)$. H acts freely on EG so $EG/H \simeq BH$. Also BH, in this representation, inherits a free K-action realizing BH as a principal K cover of $BG = EG/G$. Tensor formula (2.1) with $C^*(EG/H)$. This yields a corresponding K-isomorphism of cochain complexes,

$$\begin{aligned} C^*(EG/H) \quad &\otimes \quad \left\{ \mathbb{F}_p \oplus \left(\bigoplus_{[\sigma_i],\, i\ odd} \mathbb{F}_p[K/K_{\sigma_i}] \right) \oplus P \right\} \\ &\cong_K \quad C^*(EG/H) \otimes \left\{ \left(\bigoplus_{[\sigma_i],\, i\ even} \mathbb{F}_p[K/K_{\sigma_i} \right) \oplus Q \right\}. \end{aligned}$$

Apply the functor $H^*(K;\ -\ )$ to this equation where we regard the tensor product of two $\mathbb{F}_pK$-modules as a $\mathbb{F}_pK$-module via the usual rule $k(a \otimes b) = k(a) \otimes k(b)$ when $k \in K$. Note that for any subgroup $L \subseteq K$ we have

$$H^*(K; C^* \otimes \mathbb{F}_p[K/L]) \cong H^*(L; C^*/L).$$

On the other hand we also have the following result

LEMMA (2.3). *For any subgroup $L \subseteq K$, we have*

$$H^*(L; C^*(EG/H)) \cong H^*(\pi^{-1}(L)).$$

Proof of (2.3). The left hand side computes the cohomology of

$$EL \times_L EG/H \; = \; (EL \times EG/H)/L \simeq (EG/H)/L \simeq B\pi^{-1}(L).$$

∎

At this point the only terms which remain to be understood are $H^*(K; C^*(EG/H) \otimes P)$ and $H^*(K; C^*(EG/H) \otimes Q)$. For this step we need

LEMMA (2.4). *If P is a projective $\mathbb{F}_pK$ module and C^* is any $\mathbb{F}_pK$ cochain complex, then*

$$H^*(K; C^* \otimes P) \cong [H^*(C) \otimes P]^K.$$

Proof of (2.4). There is a spectral sequence with

$$E_2^{p,q} \; = \; H^p(K; H^q(C^* \otimes P)) \Rightarrow H^{p+q}(K; C^* \otimes P).$$

However, as P is projective we have

$$H^p(K;H^q(C\otimes P)) \cong H^p(K;H^q(C)\otimes P) \cong \begin{cases} 0 & \text{for } p>0, \\ (H^q(C)\otimes P)^K & \text{for } p=0. \end{cases}$$

Thus, the spectral sequence collapses at E_2 and (2.4) follows. ∎

We return to the proof of (2.2). Take (2.1), tensor with $H^*(H)$, and apply K-invariants to it. From the discussion above we obtain

$$\begin{aligned} H^*(H)^K \quad &\oplus \quad \left(\bigoplus_{[\sigma_i],\ i\ odd} H^*(H)^{K_{\sigma_i}}\right) \oplus (H^*(H)\otimes P)^K \\ &\cong \quad \left(\bigoplus_{[\sigma_i],\ i\ even} H^*(H)^{K_{\sigma_i}}\right) \oplus (H^*(H)\otimes Q)^K, \end{aligned}$$

or, stated in virtual terms,

$$(H^*(H)\otimes St(K))^K \cong \left(\sum(-1)^i H^*(H)^{K_{\sigma_i}}\right) - H^*(H)^K.$$

When we substitute this expression (2.2) follows. ∎

Remark. Instead of $|A_p(K)|$ we could have used any other K-complex for which (2.1) holds. These include $|S_p(K)|$ (the poset of non-trivial p-subgroups) or the Tits buildings if K is of Chevalley type, and for M_{22} the results of [RSY] give another lattice of subgroups distinct from these which also satisfy (2.1). See Example 4.4 for further details.

We now give several examples where (2.2) is useful, starting with familiar extensions. The following notation will be useful from here on: we set $V_n = (\mathbb{Z}/2)^n$ and we regard V_n as the n-dimensional vector space over the field $\mathbb{F}_2$.

Example (2.5). The symmetric group on 4 letters, Σ_4, is given as the extension

$$1 \longrightarrow K\ (\cong \mathbb{Z}/2\times\mathbb{Z}/2) \longrightarrow \Sigma_4 \longrightarrow SL_2(\mathbb{F}_2)\ (\cong \Sigma_3) \longrightarrow 1$$

where K is the Klein group. The poset space $|A_2(\Sigma_3)|$ consists of three copies of $\mathbb{Z}/2$ so the orbit space consists of a single point with isotropy group $\mathbb{Z}/2$, and we have

$$H^*(\Sigma_4) \oplus H^*(V_2)^{\mathbb{Z}/2} \cong H^*(V_2)^{SL_2(\mathbb{F}_2)} \oplus H^*(D_8).$$

Now, $H^*(V_n)^{\Sigma_n} = \mathbb{F}_2[x_1,\dots,x_n]^{\Sigma_n} = \mathbb{F}_2[w_1,w_2,\dots,w_n]$, where w_i denotes the i^{th} symmetric power of the monomials x_i, the well known symmetric algebra, and

$$\mathbb{F}_2[x_1,\dots,x_n]^{GL_n(\mathbb{F}_2)} = \mathbb{F}_2[D_{2^n-1}, D_{2^n-2^{n-2}},\dots,D_{2^n-1}],$$

the well known Dickson algebra. Here the $D_{2^n-2^i}$ are the Dickson invariants. Consequently, passing to Poincaré series we have

$$P_{\Sigma_4} = \frac{1}{(1-t^2)(1-t^3)} + \frac{1}{(1-t)^2} - \frac{1}{(1-t)(1-t^2)} = \frac{1+t+t^2+t^3}{(1-t^2)(1-t^3)}.$$

Remark. Since $H^*(D_8)$ is detected by abelian subgroups the same is true of $H^*(\Sigma_4)$ and we have that $H^*(\Sigma_4)$ is Cohen-Macaulay, i.e., freely and finitely generated over a polynomial subalgebra. In this case the subalgebra of elements which restrict to the Dickson algebras $\mathbb{F}_2[D_2, D_3]$ in both non-conjugate copies of $(\mathbb{Z}/2)^2 \subset \Sigma_4$.

Example (2.6). Example (2.5) generalizes to any extension of the form

$$1 \longrightarrow V_{2n} \longrightarrow G \longrightarrow SL_2(\mathbb{F}_{2^n}) \longrightarrow 1$$

by using the Tits building for $SL_2(\mathbb{F}_{2^n})$. Once more X consists only of isolated points all in the same orbit and stabilizer the Borel group which, in this case, is given as the semi-direct product

$$B \cong (\mathbb{Z}/2)^n \rtimes (\mathbb{Z}/2^n - 1).$$

Hence (2.2) gives

$$H^*(G) \oplus H^*(V_{2n})^B \cong H^*(V_{2n})^{SL_2(\mathbb{F}_{2^n})} \oplus H^*(Syl_2(G))^{\mathbb{Z}/2^n-1}.$$

From the geometry of the constructions above it is direct to see that this expression is more than merely a formal isomorphism. It corresponds to an exact sequence

$$0 \longrightarrow H^*(G) \longrightarrow H^*(V_{2n})^{SL_2(\mathbb{F}_{2^n})} \oplus H^*(Syl_2(G))^{\mathbb{Z}/2^n-1} \longrightarrow H^*(V_{2n})^B \longrightarrow 0$$

which identifies $H^*(G)$ as those elements $\mu \in H^*(Syl_2(G))^{\mathbb{Z}/2^n-1}$ which satisfy the condition

$$(i_{V_{2n}}^{Syl_2(G)})^*(\mu) \in H^*(V_{2n})^{SL_2(\mathbb{F}_{2^n})}.$$

Indeed, we could have stated Theorem (2.2) in terms of a split long exact sequence of the type above. This follows from the version of (2.1) which is proved in [W3]. In practice, however, we use (2.2) for Poincaré series calculations, usually preferring double coset decompositions to determine the ring structure of the resulting cohomology groups.

In the special case $SL_2(\mathbb{F}_4) \cong \mathcal{A}_5$, the invariants $H^*(V_4)^{\mathcal{A}_5}$ are studied in [AM1] as they play an important role in the calculation of $H^*(L_3(4))$ and the cohomology of the sporadic simple groups J_2, J_3, M_{22}, and $O'N$ as discussed in the introduction.

Example (2.7). There is a non-split extension of the form

$$1 \longrightarrow (\mathbb{Z}/2)^3 \longrightarrow E \longrightarrow GL_3(\mathbb{F}_2) \longrightarrow 1$$

where the action of $GL_3(\mathbb{F}_2)$ on $(\mathbb{Z}/2)^3$ is the usual one. Indeed, Alperin [A] has given a complete discussion of extensions of the form $(\mathbb{Z}/2^r)^3 \cdot GL_3(\mathbb{F}_2)$ where the action is the usual one when restricted to $(\mathbb{Z}/2)^3 \subset (\mathbb{Z}/2^r)^3$, and has shown that there are exactly two, the first which is split and one other which is non-split. The group E is the basic non-split example and is particularly interesting because $Syl_2(E) \cong Syl_2(M_{12})$, (compare (4.2)).

Using the Tits building for $GL_3(\mathbb{F}_2)$ with parabolics $P_1 \cong \Sigma_4$, $P_2 \cong \Sigma_4$ and $B = P_1 \cap P_2 \cong D_8$ we obtain

$$\begin{aligned} H^*(E) \ &\oplus \ H^*(Syl_2(E)) \oplus H^*(V_3)^{P_1} \oplus H^*(V_3)^{P_2} \\ &\cong H^*(V_3)^{GL_3(\mathbb{F}_2)} \oplus H^*(\pi^{-1}(P_1)) \oplus H^*(\pi^{-1}(P_2)) \oplus H^*(V_3)^{D_8}. \end{aligned}$$

Example (2.8). Consider the group $\mathcal{A}_6 \subsetneq Sp_4(\mathbb{F}_2) \cong \Sigma_6$. The resulting action of $\mathcal{A}_6$ on V_4 allows us to define a semi-direct product

$$G = V_4 \rtimes \mathcal{A}_6$$

which is a 2-local subgroup of the third Mathieu group M_{22}. In this case (2.1) reduces modulo projectives to

$$\mathbb{F}_2 \oplus \mathbb{F}_2[\mathcal{A}_6/D_8] \cong_{\mathcal{A}_6} \mathbb{F}_2[\mathcal{A}_6/Q_1] \oplus \mathbb{F}_2[\mathcal{A}_6/Q_2]$$

where $Q_1 \cong Q_2 \cong \Sigma_4$ are the two distinct copies of Σ_4 contained in $\mathcal{A}_6$. We have

$$\begin{aligned} H^*(G) \ &\oplus \ H^*(Syl_2(G)) \oplus H^*(V_4)^{Q_1} \oplus H^*(V_4)^{Q_2} \\ &\cong \ H^*(V_4)^{\mathcal{A}_6} \oplus H^*(V_4 \rtimes Q_1) \oplus H^*(V_4 \rtimes Q_2) \oplus H^*(V_4)^{D_8}. \end{aligned}$$

The invariants $H^*(V_4)^{\mathcal{A}_6}$ will be determined in §3.

3. The invariant subrings for subgroups of $GL_4(\mathbb{F}_2)$

The classical isomorphism of $GL_4(2)$ with $\mathcal{A}_8$ is given explicitly in [D, pp.290-292] by setting

$$E_1 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \ E_2 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \ E_3 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix},$$

(3.1)

$$E_4 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \ E_5 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ E_6 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix},$$

verifying the relations

$$E_1^3 = E_{i+1}^2 = (E_iE_{i+1})^3 = (E_iE_j)^2 = 1, \qquad (i,j = 1,\dots,6, j > i+1),$$

and setting up the correspondence

$$E_1 \sim (123),\ E_2 \sim (12)(34),\ E_3 \sim (12)(45).$$

If we make a change of basis

$$e_1 = \begin{pmatrix} 0\\0\\1\\1 \end{pmatrix},\ e_2 = \begin{pmatrix} 0\\1\\0\\1 \end{pmatrix},\ e_3 = \begin{pmatrix} 1\\1\\1\\0 \end{pmatrix},\ e_4 = \begin{pmatrix} 0\\1\\1\\1 \end{pmatrix},$$

then the first four matrices take the simpler form

$$E_1 \sim \begin{pmatrix} 0&1&0&0\\1&1&0&0\\0&0&1&1\\0&0&1&0 \end{pmatrix},\ E_2 \sim \begin{pmatrix} 1&0&0&0\\0&1&0&0\\1&0&1&0\\0&1&0&1 \end{pmatrix},$$

$$E_3 \sim \begin{pmatrix} 0&1&1&0\\1&0&0&1\\0&0&0&1\\0&0&1&0 \end{pmatrix},\ E_4 \sim \begin{pmatrix} 0&1&0&0\\1&0&0&0\\1&0&0&1\\0&1&1&0 \end{pmatrix}.$$

Consequently, if we set $\zeta = \begin{pmatrix} 0&1\\1&1 \end{pmatrix}, J = \begin{pmatrix} 0&1\\1&0 \end{pmatrix}, I = \begin{pmatrix} 1&0\\0&1 \end{pmatrix}$, we obtain

$$E_1 \sim \begin{pmatrix} \zeta&0\\0&\zeta^2 \end{pmatrix},\ E_2 \sim \begin{pmatrix} I&0\\I&I \end{pmatrix},\ E_3 \sim \begin{pmatrix} J&I\\0&J \end{pmatrix},\ E_4 \sim \begin{pmatrix} J&0\\I&J \end{pmatrix}.$$

It follows that $E_3E_4E_3 = \begin{pmatrix} 0&I\\I&0 \end{pmatrix}$ and the three matrices $(E_1, E_2, E_3E_4E_3)$ are our usual generators for $SL_2(\mathbb{F}_4) = \mathcal{A}_5$. Finally adding E_4 gives generators for the commutator subgroup of the symplectic group $Sp_4(\mathbb{F}_2) \cong \Sigma_6$, $Sp_4(\mathbb{F}_2)' = \mathcal{A}_6$.

We studied certain of these subgroups in [AM1]. In particular we studied the first Σ_4 and $SL_2(\mathbb{F}_4)$ there. The essential step was to use cohomology with $\mathbb{F}_4$ as coefficient ring so that the elements of order three could be diagonalized. In particular the main process was as follows. Set $\mathcal{A} = \zeta_3 a + \zeta_3^2 b$, $\mathcal{B} = \zeta_3^2 a + \zeta_3 b$, $\mathcal{C} = \zeta_3 c + \zeta_3^2 d$, $\mathcal{D} = \zeta_3^2 c + \zeta_3 d$ where $a, b, c, d \in Hom(\mathbb{F}_2^4, \mathbb{F}_2)$ are dual to e_1, e_2, e_3, e_4 respectively. Then the action of our generators above (or more accurately their duals) is given by

$$E_1^*(\mathcal{A}) = \zeta_3\mathcal{A},\ E_1^*(\mathcal{B}) = \zeta_3^2\mathcal{B},\ E_1^*(\mathcal{C}) = \zeta_3^2\mathcal{C},$$

$$E_2^*(\mathcal{A}) = \mathcal{A},\ E_2^*(\mathcal{B}) = \mathcal{B},\ E_2^*(\mathcal{C}) = \mathcal{A} + \mathcal{C},\ E_2^*(\mathcal{D}) = \mathcal{B} + \mathcal{D},$$

and setting $\tau = E_3E_4E_3$, then

$$\tau^*(\mathcal{A}) = \mathcal{C},\ \tau^*(\mathcal{B}) = \mathcal{D}.$$

Finally, in terms of this basis the action of the new generator is given by the formula

$$E_4^*(\mathcal{A}) = \mathcal{B},\ E_4^*(\mathcal{B}) = \mathcal{A}.$$

Incidentally, if we set $E_{12} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ then we see that with respect to the new basis,

$$E_5 \sim \begin{pmatrix} E_{12}, & 0 \\ I & E_{12} \end{pmatrix}, \qquad E_6 \sim \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}.$$

Note that $\tau = E_3E_4E_3 \sim (12)(46)$ under the correspondence with $\mathcal{A}_8$ so the span $\langle E_1, E_2, \tau, E_6 \rangle \cong \Sigma_5$. This copy of Σ_5 is obtained by extending the action of $SL_2(\mathbb{F}_4)$ via the Galois automorphism of $\mathbb{F}_4$. It also embeds in the full symplectic group $Sp_4(\mathbb{F}_2)$.

The Σ_4 subgroups of $\mathcal{A}_6$

There are two non-conjugate copies of $\Sigma_4 \subset \mathcal{A}_6$, the respective normalizers of the two non-conjugate copies of $(\mathbb{Z}/2)^2$ there.

The first, generated by (E_1, E_2, E_4) is characterized by the fact that the element E_1 of order 3 has no fixed non-zero vectors in $(\mathbb{Z}/2)^4$. The second is generated by

$$\begin{array}{lcl} S & = & E_1E_3E_4 \sim (123)(465) \\ T & = & E_2E_3E_1^2E_2E_1E_3E_2 \sim (15)(24) \\ E_3 & & \sim (12)(45) \end{array}$$

For this subgroup $Dim(((\mathbb{Z}/2)^4)^S) = 2$, and defining new coordinates by

$$A = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad D = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

the original matrices become

$$S \sim \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad T \sim \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$E_3 \sim \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The invariant subring for the first Σ_4 is discussed in [AM1], see in particular (4.10), so we don't discuss it further here. We now describe the invariants of the second $\Sigma_4 = \langle S, T, E_3 \rangle$. We set w_i equal to the i^{th} symmetric monomial in the variables $\langle x_1, \ldots, x_n \rangle$. Note that, while the w_i, $1 \le i \le 4$ are not invariants of this Σ_4 action, they are invariants for the action of the subgroup $\mathcal{A}_4$, since the change of basis above shows that $\mathcal{A}_4 = \langle S, T \rangle$ acts by the usual permutation of coordinates.

The following result is contained in [H].

LEMMA (3.2). *Let $\mathcal{A}_n$ act on $\langle x_1, \ldots, x_n \rangle$ by permuting coordinates in the usual manner, then*

$$\mathbb{F}_2[x_1, \ldots, x_n]^{\mathcal{A}_n} = \mathbb{F}_2[w_1, \ldots, w_n](1, a_{n(n-1)/2})$$

where $a_{n(n-1)/2} = \sum_{\alpha \in \mathcal{A}_n} \alpha(x_1 x_2^2 x_3^3 \ldots x_{n-1}^{n-1})$.

Now we are able to state

THEOREM (3.3). *Let $\bar{w}_3 = w_3 + w_1 w_2$, $\gamma_4 = w_2(w_2 + w_1^2)$, $\gamma_8 = w_4(w_4 + \bar{w}_3 w_1)$, $\lambda_5 = \mathrm{Sq}^2(\bar{w}_3) = w_1 w_4 + (w_2 + w_1^2)\bar{w}_3$, and $b_6 = a_6 + w_2(\gamma_4 + w_1^4 + w_4) + w_1 w_2 \bar{w}_3$. Finally, set $b_7 = w_1 b_6 + w_4 \bar{w}_3$. Then*

$$\mathbb{F}_2[x_1, x_2, x_3, x_4]^{\Sigma_4} = \mathbb{F}_2[w_1, \bar{w}_3, \gamma_4, \gamma_8](1, \lambda_5, b_7, \lambda_5 b_7)$$

where the action of Σ_4 is determined by the matrices for S, T, E_3 above.

Proof. We begin by noting that E_3 normalizes $\mathcal{A}_4$ so that it acts on

$$\mathbb{F}_2[w_1, w_2, w_3, w_4](1, a_6),$$

and the desired invariants are the elements in this ring which are fixed under E_3. We have

LEMMA (3.4).

$$\begin{aligned} E_3^*(w_1) &= w_1 \\ E_3^*(w_2) &= w_2 + w_1^2 \\ E_3^*(\bar{w}_3) &= \bar{w}_3 \\ E_3^*(w_4) &= w_4 + w_1 \bar{w}_3 \\ E_3^*(a_6) &= a_6 + w_1^2(\gamma_4 + w_1^4 + w_4) + w_3 \bar{w}_3. \end{aligned}$$

Proof (3.4). Since $E_3 = E_3^*$ we have $E_3^*(x_i) = x_{j(i)} + w_1$ for appropriate $j(i)$. Consequently $E_3(w_1) = 5w_1 = w_1$. Similarly $E_3(x_ix_j) = x_ax_b + (x_a + x_b)w_1 + w_1^2$, and summing over all pairs i,j we get all pairs a,b. Similarly $E_3(w_3) = w_3 + w_1^3$, and the above result follows for w_4.

The result for a_6 was done using a computer. ∎

COROLLARY (3.5). *We have that*

$$(E_3^* + 1)a_6 + w_2(\gamma_4 + w_1^4 + w_4 + w_1\bar{w}_3) = \bar{w}_3^2.$$

∎

We now return to the proof of (3.3).

We can write $\mathbb{F}_2[x_1, \ldots, x_4]^{\mathcal{A}_4} = \mathbb{F}_2[w_1, \ldots, w_4](1, b_6)$ and

$$\mathbb{F}_2[w_1, w_2, w_3, w_4] = \mathbb{F}_2[w_1, w_2, \bar{w}_3, w_4] = \mathbb{F}_2[w_1, \bar{w}_3, \gamma_4, \gamma_8](1, w_2, w_4, w_2w_4).$$

We set $S = \mathbb{F}_2[w_1, \bar{w}_3, \gamma_4, \gamma_8]$ and

$$S_1 = S(1, w_2, w_4),$$

which is closed under the action of E_3. There is an exact sequence of E_3 modules

$$S \longrightarrow S_1 \longrightarrow S\{w_2\} \oplus S\{w_4\},$$

which leads to the long exact sequence

$$S \longrightarrow S_1^{E_3} \longrightarrow S\{w_2\} \oplus S\{w_4\} \xrightarrow{\delta} H^1(\mathbb{Z}/2; S) = S \longrightarrow H^1(\mathbb{Z}/2; S_1) \longrightarrow \cdots.$$

Clearly $\delta(\{w_2\}) = w_1^2$ while $\delta(\{w_4\}) = w_1\bar{w}_3$ so $\mathrm{Ker}(\delta) = S(\lambda_5)$. In the H^1 part of this exact sequence we note for future use that the exact sequence becomes

$$0 \longrightarrow S/(w_1^2, w_1\bar{w}_3) \longrightarrow H^1(\mathbb{Z}/2; S_1) \longrightarrow S\lambda_5 \longrightarrow 0.$$

Next, set $S_2 = S(1, w_2, w_4, w_2w_4) = \mathbb{F}_2[w_1, w_2, w_3, w_4]$. We have an exact sequence

$$S_1 \longrightarrow S_2 \longrightarrow S\{w_2w_4\}$$

which gives us the long exact sequence

$$S_1^{E_3} \longrightarrow S_2^{E_3} \longrightarrow S\{w_2w_4\} \xrightarrow{\delta} H^1(\mathbb{Z}/2; S_1) \longrightarrow \cdots.$$

Moreover, an easy check gives that $\delta(\{w_2w_4\}) = w_1\lambda_5$, so δ is an injection and $S_1^{E_3}$ is the entire invariant subring. This also gives us $H^1(\mathbb{Z}/2; S_2)$.

Finally, we add b_6. We have the exact sequence

$$0 \longrightarrow S_2 \longrightarrow S_2(1, b_6) \longrightarrow S_2\{b_6\} \longrightarrow 0,$$

and passing to cohomology we get the long exact sequence

$$S_2^{E_3} \longrightarrow S_2(1,b_6)^{E_3} \longrightarrow S_2^{E_3}\{b_6\} \xrightarrow{\delta} H^1(\mathbb{Z}/2; S_2) \longrightarrow \cdots.$$

Here the corollary above shows that $\delta(\{b_6\}) = \bar{w}_3^2$, and our calculation of $H^1(S_2;\mathbb{Z}/2)$ shows that $\mathrm{Ker}(\delta) = S_2^{E_3} w_1\{b_6\}$. But the class $w_1 b_6 + w_4\bar{w}_3$ is an actual E_3-invariant, so it becomes b_7 and we have proved the result. ∎

The two $\mathcal{A}_5$'s

There are two copies of $\mathcal{A}_5$ in $\mathcal{A}_6$. The first, given as $\langle E_1, E_2, E_3E_4E_3\rangle$, is discussed in [AM1] and its invariant subring is determined there. The other is studied as follows. First, from the Atlas, [C], it is direct to check that there are only two non-conjugate copies of $\mathcal{A}_5 \subset \mathcal{A}_8$, and consequently the second group may be realized from the usual action of $\mathcal{A}_5$ on $\mathbb{F}_2[x_1,\dots,x_5]$ by projection onto the Σ_5 invariant subring generated by the four elements $r_1 = x_1 + x_2$, $r_2 = x_1 + x_3$, $r_3 = x_1 + x_4$, and $r_4 = x_1 + x_5$. The remaining element in this basis for $\langle x_1,\dots,x_5\rangle$ is $x_1+x_2+x_3+x_4+x_5 = w_1$, and rewriting

$$\mathbb{F}_2[x_1,\dots,x_5]^{\mathcal{A}_5} = \mathbb{F}_2[w_1]\otimes\mathbb{F}_2[r_1,\dots,r_4]^{\mathcal{A}_5}.$$

Consequently, we can apply Hewett's result so

$$\mathbb{F}_2[r_1,\dots,r_4]^{\mathcal{A}_5} \cong \mathbb{F}_2[v_2,v_3,v_4,v_5](1,b_{10})$$

and arguing similarly,

$$\mathbb{F}_2[r_1,\dots,r_4]^{\Sigma_5} \cong \mathbb{F}_2[v_2,v_3,v_4,v_5]$$

where we can easily determine the explicit forms of the v_i.

The $\mathcal{A}_6$ invariant subring

The Dickson element D_8 is given as follows in terms of the above generators for $\mathbb{F}_2[x_1,\dots,x_4]^{\Sigma_4}$.

$$D_8 = \gamma_8 + \gamma_4^2 + w_1^8 + w_1^2\bar{w}_3^2.$$

Also, the Dickson element $D_{12} = \mathrm{Sq}^4(D_8)$ has the following representation

$$r_{12}\quad D_{12} = (D_8+(w_1\bar{w}_3+\gamma_4+w_1^4)^2)(\gamma_4+w_1\bar{w}_3+w_1^4)+(w_1\lambda_5+w_1^2\gamma_4+\bar{w}_3^2)^2.$$

Finally, there is the relation

$$r_{10}\qquad \lambda_5^2 = w_1^2\bar{w}_3\lambda_5 + w_1^2 D_8 + \bar{w}_3^2(w_1^4+\gamma_4) + \gamma_4^2 w_1^2 + w_1^{10}.$$

LEMMA (3.6). *The class $\bar{w}_3$ satisfies $E_1^*(\bar{w}_3) = \bar{w}_3$ so $\bar{w}_3 \in \mathbb{F}_2[x_1,\dots,x_4]^{\mathcal{A}_6}$. Also, $\bar{w}_3$, $\mathrm{Sq}^2(\bar{w}_3) = \lambda_5$, D_8, and D_{12} are transcendentally independent so $\mathbb{F}_2[x_1,\dots,x_4]^{\mathcal{A}_6}$ contains the polynomial algebra*

$$\mathbb{F}_2[\bar{w}_3,\lambda_5,D_8,D_{12}].$$

Proof. $\bar{w}_3 = S(x_1^2x_2)$ gives a representation of $\bar{w}_3$ as a symmetric sum. Now, E_1^* is determined by the formula

$$\begin{array}{rcl} x_1 & \mapsto & (x_2+x_4) \\ x_2 & \mapsto & (x_3+x_4) \\ x_3 & \mapsto & (x_1+x_4) \\ x_4 & \mapsto & (x_1+x_2+x_3) \end{array}$$

We can write $S(x_1^2x_2) = x_1^2(x_2+x_3+x_4) + x_1(x_2+x_3+x_4)^2 + x_2^2(x_3+x_4) + x_2(x_3+x_4)^2 + x_3^2x_4 + x_3x_4^2$. Consequently

$$\begin{aligned} E_1^*(\bar{w}_3) &= (x_2+x_4)^2x_2 + (x_2+x_4)x_2^2 \\ &\quad +(x_3+x_4)^2(x_2+x_3+x_4) + (x_3+x_4)(x_2^2+x_3^2+x_4^2) \\ &\quad +(x_1+x_4)^2(x_1+x_2+x_3) + (x_1+x_4)(x_1^2+x_2^2+x_3^2), \end{aligned}$$

and one sees directly that this is again $S(x_1^2x_2)$. The transcendental independence of the generators above is evident by inspection. ∎

COROLLARY (3.7). *The 9 dimensional class $\lambda_9 = \mathrm{Sq}^4(\lambda_5)$ is also invariant under $\mathcal{A}_6$.*

The following result is our main technique in determining $\mathbb{F}_2[x_1,\dots,x_4]^{\mathcal{A}_6}$ from $\mathbb{F}_2[x_1,\dots,x_4]^{\Sigma_4}$.

LEMMA (3.8). *There exists an explicit projection operator*

$$e\colon \mathbb{F}_2[x_1,\dots,x_4]^{\Sigma_4} \longrightarrow \mathbb{F}_2[x_1,\dots,x_4]^{\mathcal{A}_6},$$

i.e., $e^2 = id$ restricted to the $\mathcal{A}_6$ invariant subring, and $\mathrm{im}(e) \subset \mathbb{F}_2[x_1,\dots,x_4]^{\mathcal{A}_6}$.

Proof. We write $\mathcal{A}_6 = \coprod_1^{15} v_i\Sigma_4$ for an explicit coset decomposition of $\mathcal{A}_6$. Then we set

$$e = \sum_1^{15} v_i.$$

Clearly, if $\alpha \in \mathbb{F}_2[x_1,\dots,x_4]^{\mathcal{A}_6}$ we have $e(\alpha) = 15\alpha = \alpha$, while for $\alpha \in \mathbb{F}_2[x_1,\dots,x_4]^{\Sigma_4}$ and $g \in \mathcal{A}_6$ we have $gv_i = v_{g(i)}s_i$ with $s_i \in \Sigma_4$ so $gv_i(\alpha) = v_{g(i)}(\alpha)$ and $g(e(\alpha)) = e(\alpha)$. ∎

Remark. The situation in (3.8) occurs for any subgroup $H \subset \mathcal{A}_6$, as long as H contains a Sylow 2-subgroup of $\mathcal{A}_6$.

Using (3.8) we obtain

THEOREM (3.9). *The ring of invariants $\mathbb{F}_2[x_1,\dots,x_4]^{\mathcal{A}_6}$ is*

$$\mathbb{F}_2[\bar{w}_3,\lambda_5,D_8,D_{12}](1,\lambda_9,b_{15},\lambda_9b_{15})$$

where $b_{15} = e(\gamma_4^2 b_7)$.

Proof. The major step in the proof is to reduce the determination of the $\mathcal{A}_6$ invariants to a finite calculation. This is a direct consequence of

LEMMA (3.10). $\mathbb{F}_2[x_1,\dots,x_4]^{\Sigma_4}$ *is freely generated over the* $\mathcal{A}_6$*-invariant polynomial subring,*

$$\mathcal{B} = \mathbb{F}_2[\bar{w}_3, \lambda_5, D_8, D_{12}]$$

on the 60 generators

$$(1, b_7)\begin{cases} 1, w_1, w_1^2, w_1^3, \dots, w_1^9 \\ \gamma_4, w_1\gamma_4, w_1^2\gamma_4, \dots, w_1^9\gamma_4 \\ \gamma_4^2, w_1\gamma_4^2, \dots, w_1^5\gamma_4^2 \\ w_1^2\gamma_4^3, w_1^3\gamma_4^3, w_1^4\gamma_4^3, w_1^5\gamma_4^3. \end{cases}$$

Proof of (3.10). We have that the Poincaré series for the free $\mathcal{B}$-module on 60 generators having the dimensions of the generators above is

$$\frac{(1+x^7)(1+x^4+x^8)(1-x^{10})}{(1-x)(1-x^3)(1-x^5)(1-x^8)(1-x^{12})} = \frac{(1+x^5)(1+x^7)}{(1-x)(1-x^3)(1-x^4)(1-x^8)}$$

which is the Poincaré series for $\mathbb{F}_2[x_1,\dots,x_4]^{\Sigma_4}$. Hence, if we can show that the $\mathcal{B}$-submodule of $\mathbb{F}_2[x_1,\dots,x_4]^{\Sigma_4}$ generated by the 60 elements above is the entire invariant ring the lemma will follow.

Let $\mathcal{A} = \mathbb{F}_2[w_1, \bar{w}_3, \gamma_4, D_8](1, \lambda_5) \subset \mathbb{F}_2[x_1,\dots,x_4]^{\Sigma_4}$. We now wish to determine $\mathrm{Tor}_0^{\mathcal{B}}(\mathcal{A}, \mathbb{F}_2)$ which determines a generating set for $\mathcal{A}$ over $\mathcal{B}$.

Let $\mathcal{R} = \mathbb{F}_2[w_1, \bar{w}_3, \gamma_4, \lambda_5, D_8, D_{12}]$ be the polynomial algebra on (formal) generators. There is an obvious surjective map $\mathcal{R} \longrightarrow \mathcal{A}$ and the kernel is the ideal generated by the two elements

$$R_{12} = D_{12} + (D_8 + (w_1\bar{w}_3 + \gamma_4 + w_1^4)^2)(\gamma_4 + \bar{w}_3 w_1 + w_1^4) + (w_1\lambda_5 + w_1^2\gamma_4 + \bar{w}_3^2)^2,$$

and

$$R_{10} = \lambda_5^2 + w_1^2\bar{w}_3\lambda_5 + w_1^{10} + (w_1^4 + \gamma_4)\bar{w}_3^2 + (D_8 + \gamma_4^2)w_1^2.$$

Consequently we obtain a resolution of $\mathcal{A}$ over $\mathcal{R}$,

$$(3.11)\qquad 0 \longrightarrow \mathcal{R}s_{22} \xrightarrow{\partial} \mathcal{R}(r_{10}, r_{12}) \xrightarrow{\partial} \mathcal{R} \longrightarrow \mathcal{A} \longrightarrow 0$$

where $\partial(r_{10}) = R_{10}$, $\partial(r_{12}) = R_{12}$, and $\partial(s_{22}) = R_{12}r_{10} + R_{10}r_{12}$.

Since $\mathcal{R}$ is free over $\mathcal{B}$ the resolution (3.11) is also a resolution of $\mathcal{A}$ over $\mathcal{B}$. Moreover

$$\mathcal{R}/\mathcal{B} = \mathbb{F}_2[w_1, \gamma_4]$$

so the complex for computing $\mathrm{Tor}_*^{\mathcal{B}}(\mathcal{A}, \mathbb{F}_2)$ becomes

$$\mathbb{F}_2[w_1,\gamma_4]s_{22} \xrightarrow{\partial} \mathbb{F}_2[w_1,\gamma_4](r_{10}, r_{12}) \xrightarrow{\partial} \mathbb{F}_2[w_1,\gamma_4] \longrightarrow \mathrm{Tor}_0^{\mathcal{B}}(\mathcal{A}, \mathbb{F}_2) \longrightarrow 0.$$

Here $\partial(r_{10}) = w_1^{10} + \gamma_4^2 w_1^2$ and $\partial(r_{12}) = (\gamma_4 + w_1^4)^3 + w_1^4\gamma_4^2$. Now a direct calculation gives that $\mathrm{Tor}_0^{\mathcal{B}}(\mathcal{A}, \mathbb{F}_2)$ has 30 generators,

$$\mathrm{Tor}_0^{\mathcal{B}}(\mathcal{A}, \mathbb{F}_2) = \begin{cases} 1, w_1, w_1^2, w_1^3, \dots, w_1^9 \\ \gamma_4, w_1\gamma_4, w_1^2\gamma_4, \dots, w_1^9\gamma_4 \\ \gamma_4^2, w_1\gamma_4^2, \dots, w_1^5\gamma_4^2 \\ w_1^2\gamma_4^3, w_1^3\gamma_4^3, w_1^4\gamma_4^3, w_1^5\gamma_4^3 \end{cases}$$

and it follows that $\mathbb{F}_2[x_1, \dots, x_4]^{\Sigma_4} = \mathcal{A}(1, b_6)$ is free over $\mathcal{B}$ on the sixty generators asserted in (3.10). ∎

The proof of (3.9) is now a direct computer computation. The projection operator e is evaluated, in turn, on each of the sixty generators above. Note that $\mathcal{B}$ and these images certainly generate $\mathbb{F}_2[x_1, \dots, x_4]^{\mathcal{A}_6}$, so the final determination of the ring became simply a matter of identifying the independent generators among these images. Again this was done using the computer. ∎

The $\mathcal{A}_7$ invariant subring

The argument for $\mathcal{A}_7$ is similar to that for $\mathcal{A}_6$. We start with the Dickson algebra $\mathcal{D} = \mathbb{F}_2[D_8, D_{12}, D_{14}, D_{15}] = \mathbb{F}_2[x_1, \dots, x_4]^{GL_2(4)}$ and show to begin that $\mathbb{F}_2[x_1, \dots, x_4]^{\mathcal{A}_6}$ is freely and finitely generated over it on 56 generators. Then we use a projector similar to the e of the previous section,

$$f\colon \mathbb{F}_2[x_1, x_2, x_3, x_4]^{\mathcal{A}_6} \longrightarrow \mathbb{F}_2[x_1, x_2, x_3, x_4]^{\mathcal{A}_7}$$

on each of the 56 generators above to obtain a generating set for $\mathbb{F}_4[x_1, \dots, x_4]^{\mathcal{A}_7}$.

LEMMA (3.12). *There exists an explicit projection operator*

$$f\colon \mathbb{F}_2[x_1, \dots, x_4]^{\mathcal{A}_6} \longrightarrow \mathbb{F}_2[x_1, \dots, x_4]^{\mathcal{A}_7},$$

i.e., $f^2 = \mathrm{id}$ *when restricted to the* $\mathcal{A}_7$ *invariant subring and* $\mathrm{im}(f) \subset \mathbb{F}_2[x_1, \dots, x_4]^{\mathcal{A}_7}$.

The proof is exactly like that of (3.8). Once more $|\mathcal{A}_7 : \mathcal{A}_6| = 7$ is odd and so $\sum_1^7 w_i = f$ where the w_i are a set of coset generators for $\mathcal{A}_7$ over $\mathcal{A}_6$.

THEOREM (3.13). $\mathbb{F}_2[x_1, \dots, x_4]^{\mathcal{A}_7}$ *is freely and finitely generated over the Dickson algebra* $\mathcal{D}$ *on eight generators,* $\{1\}$, $f(\bar{w}_3^3\lambda_9)$ *in dimension 18,* $f(\bar{w}_3^5\lambda_5)$ *in dimension 20,* $f(\bar{w}_3^7)$ *in dimension 21,* $f(\bar{w}_3^5\lambda_9)$ *in dimension 24,* $f(\bar{w}_3^5\lambda_5^2)$ *in dimension 25,* $f(\bar{w}_3^6\lambda_9)$ *in dimension 27, and* $f(\bar{w}_3^7\lambda_9\lambda_{15})$ *in dimension 45. In particular it has Poincaré series*

$$\frac{1 + x^{18} + x^{20} + x^{21} + x^{24} + x^{25} + x^{27} + x^{45}}{(1 - x^8)(1 - x^{12})(1 - x^{14})(1 - x^{15})}.$$

The proof follows that of the previous theorem on the $\mathcal{A}_6$ invariant subring quite closely. We begin with

LEMMA (3.14). *The subring of* $\mathbb{F}_2[x_1,\ldots,x_4]^{\mathcal{A}_6}$ *given as* $\mathcal{B} = \mathbb{F}_2[\bar{w}_3, \lambda_5, D_8, D_{12}](1, \lambda_9)$ *contains the Dickson algebra* $\mathcal{D}$ *and is freely and finitely generated over* $\mathcal{D}$ *on the 28 generators*

$$\begin{cases} 1, \bar{w}_3, \bar{w}_3^2, \bar{w}_3^3, \bar{w}_3^4, \bar{w}_3^5, \bar{w}_3^6, \bar{w}_3^7, \\ \lambda_9, \lambda_9\bar{w}_3, \lambda_9\bar{w}_3^2, \lambda_9\bar{w}_3^3, \lambda_9\bar{w}_3^4, \lambda_9\bar{w}_3^5, \lambda_9\bar{w}_3^6, \lambda_9\bar{w}_3^7, \\ \lambda_5, \lambda_5\bar{w}_3, \lambda_5\bar{w}_3^2, \lambda_5\bar{w}_3^3, \lambda_5\bar{w}_3^4, \lambda_5\bar{w}_3^5, \\ \lambda_5^2, \lambda_5^2\bar{w}_3, \lambda_5^2\bar{w}_3^2, \lambda_5^2\bar{w}_3^3, \lambda_5^2\bar{w}_3^4, \lambda_5^2\bar{w}_3^5. \end{cases}$$

Consequently $\mathbb{F}_2[x_1,\ldots,x_4]^{\mathcal{A}_6}$ *is free and finitely generated over* $\mathcal{D}$ *on the 56 generators consisting of the 28 generators above and their products with* λ_{15}.

Proof of (3.14). One checks directly that we have

$$\begin{aligned} D_{14} &= \lambda_5\lambda_9 + \bar{w}_3^2 D_8 + \bar{w}_3^3\lambda_5, \\ D_{15} &= \lambda_5^3 + \bar{w}_3^2\lambda_9 + \bar{w}_3^5. \end{aligned}$$

This shows that $\mathcal{D} \subset \mathcal{B}$.

Let $\mathcal{C} = \mathbb{F}_2[\bar{w}_3, \lambda_5, \lambda_9, D_8, D_{12}, D_{14}, D_{15}]$ be the polynomial algebra on 7 generators in the stated dimensions. There is a surjection $\mathcal{C} \longrightarrow \mathcal{B}$ taking the generators to their images with the same names. To determine a resolution of $\mathcal{B}$ over $\mathcal{C}$ note the three relations

$$\begin{aligned} &R_{18}: && \lambda_9^2 + D_8\lambda_5^2 + \bar{w}_3^6 + \lambda_5^3\bar{w}_3 + \lambda_9\bar{w}_3^3 + D_{12}\bar{w}_3^2 \\ &R_{15}: && D_{15} + \bar{w}_3^2\lambda_9 + \lambda_5^3 + \bar{w}_3^5 \\ &R_{14}: && D_{14} + \lambda_5\lambda_9 + \bar{w}_3^2 D_8 + \bar{w}_3^3\lambda_5. \end{aligned}$$

We clearly obtain a resolution of $\mathcal{B}$ over $\mathcal{C}$ as

$$0 \longrightarrow \mathcal{C}(s_{47}) \longrightarrow \mathcal{C}(s_{29}, s_{32}, s_{33}) \longrightarrow \mathcal{C}(s_{14}, s_{15}, s_{18}) \longrightarrow \mathcal{C} \longrightarrow \mathcal{B} \longrightarrow 0$$

where $\partial(s_{14}) = R_{14}$, $\partial(s_{15}) = R_{15}$, $\partial(s_{18}) = R_{18}$, and so on. Consequently, since $\mathcal{C}$ is free over the Dickson algebra $\mathcal{D}$ we obtain a resolution of $\mathcal{B}$ over $\mathcal{D}$ as follows. Set

$$\mathcal{E} = \mathbb{F}_2[\bar{w}_3, \lambda_5, \lambda_9] \subset \mathcal{C},$$

then a chain complex for determining $\mathrm{Tor}_*^{\mathcal{D}}(\mathcal{B}, \mathbb{F}_2)$ is given as

$$\mathcal{E}(s_{47}) \longrightarrow \mathcal{E}(s_{29}, s_{32}, s_{33}) \longrightarrow \mathcal{E}(s_{14}, s_{15}, s_{18}) \longrightarrow \mathcal{E} \longrightarrow \mathrm{Tor}_0^{\mathcal{D}}(\mathcal{B}, \mathbb{F}_2) \longrightarrow 0$$

where $\partial(s_{14}) = \lambda_5\lambda_9 + \bar{w}_3^3\lambda_5$, $\partial(s_{15}) = \bar{w}_3^2\lambda_9 + \lambda_5^3 + \bar{w}_3^5$, and $\partial(s_{18}) = \lambda_9^2 + \lambda_5^3\bar{w}_3 + \bar{w}_3^3\lambda_9 + \bar{w}_3^6$, and we have

$$\begin{aligned}\mathrm{Tor}_0^{\mathcal{D}}(\mathcal{B},\mathbb{F}_2) &= \mathbb{F}_2[\bar{w}_3,\lambda_5,\lambda_9]/(\lambda_5\lambda_9+\bar{w}_3^3\lambda_5,\bar{w}3^2\lambda_9\\ &+ \lambda_5^3+\bar{w}_3^5,\lambda_9^2+\lambda_5^3\bar{w}_3+\bar{w}_3^3\lambda_9+\bar{w}_3^6).\end{aligned}$$

Within the ideal which is being factored out note the following. First the relation for λ_9^2 has the form $\lambda_9^2+\bar{w}_3r_{15}$ where r_{15} is the second relation above. Thus $\lambda_9^2 \in$ the relation set. Similarly, $\lambda_5r_{15} = \lambda_5^4 + \bar{w}_3^2r_{14}$ so λ_5^4 is in the relation set. Also, $\lambda_9r_{14} = \bar{w}_3^3\lambda_5\lambda_9 + \lambda_5\lambda_9^2$ so $\bar{w}_3^3\lambda_5\lambda_9$ is in the relation set, and expanding out λ_5r_{18} gives that $\bar{w}_3^6\lambda_5 = 0$. Also, note that r_{14} implies that $\bar{w}_3^3\lambda_5^3 = \lambda_5^3\lambda_9$ and r_{15} implies that $\bar{w}_3^5 + \bar{w}_3^2\lambda_9 + \lambda_5^3$. Consequently

$$\bar{w}_3^3r_{15} = \bar{w}_3^5\lambda_9 + \lambda_5^3\lambda_9 + \bar{w}_3^8 = (\bar{w}_3^2\lambda_9 + \lambda_5^3)\lambda_9 + \lambda_5^3\lambda_9 + \bar{w}_3^8 = \bar{w}_3^8$$

and $\bar{w}_3^8$ is in the relation set. Thus, $\mathrm{Tor}_0^{\mathcal{D}}(\mathcal{B},\mathbb{F}_2)$ can be no larger than the set asserted in (3.14).

We now show that it also cannot be smaller. Note that the Poincaré series for the free module over $\mathcal{D}$ on generators in the stated dimensions above is

$$\frac{\frac{1-x^{24}}{1-x^3}(1+x^9)+\frac{1-x^{18}}{1-x^3}(x^5+x^{10})}{(1-x^8)(1-x^{12})(1-x^{14})(1-x^{15})}$$

but this is

$$\frac{(1+x^9)(1-x^{14})(1+x^5+x^{10})}{(1-x^3)(1-x^8)(1-x^{12})(1-x^{14})(1-x^{15})} = \frac{1+x^9}{(1-x^3)(1-x^5)(1-x^8)(1-x^{12})}$$

and (3.14) follows. ∎

The next step in the proof is contained in

LEMMA (3.15). *$\mathbb{F}_2[x_1,\dots,x_4]^{\mathcal{A}_7}$ is freely and finitely generated over $\mathcal{D}$ on eight generators.*

Proof of (3.15). An easy induction using (3.14) and the projector f shows that the ring of invariants $\mathbb{F}_2[x_1,\dots,x_4]^{\mathcal{A}_7}$ is freely and finitely generated over $\mathcal{D}$. It remains to show that the number of generators is eight. To see this we pass to quotient fields. We have that the degree of $\mathbb{F}_2(x_1,\dots,x_4)^{\mathcal{A}_7}$ over $\mathbb{F}_2(x_1,\dots,x_4)^{GL_4(2)}$ is eight by the fundamental theorem of Galois theory. On the other hand $\mathbb{F}_2[x_1,\dots,x_4]^{GL_4(2)} = \mathcal{D}$ and thus $\mathcal{QD} = \mathbb{F}_2(x_1,\dots,x_4)^{GL_4(2)}$ so

$$\mathcal{Q}(\mathbb{F}_2[x_1,\dots,x_4]^{\mathcal{A}_7}) = \mathcal{QD}(w_1,\dots,w_8)$$

where $w_1,\dots,w_8$ are the eight generators of $\mathbb{F}_2[x_1,\dots,x_4]^{\mathcal{A}_7}$ over $\mathcal{D}$. ∎

For the final step the operator f was evaluated on the 28 generators in the first lemma above using a computer, and image classes in dimensions 18, 20, 21, 24, 25, and 27 were found. The resulting table is given as follows where $Sx_1^a x_2^b x_3^c x_4^d$ denotes the symmetric sum

The class $f(\bar{w}_3^3\gamma_9)$:

$$\begin{aligned}
& Sx_1^8x_2^6x_3^2x_4^2 + Sx_1^9x_2^5x_3^2x_4^2 + Sx_1^9x_2^6x_3^3 + Sx_1^8x_2^4x_3^4x_4^2 \\
+ \;& Sx_1^9x_2^5x_3^4 + Sx_1^{10}x_2^6x_3^1x_4^1 + Sx_1^{10}x_2^4x_3^2x_4^2 + Sx_1^{10}x_2^6x_3^2 \\
+ \;& Sx_1^{10}x_2^4x_3^3x_4^1 + Sx_1^9x_2^6x_3^2x_4^1 + Sx_1^{11}x_2^4x_3^2x_4^1 + Sx_1^8x_2^4x_3^3x_4^3 \\
+ \;& Sx_1^{10}x_2^5x_3^3 + Sx_1^{12}x_2^4x_3^2 + Sx_1^9x_2^4x_3^3x_4^2 + Sx_1^8x_2^7x_3^2x_4^1 \\
+ \;& Sx_1^{12}x_2^4x_3^1x_4^1 + Sx_1^{12}x_2^2x_3^2x_4^2 + Sx_1^9x_2^4x_3^4x_4^1
\end{aligned}$$

The class $f(\bar{w}_3^5\gamma_5)$:

$$\begin{aligned}
& Sx_1^{12}x_2^4x_3^3x_4^1 + Sx_1^8x_2^6x_3^4x_4^2 + Sx_1^{10}x_2^8x_3^2 + Sx_1^{12}x_2^4x_3^2x_4^2 \\
+ \;& Sx_1^9x_2^6x_3^4x_4^1 + Sx_1^8x_2^8x_3^3x_4^1 + Sx_1^9x_2^8x_3^3 + Sx_1^{10}x_2^4x_3^4x_4^2 \\
+ \;& Sx_1^{12}x_2^6x_3^2 + Sx_1^8x_2^5x_3^5x_4^2 + Sx_1^{12}x_2^5x_3^3 + Sx_1^9x_2^5x_3^4x_4^2 \\
+ \;& Sx_1^{10}x_2^8x_3^1x_4^1 + Sx_1^{13}x_2^4x_3^2x_4^1 + Sx_1^{12}x_2^6x_3^1x_4^1 + Sx_1^8x_2^7x_3^4x_4^1 \\
+ \;& Sx_1^9x_2^6x_3^5 + Sx_1^9x_2^4x_3^4x_4^3
\end{aligned}$$

The class $f(\bar{w}_3^7)$:

$$\begin{aligned}
& Sx_1^{12}x_2^6x_3^3 + Sx_1^{10}x_2^6x_3^5 + Sx_1^{14}x_2^4x_3^2x_4^1 + Sx_1^{12}x_2^4x_3^3x_4^2 \\
+ \;& Sx_1^8x_2^6x_3^5x_4^2 + Sx_1^{10}x_2^5x_3^4x_4^2 + Sx_1^8x_2^5x_3^4x_4^4 + Sx_1^{12}x_2^5x_3^4 \\
+ \;& Sx_1^8x_2^6x_3^6x_4^1 + Sx_1^9x_2^8x_3^2x_4^2 + Sx_1^8x_2^7x_3^4x_4^2 + Sx_1^{10}x_2^8x_3^2x_4^1 \\
+ \;& Sx_1^9x_2^4x_3^4x_4^4 + Sx_1^{10}x_2^4x_3^4x_4^3 + Sx_1^{12}x_2^5x_3^2x_4^2 + Sx_1^8x_2^8x_3^3x_4^2 \\
+ \;& Sx_1^{10}x_2^6x_3^4x_4^1 + Sx_1^{12}x_2^4x_3^4x_4^1 + Sx_1^9x_2^8x_3^4 + Sx_1^8x_2^8x_3^4x_4^1 \\
+ \;& Sx_1^{12}x_2^6x_3^2x_4^1 + Sx_1^{10}x_2^8x_3^3
\end{aligned}$$

The class $f(\bar{w}_3^5\gamma_9)$:

$$\begin{aligned}
& Sx_1^{16}x_2^4x_3^2x_4^2 + Sx_1^{10}x_2^8x_3^4x_4^2 + Sx_1^{10}x_2^8x_3^5x_4^1 + Sx_1^{16}x_2^5x_3^3 \\
+ \;& Sx_1^{12}x_2^8x_3^3x_4^1 + Sx_1^{16}x_2^6x_3^2 + Sx_1^{10}x_2^9x_3^5 + Sx_1^9x_2^8x_3^5x_4^2 \\
+ \;& Sx_1^8x_2^8x_3^6x_4^2 + Sx_1^{11}x_2^8x_3^4x_4^1 + Sx_1^{12}x_2^1 0x_3^2 + Sx_1^{12}x_2^8x_3^2x_4^2 \\
+ \;& Sx_1^{16}x_2^6x_3^1x_4^1 + Sx_1^{13}x_2^8x_3^2x_4^1 + Sx_1^{12}x_2^9x_3^3 + Sx_1^{16}x_2^4x_3^3x_4^1 \\
+ \;& Sx_1^{17}x_2^4x_3^2x_4^1 + Sx_1^9x_2^9x_3^4x_4^2 + Sx_1^8x_2^8x_3^5x_4^3 + Sx_1^{12}x_2^1 0x_3^1x_4^1
\end{aligned}$$

The class $f(\bar{w}_3^5\gamma_5^2)$:

$$\begin{aligned}
& Sx_1^8x_2^8x_3^5x_4^4 + Sx_1^{18}x_2^4x_3^2x_4^1 + Sx_1^{16}x_2^4x_3^4x_4^1 + Sx_1^9x_2^8x_3^6x_4^2 \\
+ \;& Sx_1^{12}x_2^1 0x_3^3 + Sx_1^{16}x_2^5x_3^4 + Sx_1^{10}x_2^9x_3^6 + Sx_1^{16}x_2^5x_3^2x_4^2 \\
+ \;& Sx_1^{12}x_2^1 0x_3^2x_4^1 + Sx_1^{12}x_2^9x_3^4 + Sx_1^{10}x_2^9x_3^4x_4^2 + Sx_1^{14}x_2^8x_3^2x_4^1
\end{aligned}$$

$$\begin{aligned}
&+ \quad Sx_1^8x_2^8x_3^6x_4^3 + Sx_1^{10}x_2^8x_3^6x_4^1 + Sx_1^{10}x_2^1 0x_3^4x_4^1 + Sx_1^{16}x_2^6x_3^2x_4^1 \\
&+ \quad Sx_1^{16}x_2^4x_3^3x_4^2 + Sx_1^{12}x_2^8x_3^3x_4^2 + Sx_1^9x_2^8x_3^4x_4^4 + Sx_1^{12}x_2^9x_3^2x_4^2 \\
&+ \quad Sx_1^{16}x_2^6x_3^3 + Sx_1^{11}x_2^8x_3^4x_4^2
\end{aligned}$$

The class $f(\bar{w}_3^6\gamma_9)$:

$$\begin{aligned}
&\quad\;\; Sx_1^{13}x_2^8x_3^4x_4^2 + Sx_1^{16}x_2^4x_3^4x_4^3 + Sx_1^{16}x_2^6x_3^5 + Sx_1^{14}x_2^8x_3^4x_4^1 \\
&+ \quad Sx_1^{16}x_2^6x_3^4x_4^1 + Sx_1^{10}x_2^8x_3^8x_4^1 + Sx_1^{10}x_2^9x_3^4x_4^4 + Sx_1^9x_2^8x_3^6x_4^4 \\
&+ \quad Sx_1^{12}x_2^8x_3^6x_4^1 + Sx_1^{16}x_2^5x_3^4x_4^2 + Sx_1^8x_2^8x_3^8x_4^3 + Sx_1^{12}x_2^1 2x_3^2x_4^1 \\
&+ \quad Sx_1^9x_2^8x_3^8x_4^2 + Sx_1^{12}x_2^1 0x_3^5 + Sx_1^8x_2^8x_3^6x_4^5 + Sx_1^{10}x_2^9x_3^8 \\
&+ \quad Sx_1^{10}x_2^8x_3^5x_4^4 + Sx_1^{16}x_2^8x_3^3 + Sx_1^{12}x_2^8x_3^5x_4^2 + Sx_1^{12}x_2^9x_3^6 \\
&+ \quad Sx_1^{20}x_2^4x_3^2x_4^1
\end{aligned}$$

These classes are easily checked and seen to be independent over $\mathcal{D}$. Thus these seven classes, together with $\{1\}$ and one other class freely generate the $\mathcal{A}_7$ invariant subring over $\mathcal{D}$. To find the last class note that the numerator in the Poincaré series for a Cohen-Macaulay ring satisfies a symmetry condition of the form, the coefficient of x^n is equal to the coefficient of x^{l-n} for some fixed l. Here, the only possibility for l is 45, and the result follows. ∎

4. The cohomology of some sporadic simple groups

In this section we will provide explicit applications of our invariant calculations to determine the mod(2) cohomology of certain important simple groups. We begin with a "small" group.

Example (4.1). Set $G = M_{11}$, the first Mathieu group having order $11 \times 10 \times 9 \times 8 = 7920 = 2^4\ 3^2\ 5\ 11$. M_{11} has 2-rank two with one conjugacy class of groups $(\mathbb{Z}/2)^2$ and one conjugacy class of involutions. From the Atlas, [C], $N(2A) = 2\cdot\Sigma_4 = GL_2(\mathbb{F}_3)$, and we can also check that $N((\mathbb{Z}/2)^2) = \Sigma_4$. Thus the quotient $A_2(M_{11})/M_{11}$ has the form

$$\begin{tikzcd} \Sigma_4 \bullet \arrow[rr, dash, "D_8"'] & & \bullet GL_2(\mathbb{F}_3) \end{tikzcd}$$

Apply (2.1) to obtain the formula

$$H^*(M_{11}) \oplus H^*(D_8) = H^*(GL_2(3)) \oplus H^*(\Sigma_4)$$

and substitute for $H^*(\Sigma_4)$ using the formula in (2.5) to get

$$H^*(M_{11}) \oplus H^*(V_2)^{\mathbb{Z}/2} \cong H^*(GL_2(3)) \oplus H^*(V_2)^{GL_2(\mathbb{F}_2)}.$$

From Quillen's results, [Q2], the Poincaré series for $H^*(GL_2(\mathbb{F}_3))$ is

$$\frac{(1+t)(1+t^3)}{(1-t^2)(1-t^4)} = \frac{1+t+t^2+t^3+t^4+t^5}{(1-t^3)(1-t^4)}$$

and so the Poincaré series for M_{11} is (as first computed in [We1], compare [BC])

$$\frac{(1+t+t^2+t^3+t^4+t^5)+(1+t^2)-(1+t^2)(1+t+t^3)}{(1-t^3)(1-t^4)} = \frac{1+t^5}{(1-t^3)(1-t^4)}.$$

The group $GL_2(\mathbb{F}_3)$ contains a Sylow 2-subgroup of M_{11} and has its $\mathrm{mod}(2)$ cohomology detected on its elementary 2-subgroups. Consequently the same is true for M_{11}. It follows that

$$H^*(M_{11};\mathbb{F}_2) \subseteq \mathbb{F}_2[x_1,x_2]^{GL_2(\mathbb{F}_2)} = \mathbb{F}_2[D_2,D_3],$$

and since there is only one element in each of the dimensions 3, 4, and 5 in this ring we have an independent proof of the result that $H^*(M_{11}) \cong \mathbb{F}_2[D_3,D_2^2](1,D_2D_3)$. In particular note that $H^*(M_{11})$ is Cohen-Macaulay.

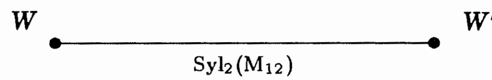
Example (4.2). We now consider the second Mathieu group M_{12} of order $95040 = 2^6\, 3^3\, 5\, 11$. Its poset space is much more complex since it has two distinct conjugacy classes of involutions as well as three distinct conjugacy classes of $(\mathbb{Z}/2)^3$'s and four conjugacy classes of $(\mathbb{Z}/2)^2$. For the details of its structure see [AMM2]. In particular, from [AMM2] we have that the Poincaré series for $H^*(M_{12};\mathbb{F}_2)$ is

$$P_{M_{12}}(t) = \frac{1+t^2+3t^3+t^4+3t^5+4t^6+2t^7+4t^8+3t^9+t^{10}+3t^{11}+t^{12}+t^{14}}{(1-t^4)(1-t^6)(1-t^7)}$$

and, in fact, $H^*(M_{12};\mathbb{Z}/2)$ is Cohen-Macaulay over the polynomial subring $\mathbb{F}_2[D_4,D_6,D_7]$. Since $H^*(BG_2;\mathbb{F}_2) \cong \mathbb{F}_2[D_4,D_6,D_7]$ as well, this gives the indication of some connections between M_{12} and the exceptional Lie group G_2. Of course $M_{12} \not\subset G_2$ so the connection is not nearly as simple as group inclusion. However, in M_{12} there are two maximal subgroups of order 192. The first is the holomorph of the quaternion group Q_8 which we write W, and the second is a split extension

$$(\mathbb{Z}/4)^2 \rtimes (\mathbb{Z}/2 \times \Sigma_3)$$

where the $\mathbb{Z}/2$ acts to invert elements in $(\mathbb{Z}/4)^2$. But W' is also seen to be the extension of the elements of order 4 in the usual torus of $G_2(q)$ for $q \cong 3,5 \bmod (8)$ by the Weyl group of G_2. These groups both contain $Syl_2(M_{12})$ so there is a configuration



contained in M_{12}. In [AMM2] it is shown that this configuration completely determines $H^*(M_{12};\mathbb{F}_2)$ as the intersection in $H^*(Syl_2(M_{12}))$ of the restriction images of $H^*(W)$ and $H^*(W')$. On the other hand, recalling the group E discussed in 2.7 and the fact that $W \cong \pi^{-1}(P_1)$, $W' \cong \pi^{-1}(P_2)$, we have a very similar configuration in E. However, the two configurations are not isomorphic and here the deviation between the cohomology of E and that of M_{12} is explained by (2.2). Indeed, applying (2.2) we have

$$H^*(E) \cong H^*(M_{12}) \oplus (H^*(V_3) \otimes St(GL_3(\mathbb{F}_2)))^{GL_3(\mathbb{F}_2)}. \tag{4.3}$$

On the other hand, using [M] we can give an independent evaluation of $H^*(E)$ using the fact that E appears as the normalizer of one of the two $(\mathbb{Z}/2)^3$ tori in $G_2(\mathbb{F}_q)$ for $q \cong 5 \bmod (8)$, [FM]. In particular the result is

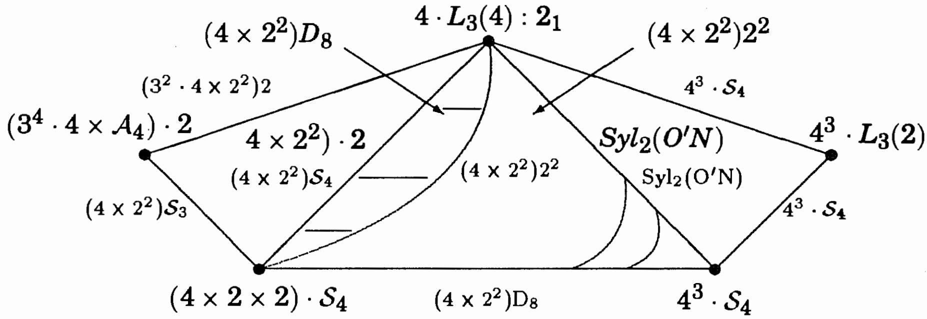
$$P_E(t) = \frac{q(t)}{(1-t^4)(1-t^6)(1-t^7)}$$

where $q(t) = 1+x^2+3x^3+2x^4+4x^5+5x^6+4x^7+5x^8+4x^9+2x^{10}+3x^{11}+x^{12}+x^{14}$. Note that E is also a subgroup of the compact 14 dimensional Lie group G_2 and $q(t)$ is the Poincaré series of the compact 14 dimensional manifold G/E, [M]. It follows that the error term $(H^*(V_3) \otimes St(GL_3(\mathbb{F}_2)))^{GL_3(\mathbb{F}_2)}$ has Poincaré series

$$e_3(t) = \frac{t^4+t^5+t^6+2t^7+t^8+t^9+t^{10}}{(1-t^4)(1-t^6)(1-t^7)},$$

a result which is useful for understanding the group $O'N$.

Example (4.4). The O'Nan group $O'N$ has order $460,815,505,920 = 2^9\,3^4\,5\,7^3\,11\,19\,31$, and in [AM2] we determine its poset space, obtaining the following picture.



From this picture some easy cancellations give

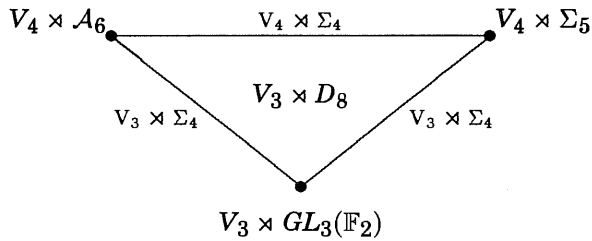
$$H^*(O'N) \oplus H^*((\mathbb{Z}/4)^3 \cdot \Sigma_4) \cong H^*((\mathbb{Z}/4)^3 \cdot GL_3(\mathbb{F}_2)) \oplus H^*(\mathbb{Z}/4 \cdot SL_3(\mathbb{F}_4) \rtimes \mathbb{Z}/2),$$

and using (2.2) this can be reduced to

$$\begin{aligned} H^*(O'N) &\oplus H^*(S)^{\mathbb{Z}/2} \cong H^*(S)^{\Sigma_3} \oplus (H^*(V_3) \otimes St))^{GL_3(\mathbb{F}_2)} \\ &\oplus H^*(\mathbb{Z}/4 \cdot SL_3(\mathbb{F}_4) \rtimes \mathbb{Z}/2) \end{aligned}$$

where $S \cong (\mathbb{Z}/4)^3 \cdot (\mathbb{Z}/2)^2$. The terms involving S are reasonably direct to evaluate while the term involving the Steinberg module is discussed in (4.2). The final term is essentially determined in [AM1]. The $\mathcal{A}_5$ invariants discussed in §3 play a key part in the work there.

Example (4.5). For the third Mathieu group M_{22} we use a sporadic geometry described in [RSY] which also satisfies the hypotheses necessary for (2.1) to remain valid. The associated complex has the form



We apply (2.1), (2.2), to this picture and, after some cancellation we have

$$\begin{aligned} H^*(M_{22}) \oplus H^*(Syl_2(M_{22})) &\cong H^*(V_4 \rtimes \Sigma_4) \oplus H^*(V_4 \rtimes \Sigma_5) \\ &\oplus (H^*(V_3) \otimes St)^{GL_3(\mathbb{F}_2)} \oplus (H^*(V_4) \otimes St)^{\mathcal{A}_6}. \end{aligned}$$

From this, the results in [AM3], and the discussion of invariants in §3 we obtain the Poincaré series for M_{22}. The formula is very messy and not too illuminating, so we defer details to a further paper.

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