ON THE K-THEORY OF THE CLASSIFYING SPACE OF A DISCRETE GROUP

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Dedicated to the memory of José Adem (1921–1991).

§0. INTRODUCTION

Let G be a finite group. By considering the action of G on itself by conjugation, one arrives at the class equation for G,

$$1 - \frac{1}{|G|} = \sum_{\substack{(g)\\ g\neq 1}} \frac{1}{|C(g)|} \tag{0.1}$$

where g ranges over conjugacy classes of elements in G, with centralizer $C(g)$.

Now assume that Γ is a discrete group of finite virtual cohomological dimension with centralizers of elements of finite order in Γ homologically finite (for example $SL_n(\mathbb{Z})$). Brown [B2] proved the generalization of (0.1) to Γ:

$$\tilde{\chi}(\Gamma) - \chi(\Gamma) = \sum_{\substack{(\gamma)\\ \gamma\neq 1}} \chi(C(\gamma)) \tag{0.2}$$

where γ ranges over conjugacy classes of elements of finite order in Γ, $\tilde{\chi}$ denotes the topological Euler characteristic and χ the usual group-theoretic version (if Γ' is a torsion–free subgroup of finite index in Γ then by definition $\chi(\Gamma) = \frac{\tilde{\chi}(\Gamma')}{[\Gamma:\Gamma']}$). The

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point of (0.2) is that the difference between $\tilde{\chi}(\Gamma)$ and $\chi(\Gamma)$ is determined by the torsion in Γ, in particular if Γ is torsion-free they coincide.

Now let us recall a well-known result in algebra: let $R(G)$ denote the character ring of a finite group G. Then

$$\dim_{\mathbb{C}} R(G)\otimes\mathbb{C}-1=\begin{array}{l}\text{No. of conjugacy}\\ \text{classes of non-trivial}\\ \text{elements in } G\,.\end{array}\tag{0.3}$$

The term on the left can be identified with $\dim_{\mathbb{C}} K_G^*(*)\otimes\mathbb{C}$, the dimension of the equivariant K–theory of a point. In this paper we will use this interpretation to outline an approach for generalizing this formula to infinite groups Γ of finite virtual cohomological dimension (with suitable finiteness conditions), motivated by (0.2).

By a result due to Serre [S], Γ will act on a finite dimensional, contractible space X, with finite isotropy, and such that $X^H\simeq *$ for all $H\subseteq\Gamma$ finite. Fix an extension $1\to\Gamma'\to\Gamma\xrightarrow{\pi}G\to 1$ with Γ' torsion-free, G finite. Then Γ' acts freely on X, G acts on X/Γ' such that the finite subgroups in Γ correspond under π to the isotropy subgroups in G.

We have

Theorem 3.1

$$K_G^*(X/\Gamma')\otimes\mathbb{C}\cong K^*(B\Gamma')^G\otimes\mathbb{C}\oplus\bigoplus_{\gamma\in\mathcal{C}-\{1\}}K^*(B(C(\gamma)\cap\Gamma'))^{H_\gamma}\otimes\mathbb{C}$$

where $\mathcal{C}$ is the set of conjugacy classes of elements of finite order in Γ, and $H_\gamma = C(\gamma)/C(\gamma)\cap\Gamma'$, a finite group. ∎

An immediate corollary of this is the formula

$$\chi(K_G^*(X/\Gamma')\otimes\mathbb{C})-\tilde{\chi}(\Gamma)=\sum_{\gamma\in\mathcal{C}-\{1\}}\tilde{\chi}(C(\gamma))\,.\tag{0.4}$$

where $\chi(K_G^*(X/\Gamma')\otimes\mathbb{C})=\dim_{\mathbb{C}}K_G^0(X/\Gamma')\otimes\mathbb{C}-\dim_{\mathbb{C}}K_G^1(X/\Gamma')\otimes\mathbb{C}$.

The ingredient which makes (0.3) useful is the fact that $R(G)^\wedge \cong K^*(BG)$ (the completion theorem). Let Γ act on $X \times EG$ diagonally and through $\Gamma \xrightarrow{\pi} G$; then $B\Gamma \simeq X/\Gamma' \times_G EG$, whence the completion theorem implies $K^*_G(X/\Gamma')^\wedge \cong K^*(B\Gamma)$, and so, up to IG-adic completion, the preceding results provide information on $K^*(B\Gamma)$. For example, if G is a p-group completion is p-adic completion, and hence (calculating over the p–adics $\mathbb{C}_p$),

$$\chi(K^*(B\Gamma) \otimes \mathbb{C}_p) - \tilde{\chi}(\Gamma) = \sum_{\gamma \in \mathcal{C} - \{1\}} \tilde{\chi}(C(\gamma))\,. \tag{0.5}$$

We point out that Γ will always contain subgroups of this form (G a p-group) with finite index.

The key technical device which we use is a result of N. Kuhn [K] expressing $K^*_G(Y) \otimes \mathbb{C}$ as a sum of $K^*(Y^{<g>}/C(g)) \otimes \mathbb{C}$, as g ranges over conjugacy classes of elements in G.

The paper is organized as follows: in §1 we describe the Γ-complex X; in §2 we outline the main properties of equivariant K-theory, in §3 we prove our result and in §4 we provide examples, as well as a p-local version of the results in §3: using p–adic K–theory we obtain an exact formula (4.2).

Formula (3.1) indicates that the K-theory of $B\Gamma$ can be calculated given enough information about its elements of finite order and their centralizers. It is interesting to compare this with results obtained by Brown (see [B3]) for computing the (high–dimensional) *cohomology* of $B\Gamma$. In the general situation there is a spectral sequence involving the cohomology of the normalizers of the finite subgroups, converging to the cohomology of Γ in sufficiently high dimensions (or in any dimension if Farrell Cohomology is used). This can be difficult to deal with, except in the rank one situation, where the spectral sequence only has one line. In contrast, $K^*(B\Gamma) \otimes \mathbb{C}$ seems to be much more accessible in terms of subgroup data, a fact which is of course evident for finite groups. One may expect that by using the Atiyah–Hirzebruch spectral sequence or other techniques, this can (in some cases) yield information on the cohomology.

§1. GROUPS OF FINITE VCD

Definition 1.1: *A discrete group Γ is said to have finite virtual cohomological dimension (vcd $\Gamma < \infty$) if there is a subgroup of $\Gamma' \subseteq \Gamma$ of finite index such that Γ' has finite cohomological dimension.* ∎

Examples of this type of group include arithmetic groups (such as $SL_n(\mathbb{Z})$) and mapping class groups.

The key geometric ingredient in the analysis of these groups is a result due to Serre [S]:

Theorem 1.2 (Serre)

If Γ is a group with vcd$\Gamma < \infty$, then there exists a finite dimensional, proper contractible Γ-complex X with the following additional property: X^H is contractible for all finite subgroups $H \subseteq \Gamma$. ∎

This can be easily summarized as follows. Fix $\Gamma' \subseteq \Gamma$ a *normal* subgroup of finite cohomological dimension. Then by the well-known result of Eilenberg and Ganea [EG] we can find a finite dimensional $K(\Gamma', 1)$ complex, whose universal cover X' is a contractible Γ'-complex. If $r = [\Gamma : \Gamma']$, then Γ acts faithfully as a group of automorphisms of the principal Γ'-bundle $\Gamma \to \Gamma/\Gamma'$ and hence embeds in the full automorphism group $\Sigma_r \times_T (\Gamma')^r$. As the latter group acts on $(X')^r$, we obtain a Γ-action on $X = (X')^r$. One checks that this satisfies all the conditions. In particular [B] if $H \subseteq \Gamma$ is finite, $X^H \cong (X')^k$, where $k = [\Gamma : \Gamma']/|H|$, hence it is contractible.

In a more abstract language, X can be regarded as an "$E_\Gamma(\Gamma')$-space." We recall what that means. Given $H \triangleleft G$, let $F_G(H)$ be the family of all subgroups $K \subseteq G$ such that $K \cap H = \{e\}$. Then a G-space X is said to be an $E_G(H)$-space if

(1) $X^K = \emptyset$ for all $K \notin F_G(H)$

(2) X^K is contractible for all $K \in F_G(H)$.

The orbit space $E_G(H)/H$ is called a $B_G H$ space, and as a G/H-space is unique up to weak G/H-equivalence.

In our situation $F_\Gamma(\Gamma')$ consists of all finite subgroups in Γ, whence Serre's construction is a particular finite dimensional model for the generalized classifying space $B_\Gamma\Gamma'$; (see [tD] for more on this).

Let us now fix an extension

$$1 \to \Gamma' \to \Gamma \to G \to 1$$

where $cd\Gamma' < \infty$, $|G| < \infty$. We will now make a detailed examination of the fixed-point sets $X^{\langle\gamma\rangle}$, where $\gamma \in \Gamma$ is an element of finite order, $\langle\gamma\rangle$ the subgroup it generates, and X a complex as in 1.2.

First we have

Lemma 1.3: *Let $g \in G$, then*

$$(X/\Gamma')^{\langle g\rangle} = \coprod_{H \in \mathcal{C}} X^H/\Gamma' \cap N(H) \simeq \coprod_{H \in \mathcal{C}} B(\Gamma' \cap N(H))$$

where $\mathcal{C}$ ranges over all Γ'-conjugacy classes of finite subgroups $H \subseteq \Gamma$ mapping onto $\langle g\rangle$.

Proof: This is explicitly described in [B], pg. 267, and follows from looking at inverse images under $\pi : \Gamma \to G$ and the additional fact that X^H is a free, contractible $\Gamma' \cap N(H)$–space. ∎

Now consider the $C(g)$-action on this fixed-point set. We describe its orbit space as follows

Lemma 1.4:

$$\coprod_{(g)} (X/\Gamma')^{<g>}/C(g) = \coprod_{(\gamma)} X^{<\gamma>}/C(\gamma)$$

where the elements range over conjugacy classes of elements of finite order in G and Γ respectively.

Proof: Let $\widetilde{C(g)} = \pi^{-1}(C(g))$, then this group acts on $\coprod_{\gamma\in\tilde{\mathcal{C}}} X^\gamma$, where $\tilde{\mathcal{C}}$ is the collection of all elements of finite order in Γ mapping onto g. If $x \in X^\gamma$, $c \in \widetilde{C(g)}$, take cx; clearly $cx \in X^{c\gamma c^{-1}}$, and $\pi(c\gamma c^{-1}) = g$.

We now take the orbit space

$$\left(\coprod_{\gamma\in\tilde{\mathcal{C}}} X^\gamma\right) / \widetilde{C(g)} = \coprod_{\gamma\in\mathcal{C}} X^\gamma / \widetilde{C(g)} \cap C(\gamma)$$

where now

$$\mathcal{C} = \begin{array}{l} \widetilde{C(g)}\text{-conjugacy classes of} \\ \gamma \in \Gamma \text{ of finite order, } \pi(\gamma) = g\,. \end{array}$$

Clearly $C(\gamma) \subseteq \widetilde{C(g)}$, hence $\widetilde{C(g)} \cap C(\gamma) = C(\gamma)$.

Next we claim that if $\gamma \in \Gamma$ of finite order, $\pi(\gamma) = g$, and if $\gamma' = \gamma_1\gamma\gamma_1^{-1}$ also satisfying $\pi(\gamma') = g$, then $\gamma_1 \in \widetilde{C(g)}$. For we have

$$\pi(\gamma_1) g \pi(\gamma_1)^{-1} = g, \text{ hence } \pi(\gamma_1) \in C(g)$$

and so $\gamma_1 \in \widetilde{C(g)}$.

As elements of finite order in Γ are mapped injectively under π, as we range over conjugacy classes of elements of G, we obtain the asserted disjoint union. ∎

We have the following cohomological corollary of 1.4.

Corollary 1.5:

$$H^*\left(\coprod_{(g)} (X/\Gamma')^{\langle g\rangle}/C(g), \mathbb{Q}\right) \cong \bigoplus_{\substack{(\gamma) \\ \text{finite order}}} H^*(BC(\gamma), \mathbb{Q})\,.$$

Proof: We have extensions

$$1 \to \Gamma' \cap C(\gamma) \to C(\gamma) \to H_\gamma \to 1$$

where H_γ is *finite*. Hence we obtain

$$H^*(BC(\gamma),\mathbb{Q}) \cong H^*(B\Gamma' \cap C(\gamma),\mathbb{Q})^{H_\gamma} \cong H^*(X^{<\gamma>}/C(\gamma),\mathbb{Q}),$$

the last equality because $X^{<\gamma>}$ is a contractible $C(\gamma)$-space with a free $C(\gamma)\cap\Gamma'$-action. ∎

§2. EQUIVARIANT K-THEORY

We recall some well-known facts about G-equivariant K-theory, for G a finite group. Our main reference is [A-S1]. In this section we assume Y is a compact G-space.

$K^*_G(Y)$ is defined by using G-vector bundles over Y. Using the natural map $Y \to *$, we have a homomorphism

$$K^*_G(*) \to K^*_G(Y)$$

with which $K^*_G(Y)$ is a (we will assume finitely generated) $K^*_G(*)$-module. Recall that $K^*_G(*) = R(G)$, the complex character ring of the finite group G. Denote by $Y_G = (Y \times EG)/G$ the usual Borel construction. If F is a G-vector bundle on Y, then $(F \times EG)/G$ is a vector bundle on Y_G; the assignment $F \mapsto (F\times EG)/G$ is additive hence it induces a homomorphism

$$\alpha : K^*_G(Y) \to K^*(Y \times_G EG)\,.$$

Here the term on the right is defined as

$$\lim_{\leftarrow} K^*_G(Y \times EG^{(n)})$$

for a suitable filtration of EG, and the map above can be alternatively constructed as follows: the natural projection $Y \times EG^n \to Y$ induces

$$K^*_G(Y) \xrightarrow{\alpha_n} K^*(Y \times_G EG^n)\,.$$

In particular, if IG^n denotes the kernel of $R(G) = K^*_G(*) \to K^*_G(EG^n)$, then α_n factors

$$K^*_G(Y)/IG^n K^*_G(Y) \xrightarrow{\alpha_n} K^*(Y \times_G EG)\,.$$

The main result due to Atiyah and Segal concerning this map is

Theorem 2.1 (Completion Theorem)

Let Y be a compact G-space such that $K^*_G(Y)$ is finite over $R(G)$. Then the homomorphisms

$$\alpha_n : K^*_G(Y)/IG^n \longrightarrow K^*_G(Y \times EG^n)$$

induce an isomorphism of pro-rings. ∎

In particular if $IG \subset R(G)$ is the augmentation ideal, and $K^*_G(Y)$ is endowed with the IG-adic topology, then 2.1 can be rephrased as saying that

$$\alpha : K^*_G(Y) \longrightarrow K^*(Y \times_G EG)$$

induces an isomorphism of the IG-adic completion of $K^*_G(Y)$ with $K^*(Y \times_G EG)$.

The principle then is that $K^*_G(Y)$ can be *approximated* by $K^*(Y \times_G EG)$.

To conclude this section we describe a more recent result due to N. Kuhn [K] relating $K^*_G(Y)$ to the K-theory of $Y^{<g>}/C(g)$. Namely, he proves that there is an isomorphism

$$K^*_G(Y) \otimes \mathbb{C} \cong \bigoplus_{(g)} K^*(Y^{<g>}/C(g)) \otimes \mathbb{C} \tag{2.2}$$

where the sum is taken over all conjugacy classes of $g \in G$ (note the case $Y = *$, we recover the formula for $\dim_{\mathbb{C}} R(G) \otimes \mathbb{C}$). This correspondence can be outlined as follows. Let F be a G-vector bundle on Y; then on $F\big|_{Y^{<g>}}$ the element g still acts, leaving points in the base fixed. Therefore $F\big|_{Y^{<g>}}$ will split as a direct sum of vector bundles corresponding to the eigenspaces of g; putting the eigenvalue in the second factor gives an element in $K(Y^{<g>}) \otimes \mathbb{C}$. $C(g)$ acts on $Y^{<g>}$, hence we can take invariants to obtain an element in $K(Y^{<g>})^{C(g)} \otimes \mathbb{C} = K(Y^{<g>}/C(g)) \otimes \mathbb{C}$; the same holds for $K^1_G(Y)$. This description is also given in [H-H] , [A-S2], where it is pointed out that this is related to work in string theory concerning orbifold Euler characteristics.

§3. APPLICATION TO $K^*(B\Gamma)$

The goal of this section will be to use the preceding results to obtain an approximation to $K^*(B\Gamma)$, for Γ a discrete group of finite v.c.d., with suitable finiteness assumptions.

As before, fix an extension

$$1 \to \Gamma' \to \Gamma \to G \to 1$$

where cd $\Gamma' < \infty$, $|G| < \infty$. Let X be an admissible Γ-complex as before, with suitable (e.g. compactness) finiteness assumptions on X/Γ'. Then, using $\Gamma \to G$, Γ acts diagonally on $X \times EG$ *without* any non–trivial isotropy. As $X \times EG \sim *$, this means

$$B\Gamma \simeq (X \times EG)/\Gamma = X/\Gamma' \times_G EG$$

and hence

$$K^*(B\Gamma) \cong K^*(X/\Gamma' \times_G EG)\,.$$

It is now evident that the completion theorem provides a method for approaching $K^*(B\Gamma)$, namely via the map

$$K^*_G(X/\Gamma') \xrightarrow{\alpha} K^*(B\Gamma)$$

which will induce an isomorphism $K^*_G(X/\Gamma')^\wedge \cong K^*(B\Gamma)$.

We have

Theorem 3.1:

$$K_G^*(X/\Gamma') \otimes \mathbb{C} \cong \bigoplus_{(\gamma)} K^*(B(C(\gamma) \cap \Gamma'))^{H_\gamma} \otimes \mathbb{C}$$

where γ ranges over conjugacy classes of elements of finite order in Γ, and $H_\gamma = C(\gamma)/C(\gamma) \cap \Gamma'$, a finite group.

Proof: Simply combine (2.2) with (1.4) and identify

$$\begin{aligned} K^*(X^{<\gamma>}/C(\gamma)) \otimes \mathbb{C} &= K^*([X^{<\gamma>}/C(\gamma) \cap \Gamma']/H_\gamma) \otimes \mathbb{C} \\ &= K^*(B(C(\gamma) \cap \Gamma'))^{H_\gamma} \otimes \mathbb{C}\,. \end{aligned}$$

∎

From this we derive (if each $C(\gamma)$ is homologically finite):

Corollary 3.2.

$$\chi(K_G^*(X/\Gamma')) = \sum_{\substack{(\gamma) \\ \text{finite order}}} \tilde{\chi}(C(\gamma))\,.$$

Proof: Use that fact that the Euler characteristic of $K^*(Y)$ equals $\chi(Y)$. ∎

What we have is an expression for $K_G^*(X/\Gamma')$ involving only centralizers of elements of finite order in Γ, which after IG-adic completion *determines* $K^*(B\Gamma)$. In some cases this completion process can be straightforward, in particular if G is a p-group, it is just p-adic completion. Computing ranks over $\mathbb{C}_p$ leads to

Theorem 3.3: *Let Γ be a discrete group of finite v.c.d. such that (a) the centralizers $C(\gamma)$ are of finite homological type for all $\gamma \in \Gamma$ of finite order and (b) there is a torsion-free subgroup Γ', normal in Γ and such that $\Gamma/\Gamma' = G$ is a finite p-group. Then*

$$\chi(K^*(B\Gamma) \otimes \mathbb{C}_p) - \tilde{\chi}(\Gamma) = \sum_{\substack{(\gamma) \\ \gamma \text{ of finite} \\ \text{order} \neq 1}} \tilde{\chi}(C(\gamma))\,.$$

∎

Remark: An elementary case of the above results occurs when Γ is torsion-free; then $X/\Gamma \simeq B\Gamma$ and hence $\chi(K^*(B\Gamma)) = \tilde{\chi}(\Gamma)$; on the right-hand side it vanishes because there is no torsion.

Recall now the usual Euler characteristic for discrete groups:

$$\chi(\Gamma) = \tilde{\chi}(\Gamma')/[\Gamma : \Gamma']$$

(we assume it is well-defined). Then the results above should be compared with a theorem due to K. Brown [B2]:

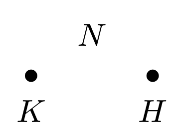
$$\tilde{\chi}(\Gamma) - \chi(\Gamma) = \sum_{\substack{(\gamma)\neq(1) \\ \gamma \text{ finite order}}} \chi(C(\gamma))\,.$$

Brown's result can be considered as the generalization of the "class equation" from finite group theory. Our result is an extension of the formula for $\dim_{\mathbb{C}} R(G) \otimes \mathbb{C}$.

§4. EXAMPLES AND A LOCAL VERSION

Assume we have a group Γ fitting into an extension $1 \to \Gamma' \to \Gamma \to G \to 1$ as described before. Then Γ will contain a subgroup $\Gamma_{(p)}$ of finite index normalizing Γ' and with $\Gamma_{(p)}/\Gamma' \cong \mathrm{Syl}_p(G)$. Hence the class of groups to which the exact formula 3.3 can be applied is a large one. Here we concentrate on familiar examples.

Example 4.1: Let $\Gamma = K *_N H$, the amalgamated product of two finite groups over a common subgroup. In this case the group is virtually free [S] and we may take Γ' to be a free group of finite index. Here X can be taken to be a tree, X/Γ' a finite graph on which $G = \Gamma/\Gamma'$ acts with orbit space



i.e. there are two orbits of vertices, with stabilizers K, H respectively, and one orbit of edges, with stabilizer N. In this case we use a spectral sequence due to Segal [Se] for computing equivariant cohomology (indeed it is a very simple Mayer–Viaetoris sequence in this case) to obtain the exact sequence:

$$0 \to K^0_G(X/\Gamma') \to R(K) \oplus R(H) \to R(N) \to K^1_G(X/\Gamma') \to 0.$$

This may be used to compute $K^*_G(X/\Gamma')$. Compare this to the formula [B]

$$\chi(\Gamma) = \frac{1}{|K|} + \frac{1}{|H|} - \frac{1}{|N|}.$$

This applies to $\Gamma = SL_2(\mathbb{Z}) = \mathbb{Z}/6 *_{\mathbb{Z}/2} \mathbb{Z}/4$ to yield

$$dim_{\mathbb{C}}\, K^*_G(X/\Gamma') \otimes \mathbb{C} = \begin{cases} 0, & \text{if } * \text{ odd;} \\ 8, & \text{if } * \text{ even.} \end{cases}$$

A similar formula can obviously be proved for any virtually free group of finite vcd, taking into account the edge and vertex stabilizers of the corresponding tree.

There is a local way of analyzing K-theory, by using p-adic K-theory, K_p (see [H]). The field $\mathbb{C}$ is replaced by $\mathbb{C}_p$, the completion of the algebraic closure of $\mathbb{Q}_p$, and instead of characters on G we specialize to class functions on

$$Tors_p(G) = \{g \in G \mid g \text{ is of order } p^n, \text{ for some } n \geq 0\}.$$

Then Atiyah's result can be reformulated as

$$K_p(BG) \otimes \mathbb{C}_p \cong \mathbb{C}_p\text{-valued class functions on } Tors_p(G).$$

In this setting we obtain a local version of our main result:

Theorem 4.2:

(i) $$K^*_p(B\Gamma) \otimes \mathbb{C}_p \cong \bigoplus_{\substack{(\gamma) \\ \gamma \in Tors_p(\Gamma)}} K^*_p(B\Gamma' \cap C(\gamma))^{H_\gamma} \otimes \mathbb{C}_p$$

(ii) $$\chi(K^*_p(B\Gamma) \otimes \mathbb{C}_p) = \sum_{\substack{(\gamma) \\ \gamma \in Tors_p(\Gamma)}} \tilde{\chi}(C(\gamma))$$

∎

Example 4.3 : Now let $\Gamma = \mathrm{SL}_3(\mathbb{Z})$, $\Gamma' = \Gamma(3)$, $G = \mathrm{SL}_3(\mathbb{F}_3)$. Note $|G| = 2^4 \cdot 3^3 \cdot 13$ but that Γ has no 13-torsion. Hence we deduce

$$K_p^*(B\Gamma) \otimes \mathbb{C} \cong \bigoplus_{\substack{(\gamma) \\ \gamma \in Tors_p(\Gamma)}} K^*(BC(\gamma) \cap \Gamma(3))^{H_\gamma} \otimes \mathbb{C}_p \quad \text{for } p = 2, 3\,.$$

In particular for $p = 3$, we see that up to conjugacy,

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

are the only elements of finite order in Γ which are in $Tors_p(\Gamma)$. Their centralizers are cyclic of order 3 and cyclic of order 6 respectively. We obtain

$$K_3^*(B\Gamma) \otimes \mathbb{C}_3 \cong K_3^*(X/\Gamma) \otimes \mathbb{C}_3 \oplus [K_3^*(*) \otimes \mathbb{C}_3]^2\,.$$

However, from the work of Soulé [So], X/Γ is homotopically trivial, hence

$$K_3^*(\mathrm{BSL}_3(\mathbb{Z})) \otimes \mathbb{C}_3 \cong [K_3^*(*) \otimes \mathbb{C}_3]^3\,.$$

Similarly, we have that the centralizers of elements of order 2 or 4 in $\mathrm{SL}_3(\mathbb{Z})$ are rationally acyclic, hence $K_2^*(\mathrm{BSL}(\mathbb{Z})) \otimes \mathbb{C}_2$ is of rank equal to the number of distinct conjugacy classes of elements in $\mathrm{SL}_3(\mathbb{Z})$ of order 2 or 4.

Example 4.4 Let k be a totally real number field with ring of integers $\mathcal{O}$, and let ζ_k denote the Dedekind zeta function associated to k. The centralizer of every finite subgroup in $\Gamma = \mathrm{SL}_2(\mathcal{O})$ is finite, except for ± 1. Let $[Tors_p(\Gamma)]$ denote the number of Γ conjugacy classes of non–trivial elements in $Tors_p(\Gamma)$. Then, for any $p \neq 2$

$$K_p^*(\mathrm{BSL}_2(\mathcal{O})) \otimes \mathbb{C}_p \cong K_p^*(X/\Gamma) \otimes \mathbb{C}_p \oplus [\mathbb{C}_p]^{[Tors_p(\Gamma)]}$$

and

$$\chi(K_p^*(\mathrm{BSL}_2(\mathcal{O}))\otimes\mathbb{C}_p)=2\zeta_k(-1)+\sum_{(H)}\big(1-\frac{2}{|H|}\big)+[Tors_p(\Gamma)]$$

where H ranges over Γ-conjugacy classes of maximal finite subgroups. For this formula we use an identity due to K. Brown [B1] for $\tilde{\chi}(\mathrm{SL}_2(\mathcal{O}))$.

Example 4.5: Let $\Gamma = GL_{p-1}(\mathbb{Z})$, p an odd prime. It is well–known that this group has no subgroups of order p^2, and furthermore the number of conjugacy classes of elements of order p in Γ is equal to the class number of p, $Cl(p)$. The centralizer of any such element γ will be isomorphic to the group of units $\mathcal{U}$ in $\mathbb{Z}[\zeta]$, where ζ is a primitive p–th root of unity. It is also well known that $\mathcal{U}$ splits as a direct product $<\gamma>\times\mathbb{Z}^{(p-3)/2}\times\mathbb{Z}/2$. Hence we obtain

$$K_p^*(BGL_{p-1}(\mathbb{Z}))\otimes\mathbb{C}_p\cong K_p^*(B\Gamma')^G\otimes\mathbb{C}_p\oplus\bigoplus_{Cl(p)}K_p^*((S^1)^{(p-3)/2})\otimes\mathbb{C}_p$$

where Γ' is a normal torsion–free subgroup of $GL_{p-1}(\mathbb{Z})$ with finite factor group G. In addition, we obtain that

$$\chi(K_p^*(BGL_{p-1}(\mathbb{Z}))\otimes\mathbb{C}_p)=\tilde{\chi}(GL_{p-1}(\mathbb{Z})).$$

We point out that using results due to Ash [A], formulae of this type can be calculated for $GL_n(\mathbb{Z})$, provided that $p-1\leq n\leq 2p-3$.

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