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The Geometry and Cohomology of the Mathieu Group M_{12}*

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0. INTRODUCTION

It is usually very complex to calculate the cohomology of a finite group G. To date the most significant results have been on the symmetric and alternating groups [N, MM, Ma, Mu, AMM], the general linear groups over a finite field [Q1], and the extra-special p-groups [Q2].

In the list above only the alternating groups and the $GL_n(\mathbb{F}_2)$ are simple. Indeed, among the simple groups only a very few, aside from those above have been understood. It is probable that away from the characteristic, p, $H^*(PSL_n(\mathbb{F}_{p^n}); \mathbb{Z}/q)$ is available, but even this is not obvious, especially for $q=2$. And certainly we currently have no information on most of the families of groups of Lie type, for example, $G_2(\mathbb{F}_q)$, $E_6(\mathbb{F}_q)$, $E_7(\mathbb{F}_q)$, $E_8(\mathbb{F}_q)$, or any of the sporadic groups but M_{11} and J_1.

About 15 years ago Quillen introduced very powerful techniques that allowed for a much deeper understanding of group cohomology. However, it is also apparent that successful calculations over $\mathbb{Z}/p$ have only occurred when the group has a well-behaved lattice of p-elementary abelian sub-

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groups. Formidable combinatorial problems arise in the general situation, as well as necessarily complicated multiplicative relations. For example, the general calculation of $H^*(GL_n(\mathbb{F}_p); \mathbb{Z}/p)$ remains quite inaccessible, and there has been only marginal progress since Quillen's landmark results.

In this paper we calculate the mod 2 cohomology of the Mathieu Group M_{12}, 1 of the 26 sporadic simple groups. It has order 95,040, and can be represented as a collineation group in $PG(5,3)$ (the projective space of five dimensions over the field of three elements), leaving invariant a configuration of 12 points. Of the finite simple groups not belonging to infinite families, only M_{11} (of order 7,920) and J_1 (of order 175,560, with elementary abelian 2-Sylow subgroup) have been studied cohomologically [BC, Ch], both cases giving fairly immediate calculations. The group M_{12}, however, is considerably more complicated, being a rank 3 group with a 2-Sylow subgroup of order 64.

The methods used here combine the classical double coset formula of Cartan and Eilenberg [CE] with some of Quillen's techniques. First we determined the poset space of elementary abelian 2-groups (as well as their normalizers) in M_{12} (Section 2 for definitions); next we found that in our list of isotropy subgroups, there were two non-isomorphic groups of order 192. The first centralizes one of the two non-conjugate involutions and one of the three conjugate classes of $(\mathbb{Z}/2)^3$'s in M_{12}. It is quite well understood. The second normalizes one of the four conjugate classes of $\mathbb{Z}/2 \times \mathbb{Z}/2$'s and the second of the three $(\mathbb{Z}/2)^3$'s. As far as we can determine it first occurs in the literature in [G] in a completely different context, realizing M_{12} as a quotient of the amalgamated free product of the two groups of order 192 above over their intersection. This is obtained by considering automorphisms of trivalent graphs. In Section 5 we discuss this approach and, in fact, prove that at the prime $p=2$, the cohomology of M_{12} is isomorphic to that of the amalgamated product (5.1).

Even from our initial point of view, the double coset structure of W, W', seemed to contain most of the cohomological information about M_{12}. This became more precise with:

THEOREM 3.1. *Let H be a 2-Sylow subgroup of M_{12}. There exist two non-isomorphic subgroups W and W' of order* 192 *in M_{12} with H as their intersection such that*

$$H^*(M_{12}; \mathbb{Z}/2) \cong \operatorname{im}(\operatorname{res}_H^W)^* \cap \operatorname{im}(\operatorname{res}_H^{W'})^*$$

in $H^*(H; \mathbb{Z}/2)$.

Remark. Using a result of [We] together with our explicit analysis of the poset space for M_{12} and the calculation of the cohomology groups of all of its isotropy groups we obtain the Poincaré series for $H^*(M_{12}; \mathbb{Z}/2)$ as well. It is

$$p(t) = \frac{(1+t^3)^2}{(1-t^2)(1-t^3)(1-t^4)}$$

though this does not reflect the actual ring structure very well. The cohomology ring turns out to be Cohen–Macaulay, i.e., freely and finitely generated over a polynomial subalgebra, in this case a polynomial algebra on generators in dimensions 4, 6, and 7 isomorphic to the Dickson algebra $H^*((\mathbb{Z}/2)^3; \mathbb{Z}/2)^{GL_3(\mathbb{Z}/2)} \subset H^*((\mathbb{Z}/2)^3; \mathbb{Z}/2)$. To reflect this we rewrite the Poincaré series above as

$$\frac{1+t^2+3t^3+t^4+3t^5+4t^6+2t^7+4t^8+3t^9+t^{10}+3t^{11}+t^{12}+t^{14}}{(1-t^4)(1-t^6)(1-t^7)}.$$

Remark. The groups G_2 $(\mathbb{F}_q)$ of order $q^6(q^2-1)(q^6-1)$ and ${}^3D_4(\mathbb{F}_q)$ of order $q^{12}(q^2-1)(q^6-1)(q^8+q^4+1)$ with q a prime power and $q \equiv 3,5 \pmod 8$ have the same 2-Sylow subgroups as M_{12}. Consequently, we would expect the complete analysis that we give for $H^*(H; \mathbb{Z}/2)$ to be useful in studying these groups as well.

This work is the first application of the poset space to explicit cohomology calculations in rank larger than 2 (see [We] for a different aspect of this). Geometrically this means that we had to deal with a 2-complex having a large number of cells. In Section 2 we provide a diagrammatic representation of the orbit space of this complex, including a complete list of isotropy groups. The intersection in $H^*(H; \mathbb{Z}/2)$ is given in Section 4, where we calculate the cohomology rings $H^*(H; \mathbb{Z}/2)$, $H^*(W; \mathbb{Z}/2)$, and $H^*(W'; \mathbb{Z}/2)$. We then determine the images of $(\mathrm{res}^W_H)^*$ and $(\mathrm{res}^{W'}_H)^*$. Using the program MACAULAY we give the intersection as a ring over the Steenrod algebra $\mathcal{A}(2)$ with 8 generators and 14 relations (Eqs. (4.15) and (4.16), Theorem 4.17, and Corollary (4.18)).

It seems that the combination of techniques exploited here will yield equally tractable results for other sporadic simple groups; in particular we are currently analyzing M_{22} and M_{23} in this way. (For M_{22} see the more detailed comments in Section 3.)

We include two appendices containing the data on the subgroup structure of M_{12} required for our calculations.

Much of the calculational work here was done on computers. Symbolic manipulation programs like Cayley are not powerful enough to handle a group the size of M_{12}. However, with efficient storage algorithms and good multiplication algorithms, modern machines can be effective for studying groups having orders less than approximately 10^7. Indeed, in the case at hand it is likely that most, if not all, of the results of Section 2 could have been obtained by hand, but it was so much easier to use the machines that there appeared to be no choice as to preferred methods.

1. M_{12} AND ITS LATTICE OF 2-SYLOW SUBGROUPS

The goal of this section is to provide the classical group-theoretic description of M_{12}, with particular attention to its lattice of 2-Sylow subgroups. A presentation of the Mathieu group M_{12} of order

$$95{,}040 = 12 \circ 11 \circ 10 \circ 9 \circ 8 = 2^6 \circ 3^3 \circ 11 \circ 5$$

is given in Hall's book [H, p. 80]; see also [S, p. 286). It has generators

$$u_1: (1, 2, 3)(4, 5, 6)(7, 8, 9)$$
$$u_2: (1, 4, 7)(2, 5, 8)(3, 6, 9)$$
$$a: (2, 4, 3, 7)(5, 6, 9, 8)$$
$$b: (2, 5, 3, 9)(4, 8, 7, 6)$$
$$x: (1, 10)(4, 5)(6, 8)(7, 9)$$
$$y: (1, 11)(4, 6)(5, 9)(7, 8)$$
$$z: (1, 12)(4, 7)(5, 6)(8, 9)$$

and the first six elements generate M_{11}, the Mathieu group of order $7{,}920 = 11 \circ 10 \circ 9 \circ 8 = 2^4 \circ 3^3 \circ 11 \circ 5$. These are both simple sporadic groups and Benson and Carlson [BC] have recently determined the cohomology groups of M_{11}. However, M_{12} seems considerably harder. We have, to begin (see also [W]):

THEOREM 1.1. *The elements a, b generate a copy of the quaternion group $\mathcal{Q}_8$. The elements x, y, z generate a copy of $\mathcal{S}_4 = \mathrm{Aut}(\mathcal{Q}_8)$ and the subgroup*

$$W = \langle a, b, x, y, z \rangle \subset M_{12} \textit{ is } \mathcal{Q}_8 \times_T \mathrm{Aut}(\mathcal{Q}_8),$$

the semi-direct product (often called the holomorph of $\mathcal{Q}_8$ in the literature).

Proof. First, verifying that $\langle a, b \rangle = \mathcal{Q}_8$ is easy. Second, we have

$$xax = (2, 5, 3, 9)(4, 8, 7, 6) = b, \qquad \text{so} \quad xbx = a,$$
$$yay = (2, 6, 3, 8)(9, 4, 5, 7) = ba, \qquad \text{so} \quad yby = b^{-1}$$
$$zbz = (2, 6, 3, 8)(7, 9, 4, 5) = ab, \qquad \text{so} \quad zaz = a^{-1},$$

so $\langle x, y, z \rangle$ normalizes $\mathcal{Q}_8$. But,

$$yz = (1, 11, 12)(4, 5, 8)(6, 7, 9)$$

also has order 3, while

$$xyz = (1, 10, 11, 12)(4, 8, 7, 6)$$

has order 4. It follows that $\langle x, y, z \rangle$ is a quotient of the group

$$\mathcal{S}_4 = \{A, B \mid A^2 = B^3 = (AB)^4 = 1\}.$$

On the other hand, projection onto the subgroup $\mathcal{S}_4 \subset \mathcal{S}_{12}$ with generators 1, 10, 11, 12 is a surjection, so $\langle x, y, z \rangle = \mathcal{S}_4$, as claimed. The result now follows. ∎

The index of W in M_{12} is 495, and $W \cap M_{11}$ is a group of order $3 \circ 16$ and has index 165 in M_{11}. W has order 192 and contains three 2-Sylow subgroups of M_{12}. In Section 2 we see that from the geometry of the poset space of elementary abelian subgroups, there arises a *distinct* subgroup W' of order 192. In Section 3 we see that W, W' play a strikingly dominant role in our cohomology calculations.

We use the particular 2-Sylow subgroup $H \subset W \subset M_{12}$. It is generated by the three elements b, z, and zz in the list below. Moreover, the quotient by the commutator subgroup is $(\mathbb{Z}/2)^3$, so there is a decomposition of H into eight cosets of H'_2, with coset generators as follows

Coset	Element Representation
1	id
2	$b = (2, 5, 3, 9)(4, 8, 7, 6)$
3	$zz = (1, 11)(4, 7)(6, 8)(10, 12)$
4	$zz \circ b = (1, 11)(2, 5, 3, 9)(4, 6, 7, 8)(10, 12)$
5	$z = (1, 12)(4, 7)(5, 6)(8, 9)$
6	$z \circ b = (1, 12)(2, 5, 4, 6, 3, 9, 7, 8)$
7	$z \circ zz = (1, 10, 12, 11)(5, 8, 9, 6)$
8	$z \circ zz \circ b = (1, 10, 12, 11)(2, 5, 7, 6, 3, 9, 4, 8)$

To obtain the commutator subgroup explicitly we check that the commutator of z and zz is $(1, 12)(5, 9)(6, 8)(10, 11)$, while the commutator of z and b is $(2, 7, 3, 4)(5, 8, 9, 6)$. These two elements generate a group $(\mathbb{Z}/2 \times \mathbb{Z}/4)$ which has order 8 and is consequently the subgroup H'.

There are two special maximal subgroups of H. The first is the subgroup consisting of the first four cosets above. We denote it H_{21}. It can also be written as the semi-direct product

$$\mathcal{Q}_8 \times_\alpha (\mathbb{Z}/2 \times \mathbb{Z}/2),$$

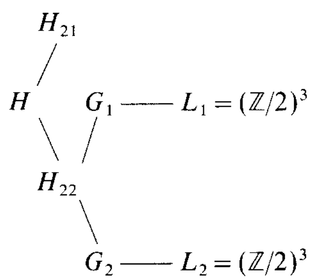
where the action is by inner automorphism. It can be described as follows: in M_{12}, let W be the centralizer of b^2. Then $|W| = 192$, and it contains three conjugates of H. The intersection of any two is H_{21}.

The second maximal subgroup consists of the first, third, fifth, and seventh cosets, and we denote it H_{22}. One can check that

$$H_{22} = H \cap \alpha H \alpha^{-1},$$

where $\alpha = (1, 6, 7)(2, 11, 5)(3, 10, 9)(4, 12, 8)$, so H_{22} is also a maximal subgroup which is given as the intersection of H with a conjugate subgroup in M_{12}.

THEOREM 1.2. *The two subgroups H_{21} and H_{22} are the only maximal subgroups of H which are obtained as the intersections of H and a second 2-Sylow subgroup of M_{12}. Moreover, besides these two groups there are only two other non-commutative subgroups, both of order 16 and contained in H_{22}, which arise as the intersection of H with a second 2-Sylow subgroup in M_{12}.*



THE INTERSECTIONS OF ORDER $\geqslant 8$

Proof. This is a computer enumeration. We began by finding and storing generators for the 1485 cosets of H in M_{12}. Then a direct calculation of the double cosets of H in M_{12} was generated. For details, see Appendix 1. ▮

Remark. We enumerate the various intersections of the groups above:

(1) $H_{21} \cap H_{22} \cong D_8 \times \mathbb{Z}/2$

(2) $G_2 \cap H_{21} = G_1 \cap H_{21} \cong (\mathbb{Z}/2)^3$

(3) $L_2 \cap H_{21} = L_1 \cap H_{21} = (\mathbb{Z}/2)^2 \subset H'$.

Using the computations in Appendix 1, we describe the conjugacy classes of elementary abelian 2-groups in M_{12}.

THEOREM 1.3. *In H there are five conjugacy classes of groups of the form* $(\mathbb{Z}/2)^3$. *In* M_{12} *these become exactly three conjugacy classes. Representatives for the conjugacy classes in H are the groups*

$$K = \langle a^2, zz, [z, zz] \rangle,$$
$$L = \langle a^2, [z, zz], b[z, zz] \rangle,$$
$$M = \langle a^2, b[z, zz]\, a, zz \circ a \rangle,$$
$$K_1 = \langle a^2, [z, zz], z \rangle,$$
$$L_1 = \langle a^2, [z, zz], zz \circ a \rangle.$$

In M_{12}, L and L_1 *are conjugate, as are* K *and* K_1. *In particular each conjugacy class of maximal elementary 2-group in* M_{12} *is represented as a subgroup of* H_{21}.

Proof. This is a direct enumeration, considerably aided by computer calculations. First, there are seven conjugacy classes of elements of order 2 in H. They are given by the table

Class	Cycle form	Number of conjugates
a^2	(2, 3)(4, 7)(5, 9)(6, 8)	1
$[z, zz]$	(1, 12)(5, 9)(6, 8)(10, 11)	2
$b[z, zz]$	(1, 12)(2, 9)(3, 5)(4, 6)(7, 8)(10, 11)	4
zz	(1, 11)(4, 7)(6, 8)(10, 12)	4
$zz \circ a$	(1, 11)(2, 7)(3, 4)(5, 8)(6, 9)(10, 12)	4
$zz \circ b \circ a$	(1, 11)(2, 8)(3, 6)(4, 5)(7, 9)(10, 12)	4
z	(1, 12)(4, 7)(5, 6)(8, 9)	8

Next the centralizers (in H) of each of the elements above except the first were constructed. The centralizer of the last was a copy of $(\mathbb{Z}/2)^3$, while the centralizers of the remaining elements were all copies of $\mathbb{Z}/2 \times D_8$ with the center generated by the element and a^2 in all cases. Also, in each case the element generated the $\mathbb{Z}/2$ split summand. Each of these groups has exactly two subgroups $(\mathbb{Z}/2)^3$, and these were enumerated. The resulting table, of necessity, had all the conjugacy classes of such subgroups of H represented. The table follows:

Representative	Group	Type
$b \circ [z, zz]$	$\langle a^2, zz, b \circ [z, zz] \rangle$	$4^3 6^4$
$b \circ [z, zz]$	$\langle a^2, zz \circ b \circ a, b \circ [z, zz] \rangle$	$4^1 6^6$
zz	$\langle a^2, [z, zz], zz \rangle$	4^7
$zz \circ a$	$\langle a^2, [z, zz], zz \circ a \rangle$	$4^3 6^4$
$zz \circ a$	$\langle a^2, zz \circ a, b \circ [z, zz] \circ a \rangle$	$4^1 6^6$
$zz \circ b \circ a$	$\langle a^2, zz \circ [z, zz], zz \circ b \circ a \rangle$	$4^3 6^4$
z	$\langle a^2, z, [z, zz] \rangle$	4^7

Here, the type symbol $4^3 6^4$ denotes the fact that of the seven non-identity elements of the group 3 have a cycle decomposition as a product of four transpositions in the embedding into $\mathcal{S}_{12}$, while four have a cycle decomposition as a product of six transpositions.

In the table above, the first and the sixth group are easily seen to be conjugate in H, as are the second and the fifth. It is also directly verified that these are the only possibilities. For example, in the fourth group all four of the elements of cycle type 6 are conjugate in H, while this is not true in the first group. Similarly, the seventh group is the centralizer of one of its elements in H, while the third is not. Thus the first statement in the theorem is verified.

The element $(1, 6, 12, 8)(5, 10, 9, 11) \in M_{12}$ conjugates the third group to the seventh, while the element $(1, 12, 11)(4, 8, 5)(6, 9, 7) \in M_{12}$ conjugates the fourth group above to the first. ∎

THEOREM 1.4. *There are four conjugacy classes of 2-groups isomorphic to $(\mathbb{Z}/2)^2$ in M_{12}. Representatives may be given as*

$$R = \langle a^2, zz \rangle \textit{ type } 4^3$$

$$R_1 = \langle zz, [z, zz] \rangle \textit{ type } 4^3$$

$$S = \langle a^2, b[z, zz] \rangle \textit{ type } 4^1 6^2$$

$$T = \langle zz \cdot b \cdot a, b[z, zz] \rangle \textit{ type } 6^3.$$

Proof. We first need to observe the result

LEMMA 1.5. *Every subgroup $G \subset H$ which is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$ is contained in a maximal elementary subgroup $X = (\mathbb{Z}/2)^3$.*

Proof. If $a^2 \in G$ then G may be assumed to contain one of the other six elements in the table of conjugacy class representatives above. The second table gives an extremal group containing G. If $a^2 \notin G$ then $\langle G, a^2 \rangle = (\mathbb{Z}/2)^3$ will serve. ∎

Continuing with the proof of the theorem, assume first that G contains an element α of type (2, 2, 2, 2); then there is a $\beta \in M_{12}$ so $\beta\alpha\beta^{-1} = a^2$ and consequently $\beta G \beta^{-1} \subset H$. Similarly, if G just consists of the identity and three elements of type (2, 2, 2, 2, 2, 2) then it is conjugate to a subgroup of the group M above. In the first case it is either conjugate to the subgroup $\langle a^2, [z, zz] \rangle$ in L, one of the three subgroups of K containing a^2, or one of the three subgroups of M containing a^2.

In the case of the subgroup of M, note that the elements of type (2, 2, 2, 2, 2, 2) form a single orbit under the action of the automorphism group. Consequently, any two subgroups containing a^2 are conjugate. It is also easy, using the particular form of the action of $\mathcal{S}_4$ on the six elements of this type in M, to show that any two subgroups consisting only of elements of this type and the identity are conjugate. Similarly, if the group is conjugate to a subgroup of L which contains a^2 and also contains an element of type (2, 2, 2, ,2, 2, 2), then there is a 2-Sylow subgroup of $\mathcal{S}_4$ which fixes a^2, but acts transitively on the four elements of this type in L. It follows again that any two subgroups of this type are conjugate. Finally, it is easy to see that the intersection of L and M is a subgroup of this type. So we have shown there are unique conjugacy classes of $(\mathbb{Z}/2)^2$ having the types of S and T.

It remains to discuss those groups G containing three elements of type (2, 2, 2, 2). The discussion above shows that we can assume that they are contained in L. The action of the group $N_{M_{12}}(L)$ on L decomposes these seven subgroups into two orbits, the first containing R and the second R_1. To verify that, in fact, R and R_1 are not conjugate in M_{12}, we used a computer program to compare all the conjugacy classes of R to R_1. ∎

The only remaining matter to discuss is the way in which these groups occur in the intersections with conjugates of H in M_{12}. This was checked on the computer, and the result was

THEOREM 1.6. *Only the conjugacy classes of $T = \langle zz \cdot b \cdot a, b[z, zz] \rangle$ among the groups isomorphic to $(\mathbb{Z}/2)^2$ in H occur as the intersections, $H \cap \alpha H \alpha^{-1}$, of H with one of its conjugates in M_{12}.*

This completes our discussion of the lattice of subgroups of H in M_{12}.

2. The Poset Space for M_{12}

Let X be a partially ordered set (poset), and denote by $|X|$ the simplicial complex associated to it. This is the simplicial complex whose vertices are the elements of X and whose simplices are the non-empty finite chains in X.

Now if G is a finite group, Brown [B] and later Quillen [Q] introduced the study of the complexes associated to the posets

$\mathscr{S}_p(G)$ = poset of non-identity p-subgroups of G

$\mathscr{A}_p(G)$ = poset of non-identity p-elementary abelian subgroups of G.

The following proposition summarizes the main properties of these poset spaces (see [Q] for details)

Proposition 2.1. *Let G be a finite group of p-rank r. Then*

(a) *$\mathscr{A}_p(G)$ is a simplical complex of dimension $r-1$, with an action of G induced by conjugation,*

(b) *the natural inclusion $\mathscr{A}_p(G) \subsetneq \mathscr{S}_p(G)$ is a homotopy equivalence,*

(c) *if G contains a non-trivial normal p-subgroup, then $\mathscr{A}_p(G)$ is contractible, and*

(d)

$$\hat{H}^*_G(\mathscr{A}_p(G), \mathbf{F}_p) \cong \hat{H}^*(G, \mathbf{F}_p). \quad \blacksquare$$

Remark. Here $\hat{H}^*_G(X)$ denotes the equivariant Tate cohomology of X; see [B] for a description. Part (d) is a particular case of a result for discrete groups proved by Brown [B, p. 293].

Proof. The first three parts of Proposition 2.1 are explicit in [Q]. For part (d) we observe that if $P \subset G$ is a p-Sylow subgroup, then the singular set of the P-action on $\mathscr{A}_p(G)$ is contractible [Q, 4.1]. Denote this singular set by $S_p(|\mathscr{A}_p(G)|)$.

We show that the dual of the augmentation $\mathbb{F}_p \xrightarrow{\varepsilon^*} \mathscr{C}^*(|\mathscr{A}_p|)$ induces an isomorphism in Tate cohomology. It fits into the short exact sequence

$$0 \longrightarrow \mathbb{F}_p \xrightarrow{\varepsilon^*} \mathscr{C}^*(|\mathscr{A}_p|) \longrightarrow \tilde{\mathscr{C}}^*(|\mathscr{A}_p|) \longrightarrow 0.$$

Consequently, it reduces to showing that $\hat{H}^*_G(\tilde{\mathscr{C}}^*(|\mathscr{A}_p|)) \equiv 0$. By the usual transfer restriction argument it suffices to show that $\hat{H}^*_P(\tilde{\mathscr{C}}^*(|\mathscr{A}_p|)) \equiv 0$ for P a p-Sylow subgroup of G. Hence we can assume that $G = P$ and prove that ε_p^* is an isomorphism. However, we have a commutative diagram with the vertical arrow an isomorphism:

$$\begin{array}{ccc} \hat{H}^*(P;\mathbb{F}_p) & \xrightarrow{\varepsilon_p^*} & \hat{H}^*(|\mathscr{A}_p(G)|) \\ & \searrow^{\bar{\varepsilon}^*} & \downarrow{\scriptstyle i_p^*} \\ & & \hat{H}_P^*(S_P(|\mathscr{A}_p(G)|)). \end{array}$$

Using the fact that $S_P(|\mathscr{A}_p(G)|)$ is contractible and a spectral sequence argument we deduce that $\bar{\varepsilon}^*$ must also be an equivalence, thus completing the proof. ▮

It is now clear from Proposition 2.1 that for cohomological considerations, the poset space is potentially most useful for groups without normal p-subgroups, e.g., simple groups. It also indicates that $H^*(G)$ can be approached using equivariant cohomology. Consequently it is important to understand the singular set of the G-action on $\mathscr{A}_p(G)$, as well as the isotropy subgroups, which in this case are the normalizers of the p-elementary abelian subgroups and the flags associated to them. For M_{12} we show that in fact two isotropy subgroups of order 192 completely control the cohomology at the prime 2. We refer to [We] for other aspects of this point of view.

We now proceed to describe $\mathscr{A}_2(M_{12})$. We start by providing a complete list of isotropy subgroups occurring for this two dimensional complex.

PROPOSITION 2.2. *The two conjugacy classes of involutions* 4^1 *and* 6^1 *in* M_{12} *satisfy*

(a) *the centralizer of* 4^1 *is* W, *the holomorph of* $\mathscr{Q}_8$ *and*

(b) *the centralizer of* 6^1 *is* $\mathbb{Z}/2 \times \mathscr{S}_5$.

Proof. (a) has already been discussed and is well known. (b) is a computer calculation, though it was known previously [W] that $|N_{M_{12}}(6^1)| = 240$. First, the elements of the centralizer of (6^1) were listed, where the specific element used was

$$(1, 11)(2, 8)(3, 6)(4, 5)(7, 9)(10, 12) = x.$$

Then the two elements $\alpha, \beta \in N_{M_{12}}(6^1)$ were found with $\alpha^2 = \beta^3 = (\alpha\beta)^5 = 1$, so $\langle \alpha, \beta \rangle = \mathscr{A}_5$. Indeed, $\alpha = a^2 = (2, 3)(4, 7)(5, 9)(6, 8)$, $\beta = (1, 10, 4)(2, 7, 3)(5, 11, 12)(6, 8, 9)$ are suitable choices. Clearly, $x \notin \mathscr{A}_5$, so $|\langle x, \alpha, \beta \rangle| = 120$, and the resulting group is normal in $N_{M_{12}}(\langle x \rangle)$ and is given as the central extension,

$$\mathbb{Z}/2 = \langle x \rangle \xrightarrow{\triangleleft} \langle x, \alpha, \beta \rangle \longrightarrow \mathscr{A}_5.$$

But there is only one (non-trivial) central extension of $\mathscr{A}_5$ by $\mathbb{Z}/2$ since $H^2(\mathscr{A}_5; \mathbb{Z}/2) = \mathbb{Z}/2$, and this is the binary icosahedral group, $\tilde{\mathscr{A}}_5$. However, the 2-Sylow subgroup of $\mathscr{A}_5$ is $\mathscr{Q}_8$, the quaternion group. Consequently, if we had the non-trivial extension we would have that $x = y^2$ for some y, but an inspection of the elements of $N_{M_{12}}(\langle x \rangle)$ shows that this is impossible, so $\langle x, \alpha, \beta \rangle = \mathbb{Z}/2 \times \mathscr{A}_5$. This implies the existence of the normal extension

$$\mathbb{Z}/2 \times \mathscr{A}_5 \overset{\lhd}{\longrightarrow} N_{M_{12}}(\langle x \rangle) \longrightarrow \mathbb{Z}/2.$$

Now, the number of extensions of $\mathbb{Z}/2 \times \mathscr{A}_5$ by $\mathbb{Z}/2$ with a given action map,

$$\mathbb{Z}/2 \to \operatorname{Out}(\mathbb{Z}/2 \times \mathscr{A}_5),$$

is given by $H^2(\mathbb{Z}/2; \mathbb{Z}(\mathbb{Z}/2 \times \mathscr{A}_5))$. The center of $\mathbb{Z}/2 \times \mathscr{A}_5$ is $\mathbb{Z}/2 = \langle x \rangle$, so this cohomology group is $\mathbb{Z}/2$. Also, $\operatorname{Out}(\mathbb{Z}/2 \times \mathscr{A}_5) = \mathbb{Z}/2$, so it follows that there are precisely four distinct extensions of the above type: the product $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathscr{A}_5$, the group $\mathbb{Z}/2 \times \mathscr{S}_5$, and two extensions in which $x = y^2$. But the product has a subgroup $(\mathbb{Z}/2)^4$ which is not a subgroup of M_{12} and we have already seen that x is not a square in $N_{M_{12}}(\langle x \rangle)$. The proposition follows. ∎

For later use we need the following

COROLLARY 2.3. *In $N_{M_{12}}(\langle x \rangle)$ there are precisely five conjugacy classes of involutions: x, α, $x\alpha$, a class t, which represents the involution $(1, 2) \in \mathscr{S}_5$, i.e., a class which extends $\mathscr{A}_5$ to $\mathscr{S}_5$, and xt. The orbit of α under conjugation in $\mathbb{Z}/2 \times \mathscr{S}_5$ contains* 15 *elements, and these are the only elements in this group of type* 4^1.

(This is clear except for the last statement. This was verified by checking that there were exactly 15 elements in $N_{M_{12}}(\langle x \rangle)$ of type 4^1 by just inspecting the table of elements.)

Next we study the normalizer of the first conjugacy class of groups $(\mathbb{Z}/2)^2$, those of type 4^2, i.e., conjugate to $\langle a^2, zz \rangle$. We have

THEOREM 2.4. *The normalizer of the group $\langle a^2, zz \rangle$ in M_{12}, $N_{M_{12}}(4^3_I)$, has order* 192 *but is not isomorphic to W. More exactly, the centralizer of $\langle a^2, zz \rangle$, $\mathbb{Z}(4^3_I)$, is given by*

$$\begin{aligned} \mathbb{Z}_{M_{12}}(\langle a^2, zz \rangle) &= (\mathbb{Z}/4 \times \mathbb{Z}/4) \times_T \mathbb{Z}/2 \\ &= \{a, d, f \mid a^4 = d^4 = f^2 = 1; ad = da, faf = a^{-1}, fdf = d^{-1}\}, \end{aligned}$$

and $N_{M_{12}}(4^3_I)$ is given as the semi-direct product

$$\mathbb{Z}_{M_{12}}(\langle a^2, zz \rangle) \times_\alpha \mathscr{S}_3$$

with the element of order 3 *acting by* $T(a)=d$, $T(d)=(ad)^{-1}$, $T(f)=a^3df$, *while an element* g *of order* 2 *acts as* $g(f)=a^2f$, $g(a)=a^{-1}$, $g(d)=ad$.

Proof. Set

$$a=(2,5,3,9)(4,8,7,6),$$

$$d=(1,12,11,10)(4,6,7,8),$$

$$f=(1,10)(4,7)(5,9)(11,12),$$

$$T=(1,4,2)(3,11,7)(5,10,6)(8,9,12),$$

$$g=(1,11)(2,7)(3,4)(5,8)(6,9)(10,12).$$

It is now systematic to verify all the claims above, except perhaps the claim that

$$N_{M_{12}}(4_I^3)\neq W.$$

But note that a 2-Sylow subgroup of $N_{M_{12}}(4_I^3)\subset W$ since f and g centralize a^2. In particular, if $N_{M_{12}}(4_I^3)$ has a center it must be $\langle a^2\rangle$, but T does not centralize a, so $N_{M_{12}}(4_I^3)$ has a trivial center and cannot be isomorphic to W.

The next normalizer that we identify is $N_{M_{12}}(4^36^4)$. We found the next result to the somewhat unexpected.

PROPOSITION 2.5. $N_{M_{12}}(4_I^3)=N_{M_{12}}(4^36^4)$.

Proof. We can check that $fa=(1,10)(2,5)(3,9)(4,6)(7,8)(11,12)$. Consequently

$$N_{M_{12}}(\langle a^2, zz\rangle) \text{ contains the subgroup } \langle a^2, zz, fa\rangle,$$

which has type 4^36^4. Moreover, $T(fa)=a^3f=fa$, and $g(fa)=a^2fa^3=fa$. Also, $dfad^{-1}=zzfa$, and $af=a^2fa$. It follows that the group $N_{M_{12}}(4_I^3)\subset N_{M_{12}}(4^36^4)$. On the other hand, $\mathbb{Z}(4^36^4)\subset\mathbb{Z}(4_I^3)$, and $N_{M_{12}}(4^36^4)/\mathbb{Z}_{M_{12}}(4^36^4)=\mathcal{S}_4$. The result follows. ∎

Remark. The normalizer of $\langle a^2, fa\rangle$ is contained in W since this group has type 4^16^2, so every element in $N_{M_{12}}(\langle a^2, fa\rangle)$ must also fix a^2. From this it is easy to check that $|N_{M_{12}}(\langle a^2, fa\rangle)|=32$. Indeed, the group is generated by d^2, g, a, and f. It is the extra-special two group

$$\mathbb{Z}/2=\langle a^2\rangle \xrightarrow{\triangleleft} N_{M_{12}}(\langle a^2, fa\rangle)\longrightarrow(\mathbb{Z}/2)^4,$$

where the latter group has generators a, d^2, g, and f. The extension data are given by $[a,d^2]=[d^2,f]=1$, $[f,g]=[g,d^2]=[a,g]=[a,f]=a^2$.

Remark. Similarly, the group $N_{M_{12}}(4^1 6^6) \subset W$, and its order is directly seen to be 96. Indeed, in checking we find that a representative group is $\langle a^2, fa, g\rangle \subset W$. Both d^2 and a normalize this group, but they are independent in their action. Moreover, as we have seen, the normalizer contains an element of order 3. Consequently we have that the 2-Sylow subgroup of $N_{M_{12}}(\langle a^2, fa, g\rangle)$ is isomorphic to the group $N_{M_{12}}(\langle a^2, fa\rangle)$ described above. Moreover, we have the group given as the normal extension

$$(\mathbb{Z}/2)^3 \xrightarrow{\triangleleft} N_{M_{12}}(4^1 6^6) \longrightarrow \mathscr{A}_4.$$

The next group to consider is $N_{M_{12}}(4^3_{II}) \cong N_{M_{12}}(\langle zz, [z, zz]\rangle)$. A computer check shows that this group has order 48 and is contained in W. In particular we have two ways of describing the resulting group:

$$(\mathbb{Z}/2)^3 = \langle a^2, zz, [z, zz]\rangle \xrightarrow{\triangleleft} N_{M_{12}}(\langle zz, [z, zz]\rangle) \longrightarrow \mathscr{S}_3,$$

and

$$\langle a^2\rangle = \mathbb{Z}/2 \xrightarrow{\triangleleft} N_{M_{12}}(\langle zz, [z, zz]\rangle) \longrightarrow \mathscr{S}_4.$$

It remains to discuss the normalizer $N_{M_{12}}(6^3)$. Since $N_{M_{12}}(6^1) = \mathbb{Z}/2 \times \mathscr{S}_5$ with a class of type 6^1 generating the $\mathbb{Z}/2$ summand, it suffices to check the centralizer of an element of order 2 and type 6^1 in the $\mathscr{S}_5$ piece. But this is given by $\mathscr{S}_3 \times \mathbb{Z}/2$ since, as we saw above, all the elements of type $4^1 \in N_{M_{12}}(\langle x\rangle)$ are contained in $\mathscr{A}_5$. Hence, the centralizer of 6^3 has order 24 and is contained in $N_{M_{12}}(6^1)$. From the last section $\mathrm{Out}_{M_{12}}(6^3) = \mathbb{Z}/3$, so $|N_{M_{12}}(6^3)| = 72$ and the group is given as the product

$$N_{M_{12}}(6^3) \cong \mathscr{S}_3 \times \mathscr{A}_4.$$

Next we analyze the flags in the 2-elementary subgroups of M_{12}

DEFINITION 2.6. Let G be a group; then a p-elementary k-flag in G is a sequence of subgroups $G_i \subset G$ and proper inclusions

$$G_1 \subset G_2 \subset \cdots \subset G_k$$

with each G_i a p-elementary group.

Two k-flags $(G_1 \subset \cdots \subset G_k)$, $(G'_1 \subset \cdots \subset G'_k)$ are conjugate if there is a $g \in G$ with $gG_ig^{-1} = G'_i$, $i = 1, 2, ..., k$. The isotropy group in G of the k-flag $(G_1 \subset \cdots \subset G_k)$ is the set of all $g \in G$ with $gG_ig^{-1} = G_i$, $i = 1, 2, ..., k$. Note that the isotropy group of a flag is the intersection of the normalizers of the G_i.

In this section we study only the 2-elementary k-flags in M_{12}, so we suppress the modifiers M_{12} and 2-elementary in what follows.

PROPOSITION 2.7. *For flags of the form* $\mathbb{Z}/2 \subset (\mathbb{Z}/2)^2$ *the conjugacy classes of the groups* G_1, G_2 *determine the conjugacy class of the flag. Consequently, we have the following table*:

Type	*Isotropy group*	*Order*
$(4^1, 4^3_{\mathrm{I}})$	H	64
$(4^1, 4^3_{\mathrm{II}})$	$D_8 \times \mathbb{Z}/2 = \langle a^2, z, zz \rangle$	16
$(4^1, 4^1 6^2)$	$N_{M_{12}}(4^1 6^2)$	32
$(6^1, 4^1 6^2)$	$\mathbb{Z}_{M_{12}}(4^1 6^2)$	16
$(6^1, 6^3)$	$\mathcal{S}_3 \times (\mathbb{Z}/2)^2$	24

(This is direct from the results of the last section.)

PROPOSITION 2.8. *For 2-flags of the form* $(\mathbb{Z}/2)^2 \subset (\mathbb{Z}/2)^3$ *we also have that the conjugacy classes of the two groups* G_1, G_2 *determine the conjugacy class of the flag. Consequently we have the following table*:

Type	*Isotropy group*	*Order*
$(4^3_{\mathrm{I}}, 4^7)$	H	64
$(4^3_{\mathrm{II}}, 4^7)$	$N_{M_{12}}(4^3_{\mathrm{II}})$	48
$(4^3_{\mathrm{I}}, 4^3 6^4)$	$N_{M_{12}}(4^3 6^4)$	192
$(4^1 6^2, 4^3 6^4)$	$N_{M_{12}}(4^1 6^2)$	32
$(4^1 6^2, 4^1 6^6)$	$N_{M_{12}}(4^1 6^2)$	32
$(6^3, 4^1 6^6)$	$\mathbb{Z}/2 \times \mathcal{A}_4$	24

There are two exceptional cases for 2-flags of type $\mathbb{Z}/2 \subset (\mathbb{Z}/2)^3$. Specifically we have

THEOREM 2.9. *There are two distinct conjugacy classes of 2-flags of the form* $(4^1, 4^7)$. *The first is represented by* $(\langle a^2 \rangle, \langle a^2, zz, [z, zz] \rangle)$ *and has isotropy group* W, *while the second is represented by* $(\langle zz \rangle, \langle a^2, zz, [z, zz] \rangle)$ *and has isotropy group* $\mathbb{Z}_{M_{12}}(\langle a^2, zz \rangle) = H_{2,2}$ *of order* 32.

Proof. That there are two distinct isotropy groups is clear since $W = N_{M_{12}}(4^7)$ fixes a^2 and makes a single orbit of the remaining terms. Then the isotropy group of $(\langle zz \rangle, \langle a^2, zz, [z, zz] \rangle)$ certainly fixes both zz and a^2 and consequently is contained in $\mathbb{Z}_{M_{12}}(\langle a^2, zz \rangle) = (\mathbb{Z}/4)^2 \times_T \mathbb{Z}/2$.

On the other hand, this centralizer, being contained in W, normalizes $\langle a^2, zz, [z, zz]\rangle$.

Finally, note that there is an element $\theta \in M_{12}$ with $\theta a^2 \theta^{-1} = zz$. Then $\theta W \theta^{-1} \cap W = \mathbb{Z}_{M_{12}}(a^2, zz\rangle)$, and it follows that there is a choice of 2-Sylow subgroup $H' \subset W$ with $H' \cap \theta H' \theta^{-1}$ also equal to this centralizer. Now, since $\mathbb{Z}_{M_{12}}(\langle a^2, zz\rangle) \nsubseteq H_{2,1}$ it must be $H_{2,2}$.

Otherwise we have

PROPOSITION 2.10. *Except for the two cases above a 2-flag of the form* $\mathbb{Z}/2 \subset (\mathbb{Z}/2)^3$ *is determined up to conjugacy by the conjugacy types of the groups* G_1, G_2. *Consequently, we have the table*

Type	*Isotropy group*	*Order*
$(4^1, 4^3 6^4)$	H	64
$(6^1, 4^3 6^4)$	$\mathbb{Z}/2 \times \mathscr{S}_4$	48
$(4^1, 4^1 6^6)$	$N_{M_{12}}(4^1 6^6)$	96
$(6^1, 4^1 6^6)$	$\mathbb{Z}/2 \times D_8$	16

Finally we consider the 3-flags. From our previous discussion we have directly

THEOREM 2.11. *There are nine conjugacy classes of* $\mathbb{Z}/2 \subset (\mathbb{Z}/2)^2 \subset (\mathbb{Z}/2)^3$ *in* M_{12}. *There are two of type* $(4^1, 4^3_I, 4^7)$ *represented by* $(\langle a^2\rangle, \langle a^2, zz\rangle, \langle a^2, zz, [z, zz]\rangle)$ *with isotropy group* H *and* $(\langle zz\rangle, \langle a^2, zz\rangle, \langle a^2, zz, [z, zz]\rangle)$ *with isotropy group* $H_{2,2}$. *The remaining classes are determined by the types of the subgroups, so we have the table*

Type	*Isotropy group*	*Order*
$(4^1, 4^3_{II}, 4^7)$	$D_8 \times \mathbb{Z}/2$	16
$(4^1, 4^3_I, 4^3 6^4)$	H	64
$(4^1, 4^1 6^2, 4^3 6^4)$	$N_{M_{12}}(4^1 6^2)$	32
$(4^1, 4^1 6^2, 4^1 6^6)$	$N_{M_{12}}(4^1 6^2)$	32
$(6^1, 4^1 6^2, 4^3 6^4)$	$\mathbb{Z}_{M_{12}}(4^1 6^2)$	16
$(6^1, 4^1 6^2, 4^1 6^6)$	$\mathbb{Z}_{M_{12}}(4^1 6^2)$	16
$(6^1, 6^3, 4^1 6^6)$	$(\mathbb{Z}/2)^3$	8

In Fig. 1 we give a schematic representation of the orbit space for $|\mathscr{A}_2(M_{12})|$, with isotropy subgroups labeled as before. One can check that $\chi(|\mathscr{A}_2(M_{12})|) = 3201$, be necessarily congruent to one modulo 64 (the order of the 2-Sylow subgroup). Given the explicit nature of our data, it might

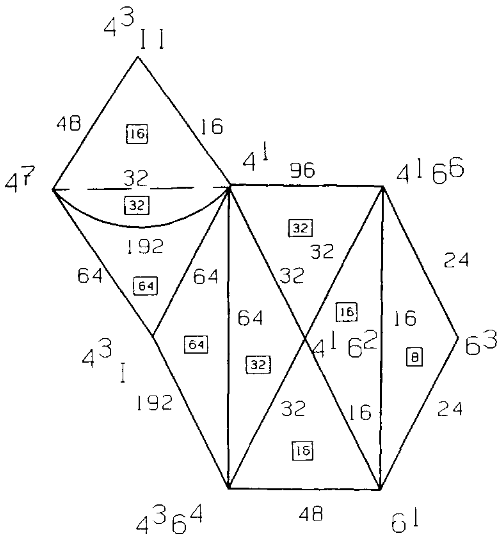


FIG. 1. The quotient of the poset space under the action of M_{12}.

be of some interest to analyze the representations $H_i(|\mathscr{A}_2(M_{12})|, \mathbb{F}_2)$, $i = 1, 2$, and verify whether they are projective $\mathbb{F}_2(M_{12})$ modules as prior evidence suggests. (See [We] for more on this.)

For readers interested in the purely computational aspects of determining the subgroups of M_{12}, we refer to [BR], where a complete list of conjugacy classes of subgroups is given.

3. The Main Result

There are two non-isomorphic groups of order 192 contained in M_{12}, as we have seen in the last section. The first, W, centralizes an involution of type $(2, 2, 2, 2)$ while the second, W', is the normalizer of a subgroup $(\mathbb{Z}/2)^3$ of type $4^3 6^4$ and of $\langle a^2, zz \rangle$. We may assume W, W' to be chosen so that $W \cap W' = H$, our usual 2-Sylow subgroup of M_{12}.

THEOREM 3.1. *The cohomology ring $H^*(M_{12}; \mathbb{Z}/2)$ is the intersection of the images*

$$(\mathrm{res}_H^W)^*: H^*(W; \mathbb{Z}/2) \subsetneq H^*(H; \mathbb{Z}/2)$$

and

$$(\mathrm{res}_H^{W'})^*: H^*(W'; \mathbb{Z}/2) \subsetneq H^*(H; \mathbb{Z}/2).$$

Proof. A key theorem of Cartan and Eilenberg, the double coset formula [CE, p. 259], identifies $H^*(G; \mathbb{Z}/p)$ as the set of stable elements in $H^*(G_p; \mathbb{Z}/p)$, where G_p is a p-Sylow subgroup and stable means the following: let $g_1, ..., g_r$ be double coset generators for G, so $G = \amalg G_p g_i G_p$, and define $G_{g_i} = G_p \cap g_i G_p g_i^{-1}$. Then $\bar{G}_{g_i} = g_i^{-1} G_{g_i} g_i \subset G_p$ and we have that any element $\varepsilon \in \text{im}(\text{res}^G_{G_p})^*$ must satisfy

$$(i_g)^* \, (\text{res}^{G_p}_{G_{g_i}})^* \, (\alpha) = (\text{res}^{G_p}_{\bar{G}_{g_i}})^* \, (\alpha), \tag{3.2}$$

where $i_g : \bar{G}_{g_i} \to G_{g_i}$ is conjugation by g_i. Conversely, an element $\alpha \in H^*(G_p; \mathbb{Z}/p)$ is stable if it satisfies (3.2) for each double coset representative g_i.

If $G_p \subset L \subset G$ and $\bar{g}_i$ are double coset representatives for L so that $G = \amalg L\bar{g}_i L$, then, first, $H^*(G; \mathbb{Z}/p) \to H^*(L; \mathbb{Z}/p)$ is an injection since L contains a copy of G_p, and, second, an element in $H^*(L; \mathbb{Z}/p)$ is in the image of $H^*(G; \mathbb{Z}/p)$ if and only if it is stable with respect to the $\bar{g}_i$; see e.g., [CE, Proposition 9.4, p. 259]. That is, we need only check the subgroups $L_{\bar{g}_i} = L \cap \bar{g}_i L \bar{g}_i^{-1}$.

In our case $G = M_{12}$ and the group L will be W. Appendix 2 gives a computer generated table listing generators for the 10 non-trivial double cosets of W (there are 11 including W itself), as well as the intersection group $\bar{W}_{g_i}$ next to it. They are arranged so that elements on the same line are conjugate by g_i. Often, it turns out that the particular choice of generator given by the computer is inconvenient to our argument. Where that happens we also list the conjugates (in W) of the group $\bar{W}_{g_i}$ which are obtained using different choices for g_i to generate the same double coset.

We now consider the elements coset by coset. The 1st element, $(1, 2, 3)(4, 5, 6)(7, 8, 9)$, acts as the identity on W_{g_1}. Hence, $i_{g_1} = id$ and (3.2) gives no restriction. The 2nd element, $g_2 = (1, 3, 11, 2)(4, 6, 8, 7)(5, 9)(10, 12)$, has $W_{g_2} = \bar{W}_{g_2}$ conjugate in W to K_1. But the element $(1, 4, 2)(3, 11, 7)(5, 10, 6)(8, 9, 12) \in W'$ takes K to a conjugate of K_1 in W'. It follows that on the intersection of the two restriction maps in Theorem 3.1, the stability condition generated by g_2 on K_1 is equivalent to a condition for K, where i_{g_2} represents the effect of an automorphism of K. But W is the normalizer of K; hence, on the image of $H^*(W; \mathbb{Z}/2)$ in $H^*(H; \mathbb{Z}/2)$, the stability condition is automatically satisfied.

The 3rd element has $i_{g_3} = id$, so there is no condition. The 4th generation gives $\bar{G}_{g_4} \neq G_{g_4}$, but the 9th conjugate of $\bar{G}_{g_4}$ does equal G_{g_4}. Hence, we can replace the generator by one which takes G_{g_4} to itself. This subgroup is $\langle a^2[z, zz], z \rangle$. In W' it is conjugate to a subgroup of K, which in W is conjugate to either $\langle a^2, zz \rangle$ or $\langle zz, [z, zz] \rangle$. But W' normalizes the 1st, while the normalizer of the 2nd is contained in W. In either case, the stability condition is automatically satisfied on the intersection in Theorem 3.1.

The 5th and 6th generators both have W_{g_i} conjugate to $\bar{W}_{g_i}$. Indeed, for the 5th the 10th conjugate of $\bar{W}_{g_5}$ is W_{g_5}, while for the 6th we use the 22nd conjugate.

For g_7 the intersection is the dihedral group D_6. Since i_g takes the 1st element to itself and it generates the 2-Sylow subgroup of the intersection, there is no condition.

For g_8 the story is the same as that for g_4. For g_9, note that $W_{g_9} \cong \mathscr{A}_4$, the alternating group on four letters. Now replace $\bar{W}_{g_9}$ by the 3rd conjugate, and note that i_g on the 2-Sylow subgroup is the identity. Hence, there is no condition.

Finally, we consider the 10th double coset generator. Here $W_{g_{10}} = H_{22}$, and by replacing $\bar{W}_{g_{10}}$ by the 3rd conjugate, we see that conjugation by the new g_{10} gives an automorphism of H_{22}. In particular, it acts as an automorphism on the center of H_{22}, which is $\langle a^2, zz \rangle$. But this means that $g_{10} \in W'$ since W' is the normalizer of $\langle a^2, zz \rangle$, and, once more, on the intersection of the two images, the stability condition is automatically satisfied.

Thus we have shown that every element in the intersection is in the image of the cohomology groups $H^*(M_{12}; \mathbb{Z}/2)$ in $H^*(H; \mathbb{Z}/2)$. But, by naturality,

$$\operatorname{im}(\operatorname{res}_H^{M_{12}})^* \subset \operatorname{im}(\operatorname{res}_H^{W})^* \cap \operatorname{im}(\operatorname{res}_H^{W'})^*,$$

and Theorem 3.1 follows. ∎

Remark. There is a similar decomposition for M_{22}. There are two subgroups K, S in M_{22} with $\langle K, S \rangle = M_{22}$, and $K \cap S$ contains a 2-Sylow subgroup. However, the analogue of Theorem 3.1 is not true in this case. The 2-Sylow subgroup H of M_{22} contains exactly two subgroups $(\mathbb{Z}/2)^4$. Both are normal in H, but the normalizer of the first (in M_{22}) is a semi-direct product $(\mathbb{Z}/2)^4 \times_\alpha \mathscr{A}_6 = K$ while the normalizer for the second is a semi-direct product $(\mathbb{Z}/2)^4 \times_\beta \mathscr{S}_5 = S$. See, e.g., [J] for further details.

A computer check shows that there are elements $v \in S$, w so that $M_{22} = K \cup KvK \cup KwK$. However, $w \notin S$, and $K \cap wKw^{-1} = \mathscr{A}_6$ is normalized by w so that the action restricts to one of the 2-Sylow subgroups of $\mathscr{A}_6$ as a non-trivial outer automorphism. From this it follows that the containment $H^*(M_{22}; \mathbb{Z}/2) \subset H^*(S; \mathbb{Z}/2) \cap H^*(K; \mathbb{Z}/2)$ is proper! A complete discussion will appear in a sequel.

In Section 5 we see how our result relates the cohomology of M_{12} to that of the amalgamated product $W *_H W'$.

4. The Calculation of $H^*(M_{12};\mathbb{Z}/2)$

In this section we use the results in Section 3 to calculate $H^*(M_{12};\mathbb{Z}/2)$. The procedure is to first study the cohomology rings of the groups H_{22}, H_{21}, then use these to obtain the ring $H^*(H;\mathbb{Z}/2)$, then use the double coset formula to obtain $H^*(W;\mathbb{Z}/2)$, $H^*(W';\mathbb{Z}/2)$. The images of these groups are next explicitly calculated in $H^*(H;\mathbb{Z}/2)$ and the generators of the low dimensional intersections are determined.

The Group H_{22}

We begin by calculating the cohomology of H_{22}. The result is

THEOREM 4.1. *There are no nilpotent elements in $H^*(H_{22};\mathbb{Z}/2)$. Indeed we have*

$$H^*(H_{22};\mathbb{Z}/2)\cong \mathbb{F}_2[r,s]\otimes \mathbb{F}_2[a,b,c]/\{a^2+ac,b^2+bc\}$$
$$\cong \mathbb{F}_2[r,s,c](1,a,b,ab),$$

where r, s are both two dimensional, while a, b, and c are one dimensional.

Proof. The center of H_{22} is $\mathbb{Z}/2\times\mathbb{Z}/2$ with generators b^2 and $m=[z,zz]$. Thus it can be given as the central extension

$$\mathbb{Z}/2\times\mathbb{Z}/2\to H_{22}\to(\mathbb{Z}/2)^3$$

and the K invariants are seen to be a^2+ac, b^2+bc, since we have that the generators for the quotient $(\mathbb{Z}/2)^3$ are the images of $l=[z,b]$, z, and zz while $[l,z]=l^2=b^2$, $[zz,l]=l^2$, $m=[z,zz]$, and $z^2=zz^2=1$.

Next, a routine calculation shows that in the resulting spectral sequence, $E_3=E_\infty$. From this the result above is immediate. ▮

From the above, an alternate description of H_{22} is as the semi-direct product

$$\mathbb{Z}/4\times\mathbb{Z}/4\times_T\mathbb{Z}/2,$$

where the generators of the $\mathbb{Z}/4\times\mathbb{Z}/4$ are l, $l\cdot z\cdot zz$, and z generates the twisting $\mathbb{Z}/2$.

The Group H_{21}

We now turn our attention to H_{21}, which can be described as the semi-direct product

$$\mathcal{Q}_8\times_\alpha(\mathbf{Z}/2)^2,$$

where the action is inner automorphism. An alternate description is as the central extension

$$\mathbf{Z}/2 \lhd H \to (\mathbf{Z}/2)^4$$

with κ-invariant $a^2+ab+b^2+ac+bd \in H^*((\mathbf{Z}/2)^4; \mathbb{Z}/2) = \mathbf{F}_2[a, b, c, d]$.

This second description shows that H_{21} is an extra-special 2-group (see [Q2]), since the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

is clearly non-singular mod 2. Consequently, the spectral sequence of the extension above collapses at E_4, and Quillen's structure theorem for the cohomology of such groups shows that

THEOREM 4.2.

$$H^*(H_{21}; \mathbb{Z}/2) = \mathbf{F}_2[\kappa_4][a, b, c, d]/(a^2+ab+b^2 + ac+bd, a^2b+ab^2+a^2c+ac^2+b^2d+bd^2).$$

Here κ_4 is the fourth power of the one dimensional generator on the fiber. However, the description above does not give us sufficient control, so we compare the calculation above with the description coming from the spectral sequence of the extension

$$\mathcal{Q}_8 \lhd H_{21} \to (\mathbf{Z}/2)^2.$$

Here, $E_2 = H^*((\mathbb{Z}/2)^2; \mathbb{Z}/2) \otimes H^*(\mathcal{Q}_8; \mathbb{Z}/2)$ since the action of $(\mathbb{Z}/2)^2$ on $\mathcal{Q}_8$ is via inner automorphisms, and these act trivially on passing to homology. A quick check using the previous calculation and the corresponding dimension counts in the first four dimensions shows that here $E_2 = E_\infty$.

Thus we have an additive description of $H^*(H_{21}; \mathbb{Z}/2)$ as

$$\mathbf{F}_2[\kappa_4, c, d](1, a, b, a^2, b^2, a^2b).$$

That is to say, it is a free module over $\mathbf{F}_2[\kappa_4, c, d]$ with six generators, as given above. However, the multiplicative structure is quite twisted. The relations are those given above. But note the particular consequences:

$$a^3 = (ac^2 + a^2d + acd), \qquad b^3 = (bd^2 + b^2c + bcd),$$

$$a^2b + b^2a = a^2c + ac^2 + b^2d + bd^2, \qquad ab = a^2 + b^2 + ac + bd.$$

The Group H

We calculate $H^*(H; \mathbb{Z}/2)$ now, using the spectral sequence associated to the (split) extension sequence

$$1 \to H_{21} \to H \to \mathbb{Z}/2 \to 1,$$

which allows us to write H as a semi-direct product $H_{21} \times_T \mathbb{Z}/2$. The E_2^{ij} term is $H^i(\mathbb{Z}/2); \{H^j(H_{21}; \mathbb{Z}/2)\})$, where the coefficients $H^j(H_{21}; \mathbb{Z}/2)$ are twisted by the action of $\mathbb{Z}/2$ given as follows: $a \leftrightarrow b$, $c \leftrightarrow d$, κ_4 fixed.

These cohomology groups are described as follows. First $E_2^{0j} = H^j(H_{21}; \mathbb{Z}/2)^{\mathbb{Z}/2}$, and, for $i > 0$ let $\tilde{H}^j = H^j(H_{21}; \mathbb{Z}/2)^{\mathbb{Z}/2}/\text{im}(1+T^*)$; then $E_2^{ij} = \tilde{H}^j \cup e^i$, where e is the non-trivial element in $E_2^{10} = H^1(\mathbb{Z}/2; \mathbb{Z}/2)$.

Explicitly, $\mathscr{M} = \mathbb{F}_2[k_4, c, d](1, a, b, a^2, b^2)$ is an $\mathbb{F}_2[\mathbb{Z}/2]$-submodule of $H^*(H_{21}; \mathbb{Z}/2)$. $\mathscr{M}$ splits as a direct sum of $\mathbb{F}_2[\mathbb{Z}/2]$ submodules:

$$\mathscr{M} \simeq \mathbb{F}_2[k_4, c, d] \oplus \mathbb{F}_2[k_4, c, d](a, b) \oplus \mathbb{F}_2[k_4, c, d](a^2, b^2).$$

The quotient module is $\mathscr{N} = H^*(H_{21}; \mathbb{Z}/2)/\mathscr{M} \simeq \mathbb{F}_2[k_4, c, d](a^2 b)$.

LEMMA 4.3. $\mathscr{M}^{\mathbb{Z}/2} = \mathbb{F}_2[k_4, c+d, cd](1, a+b, a^2+b^2, ab, a^2b+ab^2)$.

Proof. We introduce the notation $s_1 = b$, $t_1 = c+d$, $s_2 = ab$, $t_2 = cd$. Clearly $\mathbb{F}_2[k_4, c, d]^{\mathbb{Z}/2} = \mathbb{F}_2[k_4, t_1, t_2]$. Moreover, we can write $\mathbb{F}_2[k_4, c, d] = \mathbb{F}_2[k_4, t_1, t_2](1, c)$, and from this we see that $1, a+b$, $ac+bd$, a^2+b^2, a^2c+b^2d generate $\mathscr{M}^{\mathbb{Z}/2}$ freely over $\mathbb{F}_2[k_4, t_1, t_2]$. Now, note that $ab = (a+b)^2 + (ac+bd)$. Also,

$$\begin{aligned} a^2c + b^2d &= (ac+bd)(a+b) + (c+d)(ab) \\ &= (a^2+b^2+ab)(a+b) + (c+d)(ab) \\ &= s_1^3 + s_1 s_2 + t_1 s_2 \\ &= s_1^2 t_1 + s_1 t_2 + s_1 s_2 + t_1 s_2. \quad \blacksquare \end{aligned}$$

Now, consider the exact sequence of $\mathbb{F}_2[\mathbb{Z}/2]$-modules

$$1 \to \mathscr{M} \to H^*(H_{21}; \mathbb{Z}/2) \to \mathscr{N} \to 1$$

which induces a long exact sequence in cohomology

$$\begin{aligned} 1 &\longrightarrow \mathscr{M}^{\mathbb{Z}/2} \longrightarrow H^*(H_{21}; \mathbb{Z}/2)^{\mathbb{Z}/2} \longrightarrow \mathscr{N}^{\mathbb{Z}/2} \xrightarrow{\delta^0} H^1(\mathbb{Z}/2; \mathscr{M}) \\ &\longrightarrow H^1(\mathbb{Z}/2; H^*(H_{21}; \mathbb{Z}/2)) \longrightarrow \cdots. \end{aligned}$$

The action of the generator T of $\mathbb{Z}/2$ on the quotient module $\mathcal{N}$ is given by

$$T(a^2b) = ab^2 = a^2b + a^2c + ac^2 + b^2d + bd^2$$
$$\cong a^2b (\text{mod } \mathcal{M}).$$

Thus $\mathcal{N}^{\mathbb{Z}/2} \simeq \mathbb{F}_2[k_4, c+d, cd](a^2b)$. Also

$$\delta^0(a^2b) = a^2b + ab^2 = a^2c + ac^2 + b^2d + bd^2,$$

which is $(1+T)(a^2c + ac^2)$ and hence represents 0 in $H^1(\mathbb{Z}/2; \mathcal{M})$. It follows that the class

$$L_3 = a^2b + a^2c + ac^2$$

is invariant under T in $H^3(H_{21}; \mathbb{Z}/2)$ and maps onto the generator of $\mathcal{N}^{\mathbb{Z}/2}$. We have proved the first part of

THEOREM 4.4. *$H^*(H_{21}; \mathbb{Z}/2)^{\mathbb{Z}/2}$ is generated as an $\mathbb{F}_2$-algebra by $s_1 = a+b$, $t_1 = c+d$, k_4, $s_2 = ab$, $t_2 = cd$, and $L_3 = a^2b + a^2c + ac^2$. As a module over $\mathbb{F}_2[t_1, t_2, t_4]$, it is free on 1, s_1, s_1^2, s_2, s_1s_2, and L. Therefore, the following is a complete list of relations for $H^*(H_{21}; \mathbb{Z}/2)^{\mathbb{Z}/2}$:*

$$s_1^3 = s_1^2t_1 + s_1t_2$$
$$s_2^2 = s_1s_2t_1 + s_2t_1^2$$
$$s_1^2s_2 = t_1L + s_2t_1^2 + s_2t_2$$
$$s_1L = t_1L + s_1s_2t_1 + s_2t_1^2 + s_2t_2$$
$$s_2L = s_1s_2t_1^2 + s_2t_1t_2 + s_2t_1^3$$
$$L^2 = t_1t_2L + s_1s_2t_1^3 + s_2t_1^2t_2 + s_2t_1^4.$$

Proof. The relations were found using MACSYMA by applying the relations between a, b, c, d already described in $H^*(H_{21}; \mathbb{Z}/2)$. The rest of the theorem follows from the discussion preceeding it. ∎

We now turn to the calculation of $H^*(H; \mathbb{Z}/2)$ itself.

THEOREM 4.5. *$H^*(H; \mathbb{Z}/2)$ is generated by classes denoted e_1, s_1, t_1, s_2, t_2, L_3, and k_4. We can, in fact, write $H^*(H; \mathbb{Z}/2)$ as*

$$H^*(H; \mathbb{Z}/2) \simeq \mathbb{F}_2[k_4, t_1, t_2](1, s_1, s_1^2, s_2, s_1s_2, L) \oplus e_1 \circ \mathbb{F}_2[e_1, t_2, k_4](1, L).$$

Note. Upon restricting to $H^*(H_{21}; \mathbb{Z}/2)$, $e_1 \mapsto 0$ and the other classes restrict to the classes of names the same as those in the previous theorem.

Proof. The claim is really that the spectral sequence of the extension

$$1 \to H_{21} \to H \to \mathbb{Z}/2 \to 1$$

collapses. The quotient $\mathbb{Z}/2$ splits, which detects the generator e_1. $E_2^{0,*}$ was calculated in the previous theorem, and in noting the traces

$$(1+T)(a) = a+b = s_1$$

$$(1+T)(c) = c+d = t_1$$

$$(1+T)(a^2+ac) = a^2+b^2+ac+bd = ab = s_2,$$

we find that $H^1(\mathbb{Z}/2; H^*(H_{21}; \mathbb{Z}/2))$ is generated by t_2, L_3, and k_4. Indeed, it is isomorphic to $e_1 \cup \mathscr{A}$, where $\mathscr{A} \cong \mathbb{F}_2[k_4, t_2](1, L_3)$, and $H^i(\mathbb{Z}/2; H^*(H_{21}; \mathbb{Z}/2)) \cong e_1^i \mathbb{F}_2[k_4, t_2](1, L_3)$. From this it follows that the only possible differentials are determined by their values on k_4. But the class k_4 is the Stiefel–Whitney class of a representation of H, and so it must be an infinite cycle. ▮

Note that we have filtered relations $s_1 e_1 \cong t_1 e_1 \cong s_2 t_2 \cong 0$.

COROLLARY 4.6. *The Poincaré series for $H^*(H; \mathbb{Z}/2)$ has the form*

$$P(H) = \frac{1}{(1-t)^3}.$$

Proof. From the expression for $H^*(H; \mathbb{Z}/2)$ in the theorem above we can write

$$P(H) = \frac{1+t+2t^2+2t^3}{(1-t)(1-t^2)(1-t^4)} + \frac{t+t^4}{(1-t)(1-t^2)(1-t^4)}$$

but this is

$$\frac{1+2t+2t^2+2t^3+t^4}{(1-t)(1-t^2)(1-t^4)} = \frac{(1+t+t^2+t^3)(1+t)}{(1-t)(1-t^2)(1-t^4)} = \frac{1}{(1-t)^3}. \quad ▮$$

Recall that $H^*(H; \mathbb{Z}/2)$ is generated by classes e_1, s_1, t_1, s_2, t_2, L_3, and k_4, and we know the relations up to a multiple of e_1 (since we know the restrictions to $H^*(H_{21}; \mathbb{Z}/2)$):

$$
\begin{aligned}
&(1)\quad s_1e_1 = s_2e_1 = t_1e_1 = 0\\
&(2)\quad s_1^3 = s_1^2t_1 + s_1t_2\\
&(3)\quad s_2^2 = s_1s_2t_1 + s_2t_1^2\\
&(4)\quad s_1^2s_2 = t_1L + s_2t_1^2 + s_2t_2\\
&(5)\quad s_1L = t_1L + s_1s_2t_1 + s_2t_1^2 + s_2t_2\\
&(6)\quad s_2L = s_1s_2t_1^2 + s_2t_1t_2 + s_2t_1^3\\
&(7)\quad L^2 = t_1t_2L + s_1s_2t_1^3 + s_2t_1^2t_2 + s_2t_1^4.
\end{aligned}
\tag{4.7}
$$

To detect any possible missing e_1-factors we would like to compute the restriction map to $H^*(H_{22}; \mathbb{Z}/2)$. The following are straightforward:

$$
\begin{aligned}
s_1 &\mapsto 0\\
e_1 &\mapsto b\\
t_1 &\mapsto b + c\\
s_2 &\mapsto a(b + c)\\
t_2 &\mapsto s\\
k_4 &\mapsto r^2 + rs.
\end{aligned}
$$

Thus relations (1), (2), and (3) are valid. The only difficulty is that the restriction to $H^*(H_{22}; \mathbb{Z}/2)$ of the class L is not at all clear. The class L is determined up to a multiple of e_1 by $\mathrm{res}^H_{H_{21}}(L) = a^2b + a^2c + ac^2$.

To go further we consider the Gysin sequence for the extension

$$1 \longrightarrow H_{22} \longrightarrow H \longrightarrow \mathbb{Z}/2 \longrightarrow 1,$$

$$\longrightarrow H^{*-1}(H; \mathbb{Z}/2) \xrightarrow{\cup s_1} H^*(H; \mathbb{Z}/2) \xrightarrow{\mathrm{res}} H^*(H_{22}; \mathbb{Z}/2)$$

$$\xrightarrow{\mathrm{tr}^*} H^*(H; \mathbb{Z}/2) \longrightarrow .$$

From the restrictions calculated above, we see that neither class (a) nor class (r) is in the image of restriction, so the transfer map is non-zero on each. In fact we must have

$$
\begin{aligned}
\mathrm{tr}^*(a) &= e_1\\
\mathrm{tr}^*(b) &= t_2 + s_1^2 + s_1t_1
\end{aligned}
$$

since these are the only available classes in the kernel of cupping with s_1 in $H^*(H; \mathbb{Z}/2)$. Thus we also find that

$$
\begin{aligned}
\mathrm{tr}^*(br) &= e_1(t_2 + s_1^2 + s_1t_1) = t_2e_1\\
\mathrm{tr}^*(as) &= e_1t_2.
\end{aligned}
$$

So $\mathrm{tr}^*(br+as)=0$ and $br+as$ must be in the image of restriction. The class L must be involved, so that

$$\mathrm{res}^H_{H_{22}}(L)=br+as+\chi$$

with $\chi\in\mathrm{span}\langle b, c, ab+ac, s\rangle$.

We can dixcover more about the class χ by considering the restriction to $H^*(H_{22};\mathbb{Z}/2)$ of the expression

$$s_1^2s_2+t_1L+s_2t_1^2+s_2t_2.$$

We find that $\mathrm{res}^H_{H_{22}}(s_1^2s_2+t_1L+s_2t_1^2+s_2t_2)=(b+c)\chi+abc^2+ac^3$. Since this expression is a relation in $H^*(H_{21};\mathbb{Z}/2)$ the restriction to $H^*(H_{22};\mathbb{Z}/2)$ must be a multiple of e_1. Thus

$$\chi=a(b+c)\,c+bY,$$

and so $\mathrm{res}^H_{H_{22}}(L)=br+as+ac^2+abc+bY$. But recall that L was only determined up to a multiple of e_1 so we may assume $Y=0$ and

$$\mathrm{res}^H_{H_{22}}(L)=br+as+ac^2+abc.$$

We now find that relations (4), (5), and (6) of (4.7) are all valid as stated since their restrictions to $H^*(H_{22};\mathbb{Z}/2)$ are zero, but that

$$\begin{aligned}\mathrm{res}^H_{H_{22}}&(L^2+t_1t_2L+s_1s_2t_1^3+s_2t_1^2t_2+s_2t_1^4)\\&=b^2r^2+abs^2=b^2(r^2+rs)+bs(as+br+ac^2+abc)\\&=\mathrm{res}^H_{H_{22}}(e_1^2k_4+e_1t_2L),\end{aligned}$$

so we have shown

THEOREM 4.8. *$H^*(H;\mathbb{Z}/2)=\mathbb{F}_2[e_1, s_1, t_1, s_2, t_2, L_3, k_4]/\mathcal{R}$, where $\mathcal{R}$ is the set of relations*

$$\begin{aligned}&s_1e_1=s_2e_1=t_1e_1=0\\&s_1^3=s_1^2t_1+s_1t_2\\&s_2^2=s_1s_2t_1+s_2t_1^2\\&s_1^2s_2=t_1L+s_2t_1^2+s_2t_2\\&s_1L=t_1L+s_1s_2t_1+s_2t_1^2+s_2t_2\\&s_2L=s_1s_2t_1^2+s_2t_1t_2+s_2t_1^3\\&L^2=t_1t_2L+s_1s_2t_1^3+s_2t_1^2t_2+s_2t_1^4+e_1^2k_4+e_1t_2L.\end{aligned}$$

Note. We have squaring operations of the form $\mathrm{Sq}^1(s_2)=s_1s_2$, $\mathrm{Sq}^1(t_2)=(e_1+t_1)\,t_2$, $\mathrm{Sq}^1(L) = s_1s_2t_1+s_2t_1^2$, $\mathrm{Sq}^2(L)=e_1k_4+(t_2+e_1^2)\,L+t_1^2L+s_2t_1t_2$, $\mathrm{Sq}^1(k_4)=0$, $\mathrm{Sq}^2(k_4)=(t_2+e_1^2+t_1^2)\,k_4$.

COROLLARY 4.9. *The nilpotent elements of $H^*(H; \mathbb{Z}/2)$ are all contained in the ideal (s_1).*

More exactly, by using the relation $D_8 * D_8 \cong \mathcal{Q}_8 * \mathcal{Q}_8$, and the fact that $H_{21} \cong \mathcal{Q}_8 * \mathcal{Q}_8$, it is not hard to see that $H^*(H_{21}; \mathbb{Z}/2)$ actually has no nilpotents in its cohomology ring. Consequently, the same is true for $H^*(H; \mathbb{Z}/2)$.

The Group W

We now turn to the group $H^*(W; \mathbb{Z}/2)$. Since H_{21} is normal in W it follows from the double coset formula that $H^*(W; \mathbb{Z}/2) = (\mathrm{res}_H^{H_{21}})^{-1}$ $(H^*(H_{21}; \mathbb{Z}/2)^{\mathcal{S}_3})$. Hence, we begin by obtaining $H^*(H_{21}; \mathbb{Z}/2)^{\mathcal{S}_3}$.

Note that $H^*(H_{21}; \mathbb{Z}/2)^{\mathcal{S}_3} \subset H^*(H_{21}; \mathbb{Z}/2)^{\mathbb{Z}/2}$, and by writing $\Sigma_\tau = 1 + \tau + \tau^2$, where τ generates the normal $\mathbb{Z}/3 \subset \mathcal{S}_3$, we have that $T\Sigma_\tau = \Sigma_\tau T$, so

$$\Sigma_\tau H^*(H_{21}; \mathbb{Z}/2)^{\mathbb{Z}/2} = H^*(H_{21}; \mathbb{Z}/2)^{\mathcal{S}_3}.$$

To obtain this algebra explicitly, note first that $\mathbb{F}_2[k_4, t_1, t_2]^{\Sigma_3} = \mathbb{F}_2[k_4, t_2 + t_1^2, t_1 t_2]$. Moreover

$$\mathbb{F}_2[k_4, t_1, t_2] = \mathbb{F}_2[k_4, t_1, t_2]^{\Sigma_3} (1, t_1, t_1^2).$$

Consequently, $H^*(H_{21}; \mathbb{Z}/2)^{\mathbb{Z}/2}$ is free over $\mathbb{F}_2[k_4, t_2 + t_1^2, t_1 t_2]$ with the 18 generators

$$\{1, t_1, t_1^2\} \times \{1, s_1, s_1^2, s_2, s_1 s_2, L\}.$$

Let us calculate some traces:

$$\Sigma(s_1) = (a+b) + (a) + (b) = 0$$

$$\Sigma(s_1^2) = 0$$

$$\begin{aligned}\Sigma(s_2) &= (ab) + b(a+b) + (a+b)\,a = a^2 + ab + b^2 \\ &= s_2 + s_1^2\end{aligned}$$

$$\Sigma(s_1, s_2) = s_1 s_2$$

$$\begin{aligned}\Sigma(L) &= (a^2 b + a^2 c + ac^2) + (b^2(a+b) + b^2(c+d) + b(c+d)^2) \\ &\quad + ((a+b)^2\, a + (a+b)^2\, d + (a+b)\, d^2) \\ &= L + a^3 + b^3 + a^2 d + b^2 c + ad^2 + bc^2 \\ &= L + s_1^3 + s_1^2 t_1 + s_1 t_1^2 \\ &= L + s_1 t_2 + s_1 t_1^2.\end{aligned}$$

We must also calculate

$$\begin{aligned}
\Sigma(c) &= (c+d)+(d)+(c)=0\\
\Sigma(c^2) &= 0\\
\Sigma(s_1t_1) &= ac+bd=s_2+s_1^2\\
\Sigma(s_1^2t_1) &= (a+b)^2\,(c+d)+a^2d+b^2c=a^2c+b^2d\\
&= s_1^2t_1+s_1s_2+s_1t_2+s_2t_1\\
\Sigma(s_1t_1^2) &= (a+b)(c+d)^2+ad^2+bc^2=ac^2+bd^2\\
&= s_1^2t_1+s_1t_2+s_2t_1\\
\Sigma(s_1^2t_1^2) &= s_2^2+s_1^4.
\end{aligned}$$

Finally, we can use the invariance of $s_2+s_1^2$, s_1s_2, and $L+s_1t_2+s_1t_1^2$ to conclude that

$$\begin{aligned}
\Sigma(s_2t_1) &= \Sigma(s_1^2t_1)\\
\Sigma(s_2t_1^2) &= \Sigma(s_1^2t_1^2)\\
\Sigma(s_1s_2t_1) &= \Sigma(s_1s_2t_1^2)=0\\
\Sigma(Lt_1) &= \Sigma(s_1t_1t_2+s_1t_1^3)=(t_2+t_1^2)\cdot(s_2+s_1^2)\\
\Sigma(Lt_1^2) &= \Sigma(s_1t_1^2t_2+s_1t_1^4)=(t_2+t_1^2)\cdot\Sigma(s_1t_1^2).
\end{aligned}$$

Thus $H^*(H_{21};\mathbb{Z}/2)^{\Sigma_3}$ is free as a module over $\mathbb{F}_2[k_4, t_2+t_1^2, t_1t_2]$ with the 6 generators

$$\{1, s_2+s_1^2, s_1s_2, L+s_1t_2+s_1t_1^2, s_2t_1+s_1t_2+s_1^2t_1, s_2^2+s_1^4\}.$$

It now follows that

THEOREM 4.10.

$$\begin{aligned}
H^*(W;\mathbb{Z}/2) = {} & \mathbb{F}_2[k_4, t_2+t_1^2, t_1t_2](1, s_2+s_1^2, s_1s_2, L+s_1t_2\\
& +s_1t_1^2, s_1t_2+s_2t_1+s_1^2t_1, t_2^2+t_1^4)\\
& \oplus e_1\cdot\mathbb{F}_2[e_1, k_4, t_2+t_1^2](1, L+s_1t_2+s_1t_1^2)
\end{aligned}$$

with Poincaré series

$$\frac{1+t+2t^2+4t^3+2t^4+t^5+t^6}{(1-t^2)(1-t^3)(1-t^4)}.$$

The Group W'

As was the case with W, we have that $H^*(W'; \mathbb{Z}/2) = (\mathrm{res}^H_{H_{22}})^{-1} H^*(H_{22}; \mathbb{Z}/2)^{\mathscr{S}_3}$, where now, the $\mathbb{Z}/3$ action is given by the formulae of Theorem 2.4. The description of $H^*(H_{22}; \mathbb{Z}/2)$ given in Theorem 4.2 is not entirely precise for the identity of the elements r, s. However, in homology an element γ of order 3 in the normalizer of H_{22} in W' acts on duals of r, s by $\gamma(\bar{r}) = \bar{s}$, $\gamma(s) = \bar{r} + \bar{s}$. (Here, the bars denote homology duals.) Actually, once we have choosen $\bar{r}$ then the action of γ serves to determine $\bar{s}$. Similarly in cohomology we can assume $\gamma(r) = s$, $\gamma(s) = r + s$. In dimension 1 the situation is clearer and we have $\gamma(a) = b + c$, $\gamma(b) = a + b + c$, and $\gamma(c) = c$. Similarly, there is an element θ of order 2 in the normalizer which acts by $\theta(r) = r + s$, $\theta(s) = \theta\gamma(r) = \gamma^2\theta(r) = s$, while in dimension 1 this same element acts by $\theta(a) = a + b$, $\theta(b) = b$, $\theta(c) = c$:

$$\begin{array}{llll} \gamma\colon & a \mapsto b + c & \theta\colon & a \leftrightarrow a + b \\ & b \mapsto a + b + c & & r \leftrightarrow r + s \\ & c \mapsto c & & \text{fixes } b, c, s. \\ & r \mapsto s \mapsto r + s & & \end{array}$$

To calculate the Σ_3 invariants we begin studying the $\mathbb{Z}/2$ invariants. Write

$$H^*(H_{22}; \mathbb{Z}/2) = \mathbb{F}_2[r, s, c] \oplus \mathbb{F}_2[r, s, c](a, b) \oplus \mathbb{F}_2[r, s, c](ab)$$

and denote by $\mathscr{M}$ the submodule, over $\mathbb{F}_2(\mathbb{Z}/2]$,

$$\mathscr{M} = \mathbb{F}_2[r, s, c] \oplus \mathbb{F}_2[r, s, c](a, b).$$

Then the quotient module $\mathscr{N} = H^*(H_{22}; \mathbb{Z}/2)/\mathscr{M}$ is

$$\mathscr{N} \simeq \mathbb{F}_2[r, s, c](ab).$$

The exact sequence

$$0 \to \mathscr{M} \to H^*(H_{22}; \mathbb{Z}/2) \to \mathscr{N} \to 0$$

induces a long exact sequence in cohomology

$$0 \longrightarrow \mathscr{M}^{\mathbb{Z}/2} \longrightarrow (H^*(H_{22}; \mathbb{Z}/2))^{\mathbb{Z}/2} \longrightarrow \mathscr{N}^{\mathbb{Z}/2} \xrightarrow{\delta} H^1(\mathbb{Z}/2; \mathscr{M}) \longrightarrow \cdots.$$

LEMMA 4.11. $\mathscr{M}^{\mathbb{Z}/2} = \mathbb{F}_2[c, s, r^2 + rs](1, b, as + br)$.

THEOREM 4.12. *$\mathscr{N}^{\mathbb{Z}/2} \simeq \mathbb{F}_2[c, s, r^2 + rs](ab)$ and $\delta(ab) = 0$.*

Proof. $\theta(ab) = (a+b)b = ab + b^2 = ab + bc \equiv ab \bmod \mathcal{M}$. $\delta(ab) = (1+\theta)(ab) = b^2 = bc = (1+\theta)(ac) = 0 \in H^1(\mathbb{Z}/2; \mathcal{M})$. ∎

THEOREM 4.13.

$$H^*(H_{22}; \mathbb{Z}/2)^{\mathbb{Z}/2} = \mathbb{F}_2[c, s, r^2 + rs](1, b, as + br, ab + ac).$$

Proof. Clear from the above. Note that $(ab + ac)$ represents the term $(ab) \in \mathcal{N}^{\mathbb{Z}/2}$. ∎

Noting that $\mathbb{F}_2[c, s, r^2 + rs]^{\Sigma_3} = \mathbb{F}_2[c, r^2 + rs + s^2, r^2 s + rs^2]$ and $\mathbb{F}_2[c, s, f^2 + rs] = \mathbb{F}_2[c, s, r^2 + rs]^{\Sigma_3}(1, s, s^2)$, we need to consider the traces $(1 + \gamma + \gamma^2)$ on the 12 generators

$$\{1, s, s^2\} \circ \{1, b, as + br, ab + ac\}.$$

We find, in particular, that

$$(1 + \gamma + \gamma^2)(ab + ac) = (ab + ac)$$
$$(1 + \gamma + \gamma^2)(as + br) = as + cs + br$$
$$(1 + \gamma + \gamma^2)(bs^2) = br^2 + as^2 + cs^2,$$

all the others being either 0 or repetitions of those above. Consequently, we have

$$H^*(H_{22}; \mathbb{Z}/2)^{\Sigma_3} = \mathbb{F}_2[c, r^2 + rs + s^2, r^2 s + rs^2](1, ab + ac, as + cs + br, as^2 + cs^2 + br^2).$$

It now follows that

THEOREM 4.14.

$$H^*(W'; \mathbb{Z}/2) = \mathbb{F}_2[e_1 + t_1, k_4 + t_2^2, t_2 k_4](1, s_2, L + e_1 t_2 + t_1 t_2 + t_1 s_2, t_2 L + e_1 k_4 + e_1 t_2^2 + t_1 t_2^2 + t_1 s_2 t_2)$$
$$\oplus\, \mathbb{F}_2[e_1 + t_1, t_2, k_4 + t_2^2](s_1, s_1^2, s_1 s_2, s_1^2 s_2)$$

with Poincaré series

$$\frac{1 + t + 2t^2 + 3t^3 + 2t^4 + 3t^5 + 2t^6 + t^7 + t^8}{(1-t)(1-t^4)(1-t^6)}.$$

Proof. As already stated, this follows from the double coset formula. Note that $s_1^3 = s_1^2 t_1 + s_1 t_2$ and the terms on the right are already represented. Similarly $s_1 L = s_1^2 s_2 + s_1 t_1 s_2$ and these terms are represented. Thus the second summand in the given expression is all multiples of s_1 in $H^*(H; \mathbb{Z}/2)$.

The Group M_{12}

From Theorems 4.10 and 4.14 we have the image of $H^*(W;\mathbb{Z}/2)$, $H^*(W';\mathbb{Z}/2)$ in $H^*(H;\mathbb{Z}/2)$. But the intersection of these two images is $H^*(M_{12};\mathbb{Z}/2)$. We now list the generators in $H^*(M_{12};\mathbb{Z}/2)$ through dimension 7:

Name	Generator	Dimension
α_2	$s_2+s_1^2$	2
x_3	$s_1 s_2$	3
y_3	$(s_1^2+s_2)\,t_1+s_1 t_2$	3
z_3	$L+(s_1+t_1+e_1)\,t_2+s_1 t_1^2$	3
β_4	$k_4+t_2^2+e_1^4+t_1^4$	4
γ_5	$(t_2+e_1^2+t_1^2)\,L+e_1(k_4+t_2^2+e_1^2 t_2)$ $+t_1 t_2^2+t_1^3 t_2+s_1 t_2^2+s_1^2 t_1 t_2+t_1 s_2 t_2+s_1 t_1^4$	5
δ_6	$(t_2+e_1^2+t_1^2)\,k_4+(e_1^2+t_1^2)\,t_2^2$	6
ε_7	$(t_1+e_1)\,t_2 k_4$	7

(4.15)

Note that since M_{12} is simple $H^1(M_{12};\mathbb{Z}/2)=0$. We also have the Steenrod operations,

$$
\begin{array}{llll}
Sq^1(\alpha)=x, & Sq^2(\alpha)=\alpha^2, & & \\
Sq^1(x)=0, & Sq^2(x)=\alpha x, & Sq^3(x)=x^2, & \\
Sq^1(y)=\alpha^2, & Sq^2(y)=\alpha z=\alpha(x+y), & Sq^3(y)=y^2, & \\
Sq^1(z)=\alpha^2, & Sq^2(z)=\gamma, & Sq^3(z)=z^2, & \\
Sq^1(\beta)=0, & Sq^2(\beta)=\delta, & Sq^3(\beta)=\varepsilon, & Sq^4(\beta)=\beta^2, \\
Sq^1(\gamma)=z^2, & Sq^2(\gamma)=0, & Sq^3(\gamma)=0, & Sq^4(\gamma)=\beta\gamma+z\delta+\alpha\varepsilon, \\
Sq^1(\delta)=\varepsilon, & Sq^2(\delta)=0, & Sq^3(\delta)=0, & Sq^4(\delta)=\beta\delta, \\
Sq^1(\varepsilon)=0, & Sq^2(\varepsilon)=0, & Sq^3(\varepsilon)=0, & Sq^4(\varepsilon)=\beta\varepsilon,
\end{array}
$$

and the following are a complete set of relations among the generators in (4.15):

$$
\begin{array}{ll}
\alpha(x+y+z)=0, & x^3=\alpha^3 x+\alpha\beta x+x\delta, \\
xy=\alpha^3+x^2+y^2, & xz=\alpha^3+y^2, \\
x^2y=\alpha^3 z+\alpha\beta z+y\delta+\alpha\varepsilon, & yz=\alpha^3+x^2, \\
\varepsilon x=\beta x^2+\alpha^2 x^2, & \alpha\gamma=\alpha^2 y,
\end{array}
$$

$$\begin{aligned} \varepsilon y &= \alpha^2\delta + \alpha^2 y^2 + \beta x^2 + \beta y^2, & y\gamma &= \alpha y^2 \\ \varepsilon z &= \gamma^2 + \alpha^2\delta + \alpha^2 x^2 + \beta x^2 + \beta z^2, & x\gamma &= \alpha^4 + \alpha z^2, \\ z^4 &= \gamma\varepsilon + x^4 + \alpha^4\beta + z^2\delta, & \varepsilon^2 &= z^3\gamma + \alpha^2\beta\delta + \alpha^5\beta \\ & & &\quad + z\beta\varepsilon + z\delta(\gamma + \alpha z) \\ & & &\quad + \beta^2(\alpha^3 + xz + yz). \end{aligned}$$

This result was checked by the program MACAULAY, and we thank David Rusin for his help at this point. The generators in (4.15) above were analyzed by the program and shown to satisfy exactly this set of relations in $H^*(H; \mathbb{Z}/2)$. Moreover, the program calculated the Poincaré series for this algebra as

$$q(t) = \frac{1 + t^2 + 3t^3 + t^4 + 3t^5 + 4t^6 + 2t^7 + 4t^8 + 3t^9 + t^{10} + 3t^{11} + t^{12} + t^{14}}{(1 - t^4)(1 - t^6)(1 - t^7)}. \tag{4.16}$$

Then the program was used to show

THEOREM 4.17. *The algebra is Cohen–Macaulay. Specifically, it is free and finitely generated as a module over the polynomial algebra* $\mathbb{F}_2[\beta_4, \delta_6, \varepsilon_7]$.

Using a result due to Webb [We], we now derive the Poincaré series for $H^*(M_{12}; \mathbb{Z}/2)$. Write $G = M_{12}$, $X = |\mathcal{A}_2(M_{12})|$ with the conjugation action. Then Webb's result is

$$\hat{H}^*(G; \mathbb{Z}/2) \oplus \bigoplus_{\substack{\sigma_i\ i\text{-cell} \\ \text{in } X/G \\ i \text{ odd}}} \hat{H}^*(G_{\sigma_i}; \mathbb{Z}/2) \cong \bigoplus_{\substack{\sigma_i\ i\text{-cell} \\ \text{in } X/G \\ i \text{ even}}} \hat{H}^*(G_{\sigma_i}; \mathbb{Z}/2).$$

Using our explicit knowledge of the equivariant structure of X, we obtain many cancellations, effectively simplifying the formula to yield

$$H^*(M_{12}) \oplus H^*(H) \cong H^*(W) \oplus H^*(W').$$

This involves cancelling isotropy groups for vertices with edges and other isotropy groups of edges with isotropy groups of 2-simplexes. After that a few explicit calculations are required to show that the mod 2 cohomology rings of the isotropy groups of most of the remaining edges cancel off with those for remaining vertices and 2-simplices.

From our previous calculations this can be used to determine the Poincaré series for M_{12}. After some simplification we obtain

COROLLARY 4.18. *The Poincaré series for $H^*(M_{12}; \mathbb{Z}/2)$ is*

$$p(t) = \frac{1 + 3t^3 + 3t^6 + t^9}{(1-t^2)(1-t^4)(1-t^6)} = \frac{(1+t^3)^2}{(1-t^2)(1-t^3)(1-t^4)}.$$

Then we compare the polynomials in Corollary 4.18 and (4.16). It is directly seen that they are identical, and from this the proof that $H^*(M_{12}; \mathbb{Z}/2)$ is the algebra above is complete.

It would be useful to understand, in some sense, meanings for the generating classes and their relations, for example, in terms of characteristic classes for representations or images of classes from $H^*(\mathscr{S}_{12}; \mathbb{Z}/2)$, a topic we hope to consider in a sequel.

5. TRIVALENT GRAPHS AND THE AMALGAMATED PRODUCT

Given a situation such as that H, W, and W', we can find a universal completion $\Gamma = W *_H W'$ which makes the diagram below commute

$$\begin{array}{ccc} H & \hookrightarrow & W' \\ \downarrow & & \downarrow{\scriptstyle \phi_2} \\ W & \xrightarrow{\phi_1} & \Gamma \end{array}$$

and such that any Γ' which satisfies this (generated by W and W') is a quotient of Γ. Γ is called the amalgamated product of W and W' over H. It is well known (see [Se]) that an amalgamated product as above will act on a tree with finite isotropy and orbit space of the form

$$W \overset{H}{\text{——}} W'.$$

In [G], Goldschmidt analyzed the situation for actions on the cubic tree (the tree of valence 3) and obtained a classification of finite primitive amalgams of index (3, 3) (this refers to the indexes $[W:H]$, $[W':H]$). He shows that M_{12} is 1 of 15 such amalgams, necessarily a quotient of the universal one Γ.

From this we deduce the existence of the extension

$$1 \to \Gamma' \to \Gamma \to M_{12} \to 1, \tag{5.1}$$

where Γ' is a free group (it has cohomological dimension 1). Using the formula for Euler characteristics in [B], we have, on the one hand

$$\chi(\Gamma) = \frac{1}{192} + \frac{1}{192} - \frac{1}{64} = -\frac{1}{192}$$

(amalgamated product) and also

$$\chi(\Gamma)=\frac{\chi(\Gamma')}{|M_{12}|}.$$

Hence $\chi(\Gamma')=95{,}040(-1/192)=-495$ and it follows that $\Gamma'\cong *_1^{496}\mathbb{Z}$, the free group on 496 generators.

We can now state the main result of this section

THEOREM 5.2. *The map $\Gamma\to M_{12}$ induces an isomorphism*

$$H^*(M_{12};\mathbb{Z}/2)\to H^*(\Gamma;\mathbb{Z}/2).$$

Proof. Consider the map

$$\mathrm{res}_H^W\oplus\mathrm{res}_H^{W'}\colon H^*(W)\oplus H^*(W')\to H^*(H).$$

Its kernel is clearly $\mathrm{im}(\mathrm{res}_H^W)\cap\mathrm{im}(\mathrm{res}_H^{W'})\cong H^*(M_{12})$ by Theorem 3.1. On the other hand, from [We] we have seen that $H^*(W)\oplus H^*(W')\cong H^*(M_{12})\oplus H^*(H)$. Hence $\mathrm{res}_H^W\oplus\mathrm{res}_H^{W'}$ is onto. On the other hand, from the action of Γ on a tree described previously, there is a well-known long exact sequence

$$\cdots\to H^i(\Gamma)\to H^i(W)\oplus H^i(W')\to H^i(H)\to H^{i+1}(\Gamma)\to\cdots.$$

As it comes from a Mayer–Vietoris sequence the same map as before arises, and hence the sequence splits and

$$H^*(W)\oplus H^*(W')\cong H^*(\Gamma)\oplus H^*(H).$$

Consequently, by rank considerations and the fact that the finite subgroups in Γ are mapped isomorphically into M_{12} under the projections, the proof is complete. ∎

COROLLARY 5.3. *$H^1(\Gamma';\mathbb{Z}/2)$ is an M_{12}-acyclic $\mathbb{Z}/2(M_{12})$-module of rank 496 which is not projective.*

Proof. The proof follows from considering the spectral sequence over $\mathbb{Z}/2$ associated to (5.1) and the observation that 64 does not divide 496. ∎

We do not know whether this representation has been documented in the literature, but it has radically different cohomological behavior at distinct primes dividing $|M_{12}|$. For example, at $p=3$ we have a sequence

$$H^{p-2}(M_{12};H^1(\Gamma';\mathbb{Z}/3))\to H^p(M_{12};\mathbb{Z}/3)\to H^p(W;\mathbb{Z}/3)\oplus H^p(W';\mathbb{Z}/3)$$

and clearly the term on the left must be non-trivial.

APPENDIX 1

The Double Coset Decomposition of M_{12} with Respect to H

There were 44 double cosets in all. Twelve of them had only the identity in common with H, so they were ignored. Also, H, itself is its own double coset, so this too was ignored. The remaining 31 generators and the intersections with H of their conjugates are tabulated below, together with the elements which go to the intersection elements $\alpha \longrightarrow h\alpha h^{-1}$, where h is the double coset generator.

THE INTERSECTIONS OF ORDER 2

Coset generator α	h and $\alpha h \alpha^{-1}$ in intersection
(1,2,3)(4,5,6)(7,8,9)	(4,7)(5,8)(6,9)(10,11)⟶(4,7)(5,8)(6,9)(10,11).
(1,3,10,2)(4,5,9,7)	(1,10)(2,3)(6,8)(11,12)⟶(1,10)(2,3)(6,8)(11,12).
(1,4,5,9,12,10,2,8,3,6,7)	(2,4)(3,7)(6,8)(10,11)⟶(1,10)(2,3)(6,8)(11,12).
(1,8,5,9,6)(2,7,10,4,3)	(1,10)(2,9)(3,5)(4,8)(6,7)(11,12)⟶(1,10)(2,9)(3,5)(4,8)(6,7)(11,12).
(1,5,4,3,2,8,7,12)(6,10)	(1,11)(2,4)(3,7)(5,6)(8,9)(10,12)⟶(1,10)(2,9)(3,5)(4,8)(6,7)(11,12).
(1,7)(2,5,6,12,10,9,4,3)	(1,12)(2,6)(3,8)(4,5)(7,9)(10,11)⟶(1,10)(2,9)(3,5)(4,8)(6,7)(11,12).
(1,5,6,4)(2,8,3,10)	(1,10)(2,5)(3,9)(4,6)(7,8)(11,12)⟶(1,10)(2,7)(3,4)(5,6)(8,9)(11,12).
(1,7,8,3,12)(2,5,9,4,10)	(1,11)(2,7)(3,4)(5,8)(6,9)(10,12)⟶(1,10)(2,7)(3,4)(5,6)(8,9)(11,12).
(1,8,3)(2,7,5,4,12,10)(6,9)	(1,12)(2,8)(3,6)(4,9)(5,7)(10,11)⟶(1,10)(2,7)(3,4)(5,6)(8,9)(11,12).
(1,11,4,5,10,8,6,9,3,2,7)	(1,10)(2,9)(3,5)(4,8)(6,7)(11,12)⟶(1,12)(2,8)(3,6)(4,9)(5,7)(10,11).
(1,11,6,4,7,10,5,12)(2,8,9,3)	(1,11)(2,4)(3,7)(5,6)(8,9)(10,12)⟶(1,12)(2,8)(3,6)(4,9)(5,7)(10,11).
(1,11,9,3,2,5)(4,8)(7,12,10)	(1,12)(2,6)(3,8)(4,5)(7,9)(10,11)⟶(1,12)(2,8)(3,6)(4,9)(5,7)(10,11).
(1,11,8,7,5,10,3,6)(4,9)	(1,10)(4,7)(5,9)(11,12)⟶(1,12)(4,7)(5,6)(8,9).
(1,4,10,6,3,7,11,5,8,9,2)	(2,3)(5,6)(8,9)(10,11)⟶(4,7)(5,8)(6,9)(10,11).
(1,6)(2,10,9,3,7,11,5,4)	(2,7)(3,4)(5,9)(10,11)⟶(2,7)(3,4)(5,9)(10,11).
(1,8)(2,10)(5,9)(7,11)	(2,7)(3,4)(5,9)(10,11)⟶(2,7)(3,4)(5,9)(10,11).
(1,6,11,5,2,10,4,9,12,8,3)	(1,10)(2,3)(6,8)(11,12)⟶(1,12)(2,3)(5,8)(6,9).
(1,7,9,12,5,3)(4,6,11)(8,10)	(4,7)(5,8)(6,9)(10,11)⟶(1,11)(4,7)(6,8)(10,12).

THE INTERSECTIONS OF ORDER 4

The first order 4 coset generator is (1,3)(2,12,10)(4,8,7,9,5,6)

(1,12)(2,3)(4,7)(10,11) (1,10)(2,3)(6,8)(11,12)
(1,12)(4,7)(5,6)(8,9) →(2,3)(4,7)(5,9)(6,8)
(2,3)(5,6)(8,9)(10,11) (1,10)(4,7)(5,9)(11,12)

The next coset generator is (1,11,5,10,9)(2,8,4,6,7)

(2,3)(4,7)(5,9)(6,8) (2,4)(3,7)(6,8)(10,11)
(1,10)(4,7)(5,9)(11,12)→(1,12)(5,9)(6,8)(10,11)
(1,10)(2,3)(6,8)(11,12) (1,12)(2,4)(3,7)(5,9)

The third coset generator is (1,11,7,8,5)(3,4,6,12,10)

(1,12)(5,9)(6,8)(10,11) (1,12)(4,7)(5,6)(8,9)
(1,12)(2,4)(3,7)(5,9) →(2,3)(5,6)(8,9)(10,11)
(2,4)(3,7)(6,8)(10,11) (1,12)(2,3)(4,7)(10,11)

The fourth coset generator is (1,2,8,7,3,12,5,9,10,4)(6,11)

(1,11)(2,4)(3,7)(5,6)(8,9)(10,12) (1,10)(2,5)(3,9)(4,6)(7,8)(11,12)
(1,10)(2,5)(3,9)(4,6)(7,8)(11,12)→(1,12)(2,8)(3,6)(4,9)(5,7)(10,11)
(1,12)(2,6)(3,8)(4,5)(7,9)(10,11) (1,11)(2,7)(3,4)(5,8)(6,9)(10,12)

The fifth coset generator is (1,2,5,4,10,6,11,9,12,7,3)

(1,11)(2,7)(3,4)(5,8)(6,9)(10,12) (1,12)(2,8)(3,6)(4,9)(5,7)(10,11)
(1,10)(2,9)(3,5)(4,8)(6,7)(11,12)→(1,11)(2,7)(3,4)(5,8)(6,9)(10,12)
(1,12)(2,6)(3,8)(4,5)(7,9)(10,11) (1,10)(2,5)(3,9)(4,6)(7,8)(11,12)

The sixth coset generator is (1,2,7,3,10,9)(4,12,8,5,6,11)

(1,11)(2,4)(3,7)(5,6)(8,9)(10,12)		(1,11)(2,7)(3,4)(5,8)(6,9)(10,12)
(1,10)(2,9)(3,5)(4,8)(6,7)(11,12)	→	(1,10)(2,5)(3,9)(4,6)(7,8)(11,12)
(1,12)(2,8)(3,6)(4,9)(5,7)(10,11)		(1,12)(2,8)(3,6)(4,9)(5,7)(10,11)

The seventh coset generator is (1,4,6,11,8,3,12,7,5,2)(9,10)

(1,12)(2,3)(4,7)(10,11)		(1,12)(2,3)(5,8)(6,9)
(1,12)(4,7)(5,6)(8,9)	→	(1,12)(2,3)(4,7)(10,11)
(2,3)(5,6)(8,9)(10,11)		(4,7)(5,8)(6,9)(10,11)

THE INTERSECTIONS OF ORDER 8

The first order 8 coset generator is (1,11,8,9,12,10,6,5)(2,7)

(2,3)(4,7)(5,9)(6,8)		(2,4)(3,7)(6,8)(10,11)
(1,12)(5,9)(6,8)(10,11)		(1,12)(5,9)(6,8)(10,11)
(1,12)(2,3)(4,7)(10,11)		(1,12)(2,4)(3,7)(5,9)
(1,12)(2,7)(3,4)(6,8)	→	(2,7)(3,4)(5,9)(10,11)
(1,12)(2,4)(3,7)(5,9)		(2,3)(4,7)(5,9)(6,8)
(2,7)(3,4)(5,9)(10,11)		(1,12)(2,7)(3,4)(6,8)
(2,4)(3,7)(6,8)(10,11)		(1,12)(2,3)(4,7)(10,11)

The second coset generator is (1,6,11,4,9,2,10,8)(3,12,7,5)

(2,3)(4,7)(5,9)(6,8)		(1,10)(4,7)(5,9)(11,12)
(1,11)(4,7)(6,8)(10,12)		(1,10)(2,3)(6,8)(11,12)
(1,11)(2,3)(5,9)(10,12)		(2,3)(4,7)(5,9)(6,8)
(1,10)(2,5)(3,9)(4,6)(7,8)(11,12)	→	(1,11)(2,8)(3,6)(4,5)(7,9)(10,12)
(1,10)(2,9)(3,5)(4,8)(6,7)(11,12)		(1,12)(2,8)(3,6)(4,9)(5,7)(10,11)
(1,12)(2,5)(3,9)(4,8)(6,7)(10,11)		(1,12)(2,6)(3,8)(4,5)(7,9)(10,11)
(1,12)(2,9)(3,5)(4,6)(7,8)(10,11)		(1,11)(2,6)(3,8)(4,9)(5,7)(10,12)

THE INTERSECTIONS OF ORDER 16

The first order 16 coset generator is (1,9,2,12,8,10,5,3)(6,11)

(2,3)(4,7)(5,9)(6,8)		(1,10)(4,7)(5,9)(11,12)
(1,11)(2,7)(3,4)(5,8)(6,9)(10,12)		(1,11)(2,8)(3,6)(4,5)(7,9)(10,12)
(1,11)(2,4)(3,7)(5,6)(8,9)(10,12)		(1,12)(2,8)(3,6)(4,9)(5,7)(10,11)
(1,10)(2,7)(3,4)(5,6)(8,9)(11,12)		(1,12)(2,6)(3,8)(4,5)(7,9)(10,11)
(1,10)(2,4)(3,7)(5,8)(6,9)(11,12)		(1,11)(2,6)(3,8)(4,9)(5,7)(10,12)
(1,12)(5,9)(6,8)(10,11)		(1,10)(2,3)(6,8)(11,12)
(1,12)(2,3)(4,7)(10,11)		(2,3)(4,7)(5,9)(6,8)
(1,12)(4,7)(5,6)(8,9)	→	(1,12)(2,3)(4,7)(10,11)
(1,12)(2,3)(5,8)(6,9)		(1,11)(2,3)(5,9)(10,12)
(1,10,12,11)(2,7,3,4)(5,9)(6,8)		(1,10)(2,6,3,8)(4,9,7,5)(11,12)
(1,10,12,11)(2,4,3,7)		(2,6,3,8)(4,5,7,9)
(1,11,12,10)(2,7,3,4)		(2,8,3,6)(4,9,7,5)
(1,11,12,10)(2,4,3,7)(5,9)(6,8)		(1,10)(2,8,3,6)(4,5,7,9)(11,12)
(4,7)(5,8)(6,9)(10,11)		(1,11)(4,7)(6,8)(10,12)
(2,3)(5,6)(8,9)(10,11)		(1,12)(5,9)(6,8)(10,11)

The second coset generator is (1, 9, 3, 12, 5, 2, 11, 7, 10, 4)(6, 8)

(2,3)(4,7)(5,9)(6,8)	(1,12)(5,9)(6,8)(10,11)
(2,8,3,6)(4,9,7,5)	(1,11,12,10)(5,6,9,8)
(2,6,3,8)(4,5,7,9)	(1,10,12,11)(5,8,9,6)
(1,11)(4,7)(6,8)(10,12)	(2,4)(3,7)(6,8)(10,11)
(1,11)(2,3)(5,9)(10,12)	(1,12)(2,4)(3,7)(5,9)
(1,11)(2,8)(3,6)(4,5)(7,9)(10,12)	(1,11)(2,4)(3,7)(5,6)(8,9)(10,12)
(1,11)(2,6)(3,8)(4,9)(5,7)(10,12)	(1,10)(2,4)(3,7)(5,8)(6,9)(11,12)
(1,10)(4,7)(5,9)(11,12)	→(1,12)(2,3)(4,7)(10,11)
(1,10)(2,3)(6,8)(11,12)	(2,3)(4,7)(5,9)(6,8)
(1,10)(2,8,3,6)(4,5,7,9)(11,12)	(1,10,12,11)(2,3)(4,7)(5,6,9,8)
(1,10)(2,6,3,8)(4,9,7,5)(11,12)	(1,11,12,10)(2,3)(4,7)(5,8,9,6)
(1,12)(5,9)(6,8)(10,11)	(1,12)(2,7)(3,4)(6,8)
(1,12)(2,3)(4,7)(10,11)	(2,7)(3,4)(5,9)(10,11)
(1,12)(2,8)(3,6)(4,9)(5,7)(10,11)	(1,10)(2,7)(3,4)(5,6)(8,9)(11,12)
(1,12)(2,6)(3,8)(4,5)(7,9)(10,11)	(1,11)(2,7)(3,4)(5,8)(6,9)(10,12)

THE INTERSECTIONS OF ORDER 32

The first order 32 coset generator is (1, 10, 12)(4, 6, 9)(5, 7, 8)

(2,7,3,4)(5,8,9,6)	(2,5,3,9)(4,8,7,6)
(2,3)(4,7)(5,9)(6,8)	(2,3)(4,7)(5,9)(6,8)
(2,4,3,7)(5,6,9,8)	(2,9,3,5)(4,6,7,8)
(2,5,3,9)(4,8,7,6)	(2,8,3,6)(4,9,7,5)
(2,8,3,6)(4,9,7,5)	(2,7,3,4)(5,8,9,6)
(2,9,3,5)(4,6,7,8)	(2,6,3,8)(4,5,7,9)
(2,6,3,8)(4,5,7,9)	(2,4,3,7)(5,6,9,8)
(1,11)(4,7)(6,8)(10,12)	(1,10)(4,7)(5,9)(11,12)
(1,11)(2,7)(3,4)(5,8)(6,9)(10,12)	(1,10)(2,5)(3,9)(4,6)(7,8)(11,12)
(1,11)(2,3)(5,9)(10,12)	(1,10)(2,3)(6,8)(11,12)
(1,11)(2,4)(3,7)(5,6)(8,9)(10,12)	(1,10)(2,9)(3,5)(4,8)(6,7)(11,12)
(1,11)(2,5,3,9)(4,6,7,8)(10,12)	(1,10)(2,8,3,6)(4,5,7,9)(11,12)
(1,11)(2,8)(3,6)(4,5)(7,9)(10,12)	(1,10)(2,7)(3,4)(5,6)(8,9)(11,12)
(1,11)(2,9,3,5)(4,8,7,6)(10,12)	(1,10)(2,6,3,8)(4,9,7,5)(11,12)
(1,11)(2,6)(3,8)(4,9)(5,7)(10,12)	(1,10)(2,4)(3,7)(5,8)(6,9)(11,12)
(1,10)(4,7)(5,9)(11,12)	→(1,12)(5,9)(6,8)(10,11)
(1,10)(2,7)(3,4)(5,6)(8,9)(11,12)	(1,12)(2,5)(3,9)(4,8)(6,7)(10,11)
(1,10)(2,3)(6,8)(11,12)	(1,12)(2,3)(4,7)(10,11)
(1,10)(2,4)(3,7)(5,8)(6,9)(11,12)	(1,12)(2,9)(3,5)(4,6)(7,8)(10,11)
(1,10)(2,5)(3,9)(4,6)(7,8)(11,12)	(1,12)(2,8)(3,6)(4,9)(5,7)(10,11)
(1,10)(2,8,3,6)(4,5,7,9)(11,12)	(1,12)(2,7,3,4)(5,6,9,8)(10,11)
(1,10)(2,9)(3,5)(4,8)(6,7)(11,12)	(1,12)(2,6)(3,8)(4,5)(7,9)(10,11)
(1,10)(2,6,3,8)(4,9,7,5)(11,12)	(1,12)(2,4,3,7)(5,8,9,6)(10,11)
(1,12)(5,9)(6,8)(10,11)	(1,11)(4,7)(6,8)(10,12)
(1,12)(2,7,3,4)(5,6,9,8)(10,11)	(1,11)(2,5,3,9)(4,6,7,8)(10,12)
(1,12)(2,3)(4,7)(10,11)	(1,11)(2,3)(5,9)(10,12)
(1,12)(2,4,3,7)(5,8,9,6)(10,11)	(1,11)(2,9,3,5)(4,8,7,6)(10,12)
(1,12)(2,5)(3,9)(4,8)(6,7)(10,11)	(1,11)(2,8)(3,6)(4,5)(7,9)(10,12)
(1,12)(2,8)(3,6)(4,9)(5,7)(10,11)	(1,11)(2,7)(3,4)(5,8)(6,9)(10,12)
(1,12)(2,9)(3,5)(4,6)(7,8)(10,11)	(1,11)(2,6)(3,8)(4,9)(5,7)(10,12)
(1,12)(2,6)(3,8)(4,5)(7,9)(10,11)	(1,11)(2,4)(3,7)(5,6)(8,9)(10,12)

The last coset generator is (1,6,7)(2,11,5)(3,10,9)(4,12,8)

(2,7,3,4)(5,8,9,6)	(1,11,12,10)(5,6,9,8)
(2,3)(4,7)(5,9)(6,8)	(1,12)(5,9)(6,8)(10,11)
(2,4,3,7)(5,6,9,8)	(1,10,12,11)(5,8,9,6)
(1,11)(4,7)(6,8)(10,12)	(1,12)(2,7)(3,4)(6,8)
(1,11)(2,7)(3,4)(5,8)(6,9)(10,12)	(1,10)(2,7)(3,4)(5,6)(8,9)(11,12)
(1,11)(2,3)(5,9)(10,12)	(2,7)(3,4)(5,9)(10,11)
(1,11)(2,4)(3,7)(5,6)(8,9)(10,12)	(1,11)(2,7)(3,4)(5,8)(6,9)(10,12)
(1,10)(4,7)(5,9)(11,12)	(2,4)(3,7)(6,8)(10,11)
(1,10)(2,7)(3,4)(5,6)(8,9)(11,12)	(1,11)(2,4)(3,7)(5,6)(8,9)(10,12)
(1,10)(2,3)(6,8)(11,12)	(1,12)(2,4)(3,7)(5,9)
(1,10)(2,4)(3,7)(5,8)(6,9)(11,12)	(1,10)(2,4)(3,7)(5,8)(6,9)(11,12)
(1,12)(5,9)(6,8)(10,11)	(1,12)(2,3)(4,7)(10,11)
(1,12)(2,7,3,4)(5,6,9,8)(10,11)	(1,10,12,11)(2,3)(4,7)(5,6,9,8)
(1,12)(2,3)(4,7)(10,11)	(2,3)(4,7)(5,9)(6,8)
(1,12)(2,4,3,7)(5,8,9,6)(10,11)	(1,11,12,10)(2,3)(4,7)(5,8,9,6)
(1,12)(4,7)(5,6)(8,9)	⟶(1,11)(4,7)(6,8)(10,12)
(1,12)(2,7)(3,4)(6,8)	(1,12)(4,7)(5,6)(8,9)
(1,12)(2,3)(5,8)(6,9)	(1,10)(4,7)(5,9)(11,12)
(1,12)(2,4)(3,7)(5,9)	(4,7)(5,8)(6,9)(10,11)
(1,10,12,11)(5,8,9,6)	(1,11,12,10)(2,7,3,4)
(1,10,12,11)(2,7,3,4)(5,9)(6,8)	(1,12)(2,7,3,4)(5,6,9,8)(10,11)
(1,10,12,11)(2,3)(4,7)(5,6,9,8)	(1,10,12,11)(2,7,3,4)(5,9)(6,8)
(1,10,12,11)(2,4,3,7)	(2,7,3,4)(5,8,9,6)
(1,11,12,10)(5,6,9,8)	(1,10,12,11)(2,4,3,7)
(1,11,12,10)(2,7,3,4)	(2,4,3,7)(5,6,9,8)
(1,11,12,10)(2,3)(4,7)(5,8,9,6)	(1,11,12,10)(2,4,3,7)(5,9)(6,8)
(1,11,12,10)(2,4,3,7)(5,9)(6,8)	(1,12)(2,4,3,7)(5,8,9,6)(10,11)
(4,7)(5,8)(6,9)(10,11)	(1,10)(2,3)(6,8)(11,12)
(2,7)(3,4)(5,9)(10,11)	(2,3)(5,6)(8,9)(10,11)
(2,3)(5,6)(8,9)(10,11)	(1,11)(2,3)(5,9)(10,12)
(2,4)(3,7)(6,8)(10,11)	(1,12)(2,3)(5,8)(6,9)

The details on the structure of the intersections follow:

Class conjugating the Sylow 2-subgroup (1,6,7)(2,11,5)(3,10,9)(4,12,8), resulting intersection H_{22}

Class conjugating the Sylow 2-subgroup (1,9,3,12,5,2,11,7,10,4)(6,8), intersection with the commutator subgroup is the maximal elementary subgroup $\mathbb{Z}/2 \times \mathbb{Z}/2$ with generators

1, (1,12)(5,9)(6,8)(10,11), (1,12)(2,3)(4,7)(10,11), (2,3)(4,7)(5,9)(6,8).

In fact this is the intersection with the commutator subgroup of all the remaining conjugates. The intersection with the third coset is

(1,11)(2,7)(3,4)(5,8)(6,9)(10,12)

(1,10)(2,7)(3,4)(5,6)(8,9)(11,12)

(1,11)(2,4)(3,7)(5,6)(8,9)(10,12)

(1,10)(2,4)(3,7)(5,8)(6,9)(11,12).

The intersection with the fifth coset is

(1,12)(2,7)(3,4)(6,8)

(2,7)(3,4)(5,9)(10,11)

(2,4)(3,7)(6,8)(10,11)

$$(1,12)(2,4)(3,7)(5,9).$$

Finally, the intersection with the seventh coset is

$$(1,11,12,10)(5,6,9,8)$$
$$(1,10,12,11)(5,8,9,6)$$
$$(1,11,12,10)(2,3)(4,7)(5,8,9,6)$$
$$(1,10,12,11)(2,3)(4,7)(5,6,9,8).$$

This is the first of the two subgroups of order 16. It is also obtained by intersecting with the conjugate by $(1,4,10,6,5,2,11,8,9,3)(7,12)$.

The second subgroup of order 16 is obtained by intersecting with the conjugate by $(1,9,11,7,8,10,4,6)$-$(5,12)$. The intersection with the first and third cosets are the same as those for the group above. However, for the fifth coset we have that the intersection is

$$(1,12)(4,7)(5,6)(8,9)$$
$$(2,3)(5,6)(8,9)(10,11)$$
$$(4,7)(5,8)(6,9)(10,11)$$
$$(1,12)(2,3)(5,8)(6,9),$$

while the intersection with the seventh coset is

$$(1,11,12,10)(2,7,3,4)$$
$$(1,10,12,11)(2,4,3,7)$$
$$(1,11,12,10)(2,4,3,7)(5,9)(6,8)$$
$$(1,10,12,11)(2,7,3,4)(5,9)(6,8),$$

and these are manifestly different from the intersections for the first group above. This same group is also obtained by intersecting with the conjugate by $(1,4,9,11,8,12,7,5)(6,10)$.

The remaining intersections that we consider are all elementary 2-groups $(\mathbb{Z}/2)^3$. There are two which occur as the intersection of H_2 with a conjugate. Both contain the intersection with the commutator subgroup above, and both have 4 elements in common with the fifth coset. These two intersections are

$$(2,4)(3,7)(6,8)(10,11)$$
$$(1,12)(2,4)(3,7)(5,9)$$
$$(2,7)(3,4)(5,9)(10,11)$$
$$(1,12)(2,7)(3,4)(6,8)$$

obtained by conjugating with

$$(1,11,8,9,12,10,6,5)(2,7),\ (1,11,8,12,10,6)(3,4,7)(5,9),\ (1,2,8,7,6,4,9,10)(3,5,11,12),$$

and $(1,3,8,11,12,2,5,7,6,10)(4,9)$. In the other case the intersection with the fifth coset is

$$(2,3)(5,6)(8,9)(10,11)$$
$$(1,12)(4,7)(5,6)(8,9)$$
$$(4,7)(5,8)(6,9)(10,11)$$
$$(1,12)(2,3)(5,8)(6,9).$$

This case occurs when we conjugate by any one of

$$\begin{cases}(1,11,9,7)(2,6,3,8,4,12,10,5),\\ (1,11,6,7,9,4,5,2)(3,12,10,8),\\ (1,7,2,11,12,4,3,10)(5,8),\\ (1,8,3,5,2,9,7,10)(4,11,12,6).\end{cases}$$

APPENDIX 2

The Double Coset Decomposition of M_{12} with Respect to W

There are 11 double cosets of M_{12} with respect to W. The 10 non-trivial ones are described below.
Double coset generator # 1: (1,2,3)(4,5,6)(7,8,9). Intersection size is 6 . The intersection group and conjugate are

id	*id*
(4,5,6)(7,9,8)(10,12,11)	(4,5,6)(7,9,8)(10,12,11)
(4,7)(5,8)(6,9)(10,11)	(4,7)(5,8)(6,9)(10,11)
(4,6,5)(7,8,9)(10,11,12)	(4,6,5)(7,8,9)(10,11,12)
(4,8)(5,9)(6,7)(10,12)	(4,8)(5,9)(6,7)(10,12)
(4,9)(5,7)(6,8)(11,12)	(4,9)(5,7)(6,8)(11,12)

Double coset generator # 2: (1,3,11,2)(4,6,8,7)(5,9)(10,12). Intersection size is 8 . The intersection group and conjugate are

id	*id*
(1,11)(4,7)(6,8)(10,12)	(2,3)(4,6)(7,8)(10,12)
(2,3)(4,7)(5,9)(6,8)	(1,11)(4,6)(5,9)(7,8)
(1,11)(2,3)(5,9)(10,12)	(1,11)(2,3)(5,9)(10,12)
(4,8)(5,9)(6,7)(10,12)	(4,8)(5,9)(6,7)(10,12)
(1,11)(4,6)(5,9)(7,8)	(2,3)(4,7)(5,9)(6,8)
(2,3)(4,6)(7,8)(10,12)	(1,11)(4,7)(6,8)(10,12)
(1,11)(2,3)(4,8)(6,7)	(1,11)(2,3)(4,8)(6,7)

Double coset generator # 3: (1,8,3,9)(2,6,7,11)(4,5)(10,12). Intersection size is 2 . The intersection group and conjugate are

id	*id*
(1,11)(2,8)(3,6)(4,5)(7,9)(10,12)	(1,11)(2,8)(3,6)(4,5)(7,9)(10,12)

Double coset generator # 4: (1,5)(2,6,3,10,11,4,9,8). Intersection size is 4 . The intersection group and conjugate are

id	*id*
(1,10)(4,5)(6,8)(7,9)	(1,11)(2,9)(3,5)(4,7)
(1,10)(2,3)(6,8)(11,12)	(2,9)(3,5)(6,8)(10,12)
(2,3)(4,5)(7,9)(11,12)	(1,11)(4,7)(6,8)(10,12)

There are 12 conjugates of this group in W.

Conjugate number 1 .	Conjugate number 2 .	Conjugate number 3 .	Conjugate number 4 .
id	*id*	*id*	*id*
(1,11)(2,9)(3,5)(4,7)	(1,11)(2,5)(3,9)(6,8)	(2,8)(3,6)(4,7)(11,12)	(2,4)(3,7)(6,8)(10,11)
(2,9)(3,5)(6,8)(10,12)	(2,5)(3,9)(4,7)(10,12)	(1,10)(2,8)(3,6)(5,9)	(1,12)(2,4)(3,7)(5,9)
(1,11)(4,7)(6,8)(10,12)	(1,11)(4,7)(6,8)(10,12)	(1,10)(4,7)(5,9)(11,12)	(1,12)(5,9)(6,8)(10,11)

Conjugate number 5 .	Conjugate number 6 .	Conjugate number 7 .	Conjugate number 8 .
id	*id*	*id*	*id*
(2,6)(3,8)(5,9)(11,12)	(2,7)(3,4)(5,9)(10,11)	(4,9)(5,7)(6,8)(11,12)	(4,7)(5,8)(6,9)(10,11)
(1,10)(2,6)(3,8)(4,7)	(1,12)(2,7)(3,4)(6,8)	(1,10)(2,3)(4,9)(5,7)	(1,12)(2,3)(5,8)(6,9)
(1,10)(4,7)(5,9)(11,12)	(1,12)(5,9)(6,8)(10,11)	(1,10)(2,3)(6,8)(11,12)	(1,12)(2,3)(4,7)(10,11)

Conjugate number 9 .	Conjugate number 10 .	Conjugate number 11 .	Conjugate number 12 .
id	*id*	*id*	*id*
(2,3)(4,5)(7,9)(11,12)	(2,3)(5,6)(8,9)(10,11)	(1,11)(4,6)(5,9)(7,8)	(4,8)(5,9)(6,7)(10,12)
(1,10)(4,5)(6,8)(7,9)	(1,12)(4,7)(5,6)(8,9)	(2,3)(4,6)(7,8)(10,12)	(1,11)(2,3)(4,8)(6,7)
(1,10)(2,3)(6,8)(11,12)	(1,12)(2,3)(4,7)(10,11)	(1,11)(2,3)(5,9)(10,12)	(1,11)(2,3)(5,9)(10,12)

Double coset generator # 5: (1, 4, 6, 7, 10, 12, 3, 9, 11, 2, 8). Intersection size is 2 . The intersection group and conjugate are

id	*id*
(2, 3)(5, 6)(8, 9)(10, 11)	(2, 3)(4, 5)(7, 9)(11, 12)

There are 24 conjugates of this group in W.

Conjugate number 1 .	Conjugate number 2 .	Conjugate number 3 .	Conjugate number 4 .
id	*id*	*id*	*id*
(2, 3)(4, 5)(7, 9)(11, 12)	(2, 6)(3, 8)(5, 9)(11, 12)	(1, 10)(2, 3)(4, 9)(5, 7)	(1, 11)(2, 3)(4, 8)(6, 7)
Conjugate number 5 .	Conjugate number 6 .	Conjugate number 7 .	Conjugate number 8 .
id	*id*	*id*	*id*
(2, 3)(4, 6)(7, 8)(10, 12)	(1, 10)(2, 8)(3, 6)(5, 9)	(1, 11)(2, 5)(3, 9)(6, 8)	(2, 5)(3, 9)(4, 7)(10, 12)
Conjugate number 9 .	Conjugate number 10 .	Conjugate number 11 .	Conjugate number 12 .
id	*id*	*id*	*id*
(1, 11)(4, 6)(5, 9)(7, 8)	(2, 3)(5, 6)(8, 9)(10, 11)	(4, 8)(5, 9)(6, 7)(10, 12)	(1, 12)(2, 3)(5, 8)(6, 9)
Conjugate number 13 .	Conjugate number 14 .	Conjugate number 15 .	Conjugate number 16 .
id	*id*	*id*	*id*
(2, 9)(3, 5)(6, 8)(10, 12)	(1, 11)(2, 9)(3, 5)(4, 7)	(2, 7)(3, 4)(5, 9)(10, 11)	(1, 10)(2, 6)(3, 8)(4, 7)
Conjugate number 17 .	Conjugate number 18 .	Conjugate number 19 .	Conjugate number 20 .
id	*id*	*id*	*id*
(1, 12)(2, 7)(3, 4)(6, 8)	(4, 9)(5, 7)(6, 8)(11, 12)	(4, 7)(5, 8)(6, 9)(10, 11)	(1, 10)(4, 5)(6, 8)(7, 9)
Conjugate number 21 .	Conjugate number 22 .	Conjugate number 23 .	Conjugate number 24 .
id	*id*	*id*	*id*
(1, 12)(4, 7)(5, 6)(8, 9)	(1, 12)(2. 4)(3, 7)(5, 9)	(2, 8)(3, 6)(4, 7)(11, 12)	(2, 4)(3, 7)(6, 8)(10, 11)

Double coset generator # 6: (1, 4, 7, 9, 5, 3, 11, 8)(2, 6, 12, 10). Intersection size is 2 . The intersection group and conjugate are

id	*id*
(1, 12)(2, 4)(3, 7)(5, 9)	(1, 10)(4, 5)(6, 8)(7, 9)

Double coset generator # 7: (1, 4, 9, 7, 5, 12, 2, 6)(3, 10, 11, 8). Intersection size is 6 . The intersection group and conjugate are

id	*id*
(1, 12)(2, 4)(3, 7)(5, 9)	(1, 12)(4, 7)(5, 6)(8, 9)
(4, 8)(5, 9)(6, 7)(10, 12)	(1, 11)(2, 9)(3, 5)(4, 7)
(1, 10)(2, 8)(3, 6)(5, 9)	(2, 8)(3, 6)(4, 7)(11, 12)
(1, 10, 12)(2, 8, 4)(3, 6, 7)	(1, 12, 11)(2, 9, 8)(3, 5, 6)
(1, 12, 10)(2, 4, 8)(3, 7, 6)	(1, 11, 12)(2, 8, 9)(3, 6, 5)

Double coset generator # 8: (1, 8)(2, 9, 5, 10)(3, 12)(4, 7, 6, 11). Intersection size is 4 . The intersection group and conjugate are

id	*id*
(1, 11)(2, 3)(5, 9)(10, 12)	(2, 9)(3, 5)(6, 8)(10, 12)
(4, 8)(5, 9)(6, 7)(10, 12)	(1, 11)(2, 9)(3, 5)(4, 7)
(1, 11)(2, 3)(4, 8)(6, 7)	(1, 11)(4, 7)(6, 8)(10, 12)

Double coset generator # 9: (1, 9, 11, 8, 6)(2, 12, 4, 7, 10). Intersection size is 12 . The intersection group and conjugate are

id	*id*
(1,10,11)(4,9,8)(5,6,7)	(1,11,12)(4,5,8)(6,7,9)
(1,11,10)(4,8,9)(5,7,6)	(1,12,11)(4,8,5)(6,9,7)
(1,11,12)(2,8,9)(3,6,5)	(1,10,11)(2,6,9)(3,8,5)
(2,4,9)(3,7,5)(10,11,12)	(1,10,12)(2,7,9)(3,4,5)
(1,10)(2,9)(3,5)(4,8)(6,7)(11,12)	(1,10)(2,9)(3,5)(4,8)(6,7)(11,12)
(2,9,4)(3,5,7)(10,12,11)	(1,12,10)(2,9,7)(3,5,4)
(1,11)(2,4)(3,7)(5,6)(8,9)(10,12)	(1,11)(2,7)(3,4)(5,8)(6,9)(10,12)
(1,12)(2,8)(3,6)(4,9)(5,7)(10,11)	(1,12)(2,6)(3,8)(4,5)(7,9)(10,11)
(1,10,12)(2,8,4)(3,6,7)	(2,6,7)(3,8,4)(10,11,12)
(1,12,10)(2,4,8)(3,7,6)	(2,7,6)(3,4,8)(10,12,11)
(1,12,11)(2,9,8)(3,5,6)	(1,11,10)(2,9,6)(3,5,8)

There are 8 conjugates of this group in W.

Conjugate number 1 .	Conjugate number 2 .	Conjugate number 3 .
id	*id*	*id*
(1,11,12)(4,5,8)(6,7,9)	(1,11,12)(2,4,6)(3,7,8)	(1,10,11)(4,9,8)(5,6,7)
(1,12,11)(4,8,5)(6,9,7)	(1,12,11)(2,6,4)(3,8,7)	(1,11,10)(4,8,9)(5,7,6)
(1,10,11)(2,6,9)(3,8,5)	(1,10,11)(2,5,4)(3,9,7)	(1,11,12)(2,8,9)(3,6,5)
(1,10,12)(2,7,9)(3,4,5)	(1,10,12)(2,5,6)(3,9,8)	(2,4,9)(3,7,5)(10,11,12)
(1,10)(2,9)(3,5)(4,8)(6,7)(11,12)	(1,10)(2,5)(3,9)(4,6)(7,8)(11,12)	(1,10)(2,9)(3,5)(4,8)(6,7)(11,12)
(1,12,10)(2,9,7)(3,5,4)	(1,12,10)(2,6,5)(3,8,9)	(2,9,4)(3,5,7)(10,12,11)
(1,11)(2,7)(3,4)(5,8)(6,9)(10,12)	(1,11)(2,4)(3,7)(5,6)(8,9)(10,12)	(1,11)(2,4)(3,7)(5,6)(8,9)(10,12)
(1,12)(2,6)(3,8)(4,5)(7,9)(10,11)	(1,12)(2,6)(3,8)(4,5)(7,9)(10,11)	(1,12)(2,8)(3,6)(4,9)(5,7)(10,11)
(2,6,7)(3,8,4)(10,11,12)	(4,6,5)(7,8,9)(10,11,12)	(1,10,12)(2,8,4)(3,6,7)
(2,7,6)(3,4,8)(10,12,11)	(4,5,6)(7,9,8)(10,12,11)	(1,12,10)(2,4,8)(3,7,6)
(1,11,10)(2,9,6)(3,5,8)	(1,11,10)(2,4,5)(3,7,9)	(1,12,11)(2,9,8)(3,5,6)

Conjugate number 4 .	Conjugate number 5 .	Conjugate number 6 .
id	*id*	*id*
(1,12,11)(4,8,5)(6,9,7)	(1,10,11)(2,7,8)(3,4,6)	(1,12,11)(2,7,5)(3,4,9)
(1,11,12)(4,5,8)(6,7,9)	(1,11,10)(2,8,7)(3,6,4)	(1,11,12)(2,5,7)(3,9,4)
(2,5,8)(3,9,6)(10,11,12)	(1,11,12)(2,5,7)(3,9,4)	(2,6,7)(3,8,4)(10,11,12)
(1,12,10)(2,4,8)(3,7,6)	(2,5,8)(3,9,6)(10,11,12)	(1,12,10)(2,6,5)(3,8,9)
(1,11)(2,8)(3,6)(4,5)(7,9)(10,12)	(1,10)(2,5)(3,9)(4,6)(7,8)(11,12)	(1,11)(2,6)(3,8)(4,9)(5,7)(10,12)
(1,10,12)(2,8,4)(3,6,7)	(2,8,5)(3,6,9)(10,12,11)	(1,10,12)(2,5,6)(3,9,8)
(1,10)(2,4)(3,7)(5,8)(6,9)(11,12)	(1,11)(2,7)(3,4)(5,8)(6,9)(10,12)	(1,10)(2,7)(3,4)(5,6)(8,9)(11,12)
(1,12)(2,5)(3,9)(4,8)(6,7)(10,11)	(1,12)(2,8)(3,6)(4,9)(5,7)(10,11)	(1,12)(2,5)(3,9)(4,8)(6,7)(10,11)
(1,10,11)(2,5,4)(3,9,7)	(1,10,12)(4,6,9)(5,7,8)	(1,10,11)(4,9,8)(5,6,7)
(1,11,10)(2,4,5)(3,7,9)	(1,12,10)(4,9,6)(5,8,7)	(1,11,10)(4,8,9)(5,7,6)
(2,8,5)(3,6,9)(10,12,11)	(1,12,11)(2,7,5)(3,4,9)	(2,7,6)(3,4,8)(10,12,11)

Conjugate number 7 .	Conjugate number 8 .
id	*id*
(4,6,5)(7,8,9)(10,11,12)	(1,12,11)(2,6,4)(3,8,7)
(4,5,6)(7,9,8)(10,12,11)	(1,11,12)(2,4,6)(3,7,8)
(1,12,11)(2,9,8)(3,5,6)	(2,4,9)(3,7,5)(10,11,12)
(1,10,11)(2,7,8)(3,4,6)	(1,12,10)(4,9,6)(5,8,7)
(1,11)(2,8)(3,6)(4,5)(7,9)(10,12)	(1,11)(2,6)(3,8)(4,9)(5,7)(10,12)
(1,11,10)(2,8,7)(3,6,4)	(1,10,12)(4,6,9)(5,7,8)
(1,10)(2,7)(3,4)(5,6)(8,9)(11,12)	(1,10)(2,4)(3,7)(5,8)(6,9)(11,12)
(1,12)(2,9)(3,5)(4,6)(7,8)(10,11)	(1,12)(2,9)(3,5)(4,6)(7,8)(10,11)
(1,12,10)(2,9,7)(3,5,4)	(1,10,11)(2,6,9)(3,8,5)
(1,10,12)(2,7,9)(3,4,5)	(1,11,10)(2,9,6)(3,5,8)
(1,11,12)(2,8,9)(3,6,5)	(2,9,4)(3,5,7)(10,12,11)

Double coset generator # 10: (1,3,10,2,11,4,6,12,7,8)(5,9). Intersection size is 32 . The intersection group and conjugate are

id	*id*
(1,11)(4,7)(6,8)(10,12)	(2,8)(3,6)(4,7)(11,12)
(1,12)(4,7)(5,6)(8,9)	(4,9)(5,7)(6,8)(11,12)
(2,3)(4,7)(5,9)(6,8)	(1,10)(4,7)(5,9)(11,12)
(1,10,12,11)(5,8,9,6)	(2,8,3,6)(4,9,7,5)
(1,11,12,10)(5,6,9,8)	(2,6,3,8)(4,5,7,9)
(1,11)(2,3)(5,9)(10,12)	(1,10)(2,8)(3,6)(5,9)
(1,12)(2,3)(5,8)(6,9)	(1,10)(4,5)(6,8)(7,9)
(1,12)(2,4)(3,7)(5,9)	(1,12)(5,9)(6,8)(10,11)
(4,7)(5,8)(6,9)(10,11)	(2,3)(4,5)(7,9)(11,12)
(1,10)(4,7)(5,9)(11,12)	(2,6)(3,8)(5,9)(11,12)
(1,10,12,11)(2,3)(4,7)(5,6,9,8)	(1,10)(2,8,3,6)(4,5,7,9)(11,12)
(1,11,12,10)(2,3)(4,7)(5,8,9,6)	(1,10)(2,6,3,8)(4,9,7,5)(11,12)
(1,10,12,11)(2,7,3,4)(5,9)(6,8)	(1,11,10,12)(2,8,3,6)(4,7)(5,9)
(1,11,12,10)(2,4,3,7)(5,9)(6,8)	(1,12,10,11)(2,6,3,8)(4,7)(5,9)
(2,4,3,7)(5,6,9,8)	(1,12,10,11)(4,5,7,9)
(1,12)(5,9)(6,8)(10,11)	(2,3)(4,7)(5,9)(6,8)
(2,7,3,4)(5,8,9,6)	(1,11,10,12)(4,9,7,5)
(1,11,12,10)(2,7,3,4)	(1,11,10,12)(2,6,3,8)
(1,12)(2,7)(3,4)(6,8)	(1,11)(4,7)(6,8)(10,12)
(2,3)(5,6)(8,9)(10,11)	(1,10)(2,3)(4,9)(5,7)
(1,10)(2,3)(6,8)(11,12)	(1,10)(2,6)(3,8)(4,7)
(2,7)(3,4)(5,9)(10,11)	(1,11)(2,3)(5,9)(10,12)
(1,11)(2,7)(3,4)(5,8)(6,9)(10,12)	(1,11)(2,8)(3,6)(4,5)(7,9)(10,12)
(1,10)(2,4)(3,7)(5,8)(6,9)(11,12)	(1,12)(2,6)(3,8)(4,5)(7,9)(10,11)
(1,11)(2,4)(3,7)(5,6)(8,9)(10,12)	(1,12)(2,8)(3,6)(4,9)(5,7)(10,11)
(2,4)(3,7)(6,8)(10,11)	(1,12)(2,3)(4,7)(10,11)
(1,10)(2,7)(3,4)(5,6)(8,9)(11,12)	(1,11)(2,6)(3,8)(4,9)(5,7)(10,12)
(1,12)(2,7,3,4)(5,6,9,8)(10,11)	(1,11,10,12)(2,3)(4,5,7,9)(6,8)
(1,10,12,11)(2,4,3,7)	(1,12,10,11)(2,8,3,6)
(1,12)(2,3)(4,7)(10,11)	(1,10)(2,3)(6,8)(11,12)
(1,12)(2,4,3,7)(5,8,9,6)(10,11)	(1,12,10,11)(2,3)(4,9,7,5)(6,8)

There are 3 conjugates of this group in W.

Conjugate number 1 .	Conjugate number 2 .	Conjugate number 3 .
id	*id*	*id*
(2,8)(3,6)(4,7)(11,12)	(1,11)(2,9)(3,5)(4,7)	(2,4)(3,7)(6,8)(10,11)
(4,9)(5,7)(6,8)(11,12)	(1,11)(4,6)(5,9)(7,8)	(4,7)(5,8)(6,9)(10,11)
(1,10)(4,7)(5,9)(11,12)	(1,11)(4,7)(6,8)(10,12)	(1,12)(5,9)(6,8)(10,11)
(2,8,3,6)(4,9,7,5)	(2,9,3,5)(4,6,7,8)	(2,4,3,7)(5,6,9,8)
(2,6,3,8)(4,5,7,9)	(2,5,3,9)(4,8,7,6)	(2,7,3,4)(5,8,9,6)
(1,10)(2,8)(3,6)(5,9)	(2,9)(3,5)(6,8)(10,12)	(1,12)(2,4)(3,7)(5,9)
(1,10)(4,5)(6,8)(7,9)	(4,8)(5,9)(6,7)(10,12)	(1,12)(4,7)(5,6)(8,9)
(1,12)(5,9)(6,8)(10,11)	(1,12)(5,9)(6,8)(10,11)	(1,10)(4,7)(5,9)(11,12)
(2,3)(4,5)(7,9)(11,12)	(1,11)(2,3)(4,8)(6,7)	(2,3)(5,6)(8,9)(10,11)
(2,6)(3,8)(5,9)(11,12)	(1,11)(2,5)(3,9)(6,8)	(2,7)(3,4)(5,9)(10,11)
(1,10)(2,8,3,6)(4,5,7,9)(11,12)	(1,11)(2,9,3,5)(4,8,7,6)(10,12)	(1,12)(2,4,3,7)(5,8,9,6)(10,11)
(1,10)(2,6,3,8)(4,9,7,5)(11,12)	(1,11)(2,5,3,9)(4,6,7,8)(10,12)	(1,12)(2,7,3,4)(5,6,9,8)(10,11)
(1,11,10,12)(2,8,3,6)(4,7)(5,9)	(1,12,11,10)(2,9,3,5)(4,7)(6,8)	(1,11,12,10)(2,4,3,7)(5,9)(6,8)
(1,12,10,11)(2,6,3,8)(4,7)(5,9)	(1,10,11,12)(2,5,3,9)(4,7)(6,8)	(1,10,12,11)(2,7,3,4)(5,9)(6,8)
(1,12,10,11)(4,5,7,9)	(1,10,11,12)(4,8,7,6)	(1,10,12,11)(5,8,9,6)
(2,3)(4,7)(5,9)(6,8)	(2,3)(4,7)(5,9)(6,8)	(2,3)(4,7)(5,9)(6,8)
(1,11,10,12)(4,9,7,5)	(1,12,11,10)(4,6,7,8)	(1,11,12,10)(5,6,9,8)
(1,11,10,12)(2,6,3,8)	(1,12,11,10)(2,5,3,9)	(1,11,12,10)(2,7,3,4)
(1,11)(4,7)(6,8)(10,12)	(1,10)(4,7)(5,9)(11,12)	(1,11)(4,7)(6,8)(10,12)
(1,10)(2,3)(4,9)(5,7)	(2,3)(4,6)(7,8)(10,12)	(1,12)(2,3)(5,8)(6,9)
(1,10)(2,6)(3,8)(4,7)	(2,5)(3,9)(4,7)(10,12)	(1,12)(2,7)(3,4)(6,8)
(1,11)(2,3)(5,9)(10,12)	(1,10)(2,3)(6,8)(11,12)	(1,11)(2,3)(5,9)(10,12)
(1,11)(2,8)(3,6)(4,5)(7,9)(10,12)	(1,10)(2,9)(3,5)(4,8)(6,7)(11,12)	(1,11)(2,4)(3,7)(5,6)(8,9)(10,12)
(1,12)(2,6)(3,8)(4,5)(7,9)(10,11)	(1,12)(2,5)(3,9)(4,8)(6,7)(10,11)	(1,10)(2,7)(3,4)(5,6)(8,9)(11,12)
(1,12)(2,8)(3,6)(4,9)(5,7)(10,11)	(1,12)(2,9)(3,5)(4,6)(7,8)(10,11)	(1,10)(2,4)(3,7)(5,8)(6,9)(11,12)
(1,12)(2,3)(4,7)(10,11)	(1,12)(2,3)(4,7)(10,11)	(1,10)(2,3)(6,8)(11,12)
(1,11)(2,6)(3,8)(4,9)(5,7)(10,12)	(1,10)(2,5)(3,9)(4,6)(7,8)(11,12)	(1,11)(2,7)(3,4)(5,8)(6,9)(10,12)
(1,11,10,12)(2,3)(4,5,7,9)(6,8)	(1,12,11,10)(2,3)(4,8,7,6)(5,9)	(1,11,12,10)(2,3)(4,7)(5,8,9,6)
(1,12,10,11)(2,8,3,6)	(1,10,11,12)(2,9,3,5)	(1,10,12,11)(2,4,3,7)
(1,10)(2,3)(6,8)(11,12)	(1,11)(2,3)(5,9)(10,12)	(1,12)(2,3)(4,7)(10,11)
(1,12,10,11)(2,3)(4,9,7,5)(6,8)	(1,10,11,12)(2,3)(4,6,7,8)(5,9)	(1,10,12,11)(2,3)(4,7)(5,6,9,8)

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