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THE COHOMOLOGY OF THE MATHIEU GROUP M_{22}

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In this paper we determine the mod (2) cohomology of the sporadic simple group M_{22}, a group first described by E. Mathieu [10] in 1873. It is a group of rank four at $p = 2$, and of order $443{,}520 = 2^7\, 3^2\, 5\, 7\, 11$.

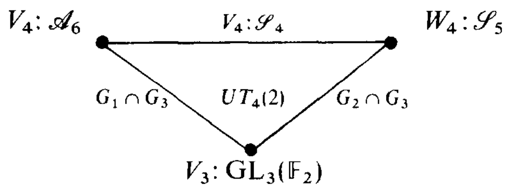
Our approach is to first determine the image of the restriction map from $H^*(M_{22}; \mathbb{F}_2)$ to the cohomology of its conjugacy classes of extremal 2-elementary subgroups. By a theorem due to Quillen and Venkov [2] this determines the cohomology up to nilpotence. Although for many groups this is actually an injection, for M_{22} there is a non-trivial kernel (the radical), denoted $Rad(M_{22})$. We explicitly determine this ideal, which fits into an exact sequence

$$0 \to Rad(M_{22}) \to H^*(M_{22}; \mathbb{F}_2) \to \mathscr{M} \to 0$$

where $\mathscr{M}$ is the image above. Of course, even though we determine $\mathscr{M}$ essentially completely as a ring, there is an extension problem in determining the ring structure of $H^*(M_{22}; \mathbb{F}_2)$ from the exact sequence, and we will see that the sequence is non-split.

Our techniques are, in fact, sufficient to determine the extension data completely, but they require a precise description of 5 classes in $\mathscr{M}$ which would require a considerable amount of computer time to obtain so we leave the description slightly incomplete, determining *all of the elements in* $H^*(M_{22}; \mathbb{F}_2)$, most of the cup product information, and most of the action of the Steenrod algebra.

The 2-local structure of M_{22} is well understood. There are three conjugacy classes of extremal 2-elementary subgroups, $V_4 \cong W_4 \cong (\mathbb{Z}/2)^4$ and $V_3 \cong (\mathbb{Z}/2)^3$, each self-centralizing in M_{22}. Their normalizers are semi-direct products $G_1 = V_4\colon \mathscr{A}_6$, $G_2 = W_4\colon \mathscr{S}_5$ and $G_3 = V_3\colon \mathrm{GL}_3(\mathbb{F}_2)$ and we obtain the diagram of subgroups contained in M_{22}:



(*)

where $G_1 \cap G_2 = V_4\colon \mathscr{S}_4$, $G_1 \cap G_3 = V_3\colon\ \mathscr{S}_4$, $G_2 \cap G_3 = V_3\colon \mathscr{S}_4$, and $G_1 \cap G_2 \cap G_3 = UT_4(2)$, the subgroup of upper triangular matrices in $L_4(2)$.

This diagram corresponds exactly to a sporadic geometry obtained by Ronan and Smith [13] to which the local methods of Webb [17] can be applied (see [14]). Hence in principle

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we could obtain $H^*(M_{22}; \mathbb{F}_2)$ from the cohomologies of the G_i's and their various intersections. However, it turns out that it is at least as hard to obtain the cohomology of these groups as it is to get the cohomology directly, and it is difficult to obtain the explicit ring structure from this. For these reasons we apply more classical (and direct) methods in this paper.

Recall that if we have a triple of groups $H \subset K \subset G$, then H is weakly closed in K if every subgroup of K which is conjugate to H in G is already conjugate to H in K. When H is p-elementary and K contains a Sylow p-subgroup of G then the Cardenas–Kuhn theorem [2, 7], asserts that

$$\begin{aligned}\operatorname{im}(\operatorname{res}^*: &H^*(G; \mathbb{F}_p) \to H^*(H; \mathbb{F}_p))\\ &= \operatorname{im}(\operatorname{res}^*: H^*(K; \mathbb{F}_p) \to H^*(H; \mathbb{F}_p)) \cap H^*(H; \mathbb{F}_p)^{W_G(H)}\end{aligned}$$

where $W_G(H) = N_G(H)/C_G(H)$ is the Weyl group of H in G. One of our critical facts is the following theorem

THEOREM 2.8: *Each of the extremal subgroups V_4, W_4, V_3, is weakly closed in $\mathrm{Syl}_2(M_{22}) \subset M_{22}$.*

Therefore, to understand the quotient $H^*(M_{22}; \mathbb{F}_2)/\mathrm{Rad}$ it is necessary to first understand $H^*(\mathrm{Syl}_2(M_{22}); \mathbb{F}_2)$, the image of

$$\oplus_i \operatorname{res}^*: H^*(\mathrm{Syl}_2(M_{22}); \mathbb{F}_2) \to H^*(V_4; \mathbb{F}_2) \oplus H^*(W_4; \mathbb{F}_2) \oplus H^*(V_3; \mathbb{F}_2)$$

and the invariants under the action of $(\mathscr{A}_6, \mathscr{S}_5, L_3(2))$ on the three summands.

The structure of $H^*(\mathrm{Syl}_2(M_{22}); \mathbb{F}_2)$ was announced in [1] but the details of the proof were deferred to the present paper. The Dickson algebra

$$\mathbb{F}_2[x_1, x_2, x_3]^{L_3(2)} = \mathbb{F}_2[d_4, d_6, d_7],$$

is well known. Here d_4 is the Dickson element, the symmetric sum $Sx_i^4 + Sx_i^2x_j^2 + Sx_i^2x_jx_k$, $Sq^2(d_4) = d_6$, $Sq^1(d_6) = d_7$ and $Sq^4(d_6) = d_4d_6$. As a consequence $Sq^4(d_7) = Sq^4Sq^1(d_6) = Sq^1Sq^4(d_6) = d_4d_7$ since $Sq^2Sq^3 = Sq^4Sq^1 + Sq^1Sq^4$ and $Sq^3(d_6) = 0$. The $\mathscr{A}_6$ invariant subring is determined in [3] as

$$\mathbb{F}_2[a, b, c, d]^{\mathscr{A}_6} = \mathbb{F}_2[\bar{w}_3, \gamma_5, d_8, d_{12}](1, \gamma_9, b_{15}, \gamma_9 b_{15})$$

where $\bar{w}_3$ is the symmetric sum $Sx_i^2x_j$, as the x_i, x_j run over a, b, c and d. Also, $\gamma_5 = Sq^2(\bar{w}_3)$, $\gamma_9 = Sq^4(\gamma_5)$, $d_{12} = Sq^4(d_8)$ and d_8 is the Dickson element, the symmetric sum $Sx_i^8 + Sx_i^4x_j^4 + Sx_i^2x_j^2x_k^4 + Sx_ix_jx_k^2x_l^4 + (x_1x_2x_3x_4)^2$. We should notice here that $Sq^1(\gamma_5) = \bar{w}_3^2$ and $Sq^1(\gamma_9) = \gamma_5^2$. Finally, the $\mathscr{S}_5$ invariant subring is determined in [1], but in a form not well adapted to our needs here. Consequently we discuss the ring further in Section 5 and we obtain

$$\mathbb{F}_2[a, b, c, d]^{\mathscr{S}_5} = \mathbb{F}_2[\bar{w}_3, \gamma_5, d_8, d_{12}](1, n_6, n_8, \gamma_9, n_{10}, n_{12}, x_{12}, x_{14}, x_{15}, x_{16}, x_{18}, x_{24})$$

where $Sq^2(n_6) = n_8$, $Sq^4(n_6) = n_{10}$, $n_{12} = n_6^2$, $x_{12} = Sq^4(n_8)$ and $x_{14} = n_6n_8$. Moreover, in the invariant subring

$$\begin{aligned}Sq^1(n_6) &= 0\\ Sq^1(n_8) &= \bar{w}_3n_6\\ Sq^1(n_{10}) &= \bar{w}_3n_8 + \gamma_5n_6\\ Sq^1(x_{12}) &= \bar{w}_3n_{10} + \gamma_5n_8\\ Sq^1(x_{14}) &= \bar{w}_3n_6^2.\end{aligned}$$

Consequently, $Sq^1(\bar{w}_3 x_{12} + \gamma_5 n_{10}) = \gamma_5^2 n_6$. The exact structure of the elements x_{15}, x_{16}, x_{18} and x_{24} are not known to us currently, but could be determined by extending the analysis of (5.1).

The image of restriction in each of $H^*(V_4; \mathbb{F}_2)$, $H^*(W_4; \mathbb{F}_2)$ is the entire invariant subring while the image in $H^*(V_3; \mathbb{F}_2)$ is $\mathbb{F}_2[d_4^2, d_6, d_7](1, d_4 d_6, d_4 d_7)$. Thus, to describe the image of $H^*(M_{22}; \mathbb{F}_2)$ in the direct sum $H^*(V_4; \mathbb{F}_2) \oplus H^*(W_4; \mathbb{F}_2) \oplus H^*(V_3; \mathbb{F}_2)$ we need to describe the multiple image classes, i.e. those classes which have non-trivial image in more than one of the three rings. It turns out that they are generated by $(\bar{w}_3, \bar{w}_3, 0)$, $(0, n_6, d_6)$, $(0, n_{10}, d_4 d_6)$ together with the polynomial ring $\mathbb{F}_2[d_8, d_{12}]$, where $d_8 \mapsto (d_8, d_8, d_4^2)$, $d_{12} \mapsto (d_{12}, d_{12}, d_6^2)$. In fact the above completely describes the multiple image classes when we note that $(\bar{w}_3^2, 0, 0)$, $(\gamma_5, 0, 0)$ and $(0, 0, d_7)$ are also in the restriction image. It is important to notice also that the multiple image property changes the Sq^1 operation on the elements which restrict, respectively, to $(0, n_6, d_6)$ and $(0, n_{10}, d_4 d_6)$, so in $H^*(M_{22}; \mathbb{F}_2)$ we have $Sq^1(n_6) = (0, 0, d_7)$ while $Sq^1(n_{10}) = (0, \bar{w}_3 n_8 + \gamma_5 n_6, d_4 d_7)$.

In summary, we can describe the non-nilpotent part of $H^*(M_{22}; \mathbb{F}_2)$ as the direct sum

$$H^*(V_4; \mathbb{F}_2)^{\mathscr{A}_6} \oplus H^*(W_4; \mathbb{F}_2)^{\mathscr{S}_5} \oplus d_7 \mathbb{F}_2[d_4, d_6, d_7]$$

where the two copies of $\mathbb{F}_2[d_8, d_{12}](1, \bar{w}_3)$ in the first two rings are identified. The key technical step in this determination, after we have proved (2.8), is to show that $(b_{15}, 0, 0)$ is in the image of the restriction map from $H^*(M_{22}; \mathbb{F}_2)$.

Finally, the radical is discussed at the end of Section 5 and shown to have the form

$$\mathbb{F}_2[d_8, d_{12}](a_2, a_7, a_{11}, a_{14})$$

where the mod 4 Bockstein $\beta_4(a_2) = \bar{w}_3$, while some higher Bockstein of a_7 is d_8, and a higher Bockstein of a_{11} is d_{12}. There are further higher Bocksteins which we do not completely understand at this time, but aside from that our results give a complete determination of $\mathrm{H}^*(M_{22}; \mathbb{F}_2)$, though there does remain an extension problem for determining the ring structure.

This extension problem can be handled in the following way. It turns out, (Proposition 5.4 in the text), that the restriction map $H^*(M_{22}; \mathbb{F}_2) \to H^*(Syl_2(L_3(4)); \mathbb{F}_2)$ is injective on the radical. Here $L_3(4) = M_{21}$ and $H^*(Syl_2(L_3(4)); \mathbb{F}_2)$ is completely determined in [1]. Thus, we can use the results there to completely determine the structure of the extension data. In particular, we have the following representations for a_7, a_{11} and a_{14} from [1, p. 197, line 5]:

$$\begin{aligned}
a_7 &= \mathscr{F}\mathscr{A}^2 v + \mathscr{A}\mathscr{F}^2 w = \mathscr{A}\mathscr{F}(\mathscr{A}v + \mathscr{F}w) \\
&= a_2 \gamma_5(1) \\
a_{11} &= \mathscr{F}\mathscr{A}^2 w^2 + \mathscr{A}\mathscr{F}^2 v^2 = \mathscr{A}\mathscr{F}(\mathscr{A}w^2 + \mathscr{F}v^2) \\
&= a_2 \gamma_9(1) \\
a_{14} &= \mathscr{A}\mathscr{F} v^3 + \mathscr{B}\mathscr{E} w^3 \\
&= \gamma_5(1)\gamma_9(2)
\end{aligned}$$

and this last relation shows the extension is non-trivial. By a similar calculation we also have

$$\gamma_5(1)\gamma_5(2) = a_2 d_8$$

and we check that

$$\gamma_5(1) n_6 = 0.$$

This gives all the extension information in dimensions less than about 16 and if our understanding of $b_{15}, x_{15}, x_{16}, x_{18}$ and x_{24} were better we could complete the determination of the extension data, but, as the results above show, the extension data is highly non-trivial.

Remark. In our original discussion we neglected to discuss the nature of the extension, and we thank the referee for energetically pointing this oversight out to us.

Remark. Since $H^*(M_{22}; \mathbb{F}_2)$ is not Cohen–Macaulay the Poincaré series, expressed as a rational function, gives us very little insight into the structure of the cohomology groups. Hence we have chosen to omit it from our paper.

It is worthwhile to note that M_{22} can be expressed as a quotient of the direct limit Γ of the triangle of subgroups $(*)$, also known as the amalgamated free product of G_1, G_2, G_3 along their intersections. Shpectorov [15] has in fact proved that this is a group isomorphism. From this and our previous work [4], it turns out that all the Mathieu groups we have analyzed $(M_{11}, M_{12}, M_{21}, M_{22})$ are quotients of amalgamated free products (of proper subgroups) which are cohomologous to them at $p = 2$. This appears to be a phenomenon which has interesting geometric consequences.

Another interesting consequence of the calculation presented here is the very recent determination of the mod 2 cohomology of the next Mathieu group, M_{23}, by Milgram. Remarkably the classifying space of this finite group turns out to be homologically 4-connected, thus disproving a conjecture due to Giffen.

Coefficients in $\mathbb{F}_2$ are assumed throughout, so they will be suppressed. We would like to thank Smith for useful comments and Ivanov for kindly pointing out Shpectorov's result to us.

1. M_{22} AND ITS SUBGROUPS

M_{22} is one of the Mathieu groups, a sporadic simple group of order $443{,}520 = 2^7\,3^2\,5\,7\,11$. It can be given as the subgroup of $\mathcal{S}_{22}$ generated by the permutations.

$$X = (1\,2\,3\,4\,5\,6\,7\,8\,9\,10\,11)(12\,13\,14\,15\,16\,17\,18\,19\,20\,21\,21\,22)$$

$$Y = (1\,4\,5\,3)(2\,8\,10\,7\,6)(12\,15\,16\,20\,14)(13\,19\,21\,18\,17)$$

$$Z = (11\,22)(8\,14)(4\,5\,3\,9)(13\,18\,17\,19)(2\,16\,10\,15)(7\,20\,6\,12).$$

We will be interested in $\mathrm{Syl}_2(M_{22})$ and in certain subgroups which contain $\mathrm{Syl}_2(M_{22})$. $\mathrm{Syl}_2(M_{22})$ has center $\mathbb{Z}/2$ and is given as a central extension

$$\mathbb{Z}/2 \to \mathrm{Syl}_2(M_{22}) \to UT_4(2)$$

where $UT_4(2)$ is the Sylow 2-subgroup of $L_4(2) \cong \mathcal{A}_8$. In Section 4, where we determine $H^*(\mathrm{Syl}_2(M_{22}))$, we will also make extensive use of two index 2 subgroups of $\mathrm{Syl}_2(M_{22})$ which are also isomorphic to $UT_4(2)$. In particular, there are also two representations of $\mathrm{Syl}_2(M_{22})$ as semi-direct products $UT_4(2){:}\,2$.

But now we describe the normalizers of the 2-elementaries, four of which contain $\mathrm{Syl}_2(M_{22})$. Recall that there is an isomorphism $\mathcal{S}_6 \cong \mathrm{Sp}_2(\mathbb{F}_2)$, hence $\mathcal{A}_6 \cong \mathrm{Sp}_4(\mathbb{F}_2)'$ acts via this inclusion on $V_4 = (\mathbb{Z}_2)^4$. Let G_1 denote the corresponding semi-direct product

$$G_1 = V_4\colon \mathscr{A}_6.$$

Similarly we have an isomorphism $\mathscr{A}_5 \cong \mathrm{SL}_2(\mathbb{F}_4)$, which induces an $\mathscr{A}_5$-action on V_4 (regarded as $(\mathbb{F}_4)^2$) that extends uniquely to an $\mathscr{S}_5$ action on W_4 via the action of the group $\mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2) \cong \mathbb{Z}/2$ on the coefficients of the matrices. Let G_2 denote the corresponding semi-direct product

$$G_2 = W_4\colon \mathscr{S}_5.$$

Janko [6] has shown that there are exactly two elementary abelian 2-subgroups of rank 4 in $\mathrm{Syl}_2(M_{22})$, V_4 and W_4, that $N_{M_{22}}(V_4) \cong G_1$, $N_{M_{22}}(W_4) \cong G_2$, and both are maximal in M_{22}. Furthermore, representative subgroups G_1 and G_2 may be chosen so that

$$G_1 \cap G_2 \cong V_4\colon \mathscr{S}_4.$$

There is a second $\mathscr{S}_4 \subset \mathscr{A}_6$. (If the first is $\langle(1\,2)(5\,6),(1\,2\,3),(1\,2)(3\,4)\rangle$ then the second is $\langle(1\,2)(3\,4),(1\,3\,5)(2\,4\,6),(1\,3)(2\,4)\rangle$.) The resulting extension $V_4\colon \mathscr{S}_4$ is the centralizer of an involution in M_{22}, [6].

There is another extremal subgroup of interest to us, the semi-direct product

$$G_3 = V_3\colon \mathrm{GL}_3(\mathbb{F}_2)$$

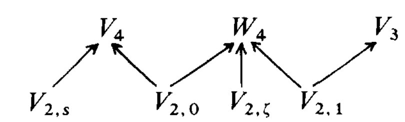
where $V_3 \cong (\mathbb{Z}_2)^3$ represents a maximal elementary abelian subgroup in M_{22}, with normalizer G_3. We can choose G_3 so that

$$G_1 \cap G_3 \cong V_3\colon \mathscr{S}_4$$
$$G_2 \cap G_3 \cong V_3\colon \mathscr{S}_4$$
$$G_1 \cap G_2 \cap G_3 = V_3\colon D_8$$

where the $\mathscr{S}_4$'s are distinct parabolics in $\mathrm{GL}_3(\mathbb{F}_2)$, with intersection D_8.

There are two further classes of 2^3's in M_{22}, one in V_4 and one in W_4. Indeed, the actions of $\mathscr{A}_6$ and $\mathscr{S}_5$ are both transitive on the 2^3's in V_4 and W_4.

Finally, there are four classes of 2^2's in M_{22}. $\mathscr{A}_6$ acting on V_4 gives two non-conjugate 2^2's, $V_{2,s}$ and $V_{2,0}$ in V_4, while $\mathscr{S}_5$ acting on W_4 gives three non-conjugate 2^2's in W_4. The classes of 2^2's in W_4 are distinguished by the determinant of a basis as an element in the orbit set $\mathbb{F}_4/\mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)$. Specifically, start with a basis of $\mathbb{F}_4^2$ over $\mathbb{F}_4$, then representatives for the classes are the $\mathbb{F}_2$-vector spaces, $\langle e_1, \zeta_3 e_1\rangle$ with determinant 0, $\langle e_1, e_2\rangle$ with determinant 1, and $\langle e_1, \zeta_3 e_2\rangle$ with determinant ζ_3. The intersection of V_3 with W_4 is a copy of 2^2 with determinant 1, while $V_{2,0} = W_4 \cap V_4$ has determinant 0. To see that these groups are distinct we check in (2.11) that they have distinct centralizers: $C(V_{2,s}) = V_4\colon \mathbb{Z}/3$, $C(V_{2,0}) = \mathrm{Syl}_2(L_3(4))$, $C(V_{2,1}) = W_4\colon \mathbb{Z}/2$, $C(V_{2,\zeta}) = W_4$. However, for all four we have $N(2^2)/C(2^2) \cong L_2(2) = \mathscr{S}_3$. The following is a picture of the containments for the four classes of 2^2's.



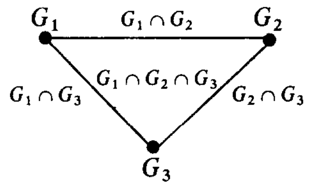
(1.1)

There is a double coset decomposition

$$M_{22} = G_1 \sqcup G_1 v G_1 \sqcup G_1 w G_1 \tag{1.2}$$

where $v \in G_2$, but $w \notin G_2$. In addition $G_1 \cap wG_1w^{-1} = \mathscr{A}_6$ is normalized by w, so that the action restricts to one of the 2-Sylow subgroups of $\mathscr{A}_6$ as a *non-trivial* outer automorphism. These details of the structure of the decomposition were determined by Overton using a Sun 3/280 computer. However, it follows that this automorphism cannot introduce any non-trivial fusion among the 2^2's, and it cannot even introduce any non-trivial stability condition on cohomology since the Weyl group of each of the 2^2's is a copy of $L_2(2)$.

Identifying subgroups conjugate in M_{22}, we can describe G_1, G_2, G_3 and their intersections by the diagram



(1.3)

It is perhaps worthwhile to point out that some of these subgroups occur in a rather special way as automorphisms of compact complex surfaces. In fact Mukai [11] has shown that $G_1 \cap G_2 = V_4 \colon \mathscr{S}_4$ and $H = V_4 \colon \mathscr{A}_5$ occur as maximal symplectic automorphism groups of a K3 surface. For the first group the associated K3 surface $S \subsetneq \mathbb{P}^3$ is given by the equation $X^4 + Y^4 + Z^4 + T^4 = 0$; for the second group it is given by the equation $X^4 + Y^4 + Z^4 + T^4 + 12XYZT = 0$. In addition, he proves that $\mathrm{Syl}_2(M_{22})$ is the unique symplectic automorphism group of a K3 surface of order 2^7, and the largest 2-group which occurs in this way. Aside from providing a concrete description of these groups, these results indicate that they contain geometric information which may be reflected in their cohomology.

2. THE SYLOW SUBGROUP OF M_{22}

The 2-Sylow subgroup of M_{22} is especially interesting as it is also the Sylow subgroup of three other simple groups, two of which are sporadic, $U_4(3), M_{23}, M^cL$, and it is closely connected to the Lyons group, $\mathrm{Syl}_2(Ly) = \mathrm{Syl}_2(M_{22})\colon 2$.

We pointed out in Section 1 that $\mathrm{Syl}_2(M_{22})$ can be given as an extension

$$1 \to UT_4(2) \overset{\triangleleft}{\to} \mathrm{Syl}_2(M_{22}) \to \mathbb{Z}/2 \to 1$$

and as a central extension

$$1 \to \mathbb{Z}/2 \overset{\triangleleft}{\to} \mathrm{Syl}_2(M_{22}) \to UT_4(2) \to 1.$$

We make these extensions explicit now and use them, together with results from [1], where we studied yet a third extension

$$1 \to \mathrm{Syl}_2(L_3(4)) \overset{\triangleleft}{\to} \mathrm{Syl}_2(M_{22}) \to \mathbb{Z}/2 \to 1$$

to determine $H^*(\mathrm{Syl}_2(M_{22}))$ in (4.1).

The group $UT_4(2) \cong V_4 \times_\lambda (\mathbb{Z}/2)^2$ has index two in $\mathrm{Syl}_2(M_{22})$ and is also the quotient of $\mathrm{Syl}_2(M_{22})$ by its center $\mathbb{Z}/2$. $UT_4(2)$ is generated by the six elements A, B, C, D, α and β, each of order two where $\langle A, B, C, D \rangle = V_4$ and $\langle \alpha, \beta \rangle = (\mathbb{Z}/2)^2$. Moreover, the action of α and

β on V_4 can be described via the diagram

$$\begin{array}{cccc} & A & \leftrightarrow & C \\ \alpha & \updownarrow & & \updownarrow \\ & B & \leftrightarrow & D \\ & & \beta & \end{array} \tag{2.1}$$

where α acts to exchange the rows while β acts to exchange the columns. The symmetry between row and column results in the existence of an outer automorphism, ϕ, of $UT_4(2)$ given as $\phi:\alpha \leftrightarrow \beta$. $A \leftrightarrow B$, while ϕ fixes B and C.

From this point of view we can think of $\mathrm{Syl}_2(M_{22})$ as the semi-direct product

$$\mathrm{Syl}_2(M_{22}) = UT_4(2):2 = \langle UT_4(2), E\rangle$$

where E interchanges α, β, but $E(A) = BCD$, $E(B) = ABD$, $E(C) = ACD$ and $E(D) = ABC$.

The center of $UT_4(2)$ is $\mathbb{Z}/2 = \langle ABCD\rangle$, and, since E also fixes this subgroup, it is the center of $\mathrm{Syl}_2(M_{22})$ as well. Consequently, as indicated, $\mathrm{Syl}_2(M_{22})$ can be described as a central extension $2 \cdot UT_4(2)$, where, in the quotient by $\langle ABCD\rangle$, the identification with $UT_4(2)$ is given by the correspondence

$$\begin{array}{cccc} & AB\alpha & \leftrightarrow & AC\beta \\ A & \updownarrow & & \updownarrow \\ & \alpha & \leftrightarrow & \beta \\ & & E & \end{array} . \tag{2.2}$$

It follows that ϕ on the quotient above lifts to an automorphism of $\mathrm{Syl}_2(M_{22})$ which we again denote by ϕ,

$$\phi : A \leftrightarrow E$$
$$\alpha \leftrightarrow \alpha$$
$$\beta \leftrightarrow AB\alpha$$

so $\phi(B) = E\alpha\beta$, $\phi(C) = EBC\alpha\beta$, $\phi(D) = EAD$. In particular, the image $\phi(UT_4(2))$ is a second copy of $UT_4(2)$ contained in $\mathrm{Syl}_2(M_{22})$,

$$\begin{array}{cccc} & E & \leftrightarrow & EBC\alpha\beta \\ \alpha & \updownarrow & & \updownarrow \\ & E\alpha\beta & \leftrightarrow & EAD \\ & & AB\alpha & \end{array} . \tag{2.3}$$

Thus we have constructed two distinct copies of the elementary two group 2^4 contained in $\mathrm{Syl}_2(M_{22})$. Moreover, from [6], these are the only copies of 2^4 contained in $\mathrm{Syl}_2(M_{22})$.

Remark 2.4. This outer automorphism, ϕ, is used to construct the 2-Sylow subgroup of the sporadic group Ly.

$M_{21} = L_3(4)$ and $\mathrm{Syl}_2(L_3(4){:}\,2_2) = \mathrm{Syl}_2(M_{22})$, where 2_2 is the automorphism induced by the non-trivial element in the group $\mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)$. An embedding

$$\langle E, \alpha\beta, A, B, AD, BC\rangle \subset L_3(4)$$

is given by

$$A \mapsto \begin{pmatrix} 1 & \zeta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B \mapsto \begin{pmatrix} 1 & \zeta^2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad AD \mapsto \begin{pmatrix} 1 & 0 & \zeta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$BC \mapsto \begin{pmatrix} 1 & 0 & \zeta^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \alpha\beta \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad E \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \zeta^2 \\ 0 & 0 & 1 \end{pmatrix}$$

and $\alpha \mapsto 2_2$. Under this isomorphism one of the elements of order three, τ, in the intersection

$$N_{M_{22}}(V_4) \cap N_{M_{22}}(W_4) = V_4 : \mathcal{S}_4$$

corresponds to the matrix

$$\begin{pmatrix} \zeta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & \zeta^{-1} \end{pmatrix}$$

so it acts on $\mathrm{Syl}_2(L_3(4))$ via the rule $\tau(\alpha\beta) = \alpha\beta E$, $\tau(A) = B$, $\tau(B) = AB$, $\tau(AD) = ABCD$, $\tau(BC) = AD$ and $\alpha\tau\alpha = \tau^{-1}$.

We define 6 elementary 2-subgroups in $\mathrm{UT}_4(2)$ as follows:

$$\begin{aligned} \mathcal{B} &= \langle \alpha, \beta, ABCD \rangle \\ \bar{\mathcal{B}} &= \langle AB\alpha, AC\beta, ABCD \rangle \\ W_\alpha &= \langle \alpha, AB, CD \rangle \\ W_{\alpha\beta} &= \langle \alpha\beta, AD, BC \rangle \\ W_\beta &= \langle \beta, AC, BD \rangle \\ V &= \langle A, B, C, D \rangle. \end{aligned} \tag{2.5}$$

One can think of these groups as inverse images in $UT_4(2)$ from the quotient $UT_4(2)$ obtained by factoring out the center, since they all contain the center. In this way $\mathcal{B}$ and $\bar{\mathcal{B}}$ arise from the two rows while the group V comes from A and the product of the two rows, and the W_θ come from the two columns and their product. In particular, since both rows and columns are in the intersection of $UT_4(2)$ and $\phi(UT_4(2))$ it follows that the W_θ and the $\mathcal{B}$, $\bar{\mathcal{B}}$ groups are contained in this intersection, but since ϕ interchanges rows and columns, their roles are interchanged.

The group $\mathcal{A}_6$ contains two conjugacy classes of elements of order three, one of which acts without fixed vectors on V_4 and the other of which has a $(\mathbb{Z}/2)^2$ fixed space. We already determined the action of τ on $\mathrm{Syl}_2(L_3(4))$. The action of an element in the other conjugacy class, T, on $UT_4(2)$ is given by $T(\alpha) = \beta$, $T(\beta) = \alpha\beta$, and

$$T(A) = B, \qquad T(B) = C, \qquad T(C) = A, \qquad T(D) = D$$

while $ETE = T^{-1}$. Clearly T cyclically permutes the three groups W_α, W_β and $W_{\alpha\beta}$ in (2.5), while it normalizes $\mathcal{B}$ and V. Again $\tau T = T^2\tau$ and $\langle \tau, T \rangle \cong \mathcal{S}_3$.

Incidentally, there is only one conjugacy class of involutions in M_{22} and

$$\langle UT_4(2), T, E \rangle \cong V_4 : \mathcal{S}_4$$

is the centralizer of $ABCD$ in M_{22}, [6]. It corresponds to the second conjugacy class $\mathcal{S}_4 \subset \mathcal{A}_6$, and is not isomorphic to $G_1 \cap G_2$.

We now identify a representative of the extremal 2^3, V_3, with $\langle \alpha, \beta, ABCD \rangle$.

PROPOSITION 2.6. *Each of the groups* $W_\alpha, W_{\alpha\beta}, W_\beta$ *in* (2.5) *is contained in a* $2^4 \subset M_{22}$.

Proof. $W_{\alpha\beta} \subset \langle E, \alpha\beta, AD, BC \rangle$ and the element T described before (2.6) is contained in $V: \mathscr{A}_6 \subset M_{22}$. But T acts transitively on the three groups W_θ. ∎

Next we have the following lemma.

LEMMA 2.7. *If* $L \subset \mathrm{Syl}_2(M_{22})$ *is conjugate to* V_3 *in* M_{22} *then* $L = \langle \alpha, \beta, ABCD \rangle$ *or* $\langle \alpha AB, \beta AC, ABCD \rangle$ *and both groups are already conjugate in* $\mathrm{Syl}_2(M_{22})$.

Proof. $L \not\subset V_4$, so the projection $\pi: V_4 : D_8 \to D_8$, when restricted to L, has image either a copy of $\mathbb{Z}/2 \subset D_8$, or one of the two $(\mathbb{Z}/2)^2$'s in $D_8, \langle \alpha, \beta \rangle$ or $\langle E, \alpha\beta \rangle$.

There are five copies of $\mathbb{Z}/2$ in D_8, $\langle \alpha \rangle$, $\langle \alpha\beta \rangle$, $\langle \beta \rangle$, $\langle E \rangle$ and $\langle E\alpha\beta \rangle$. Suppose that $\pi(L) = \langle \alpha \rangle$. Then there is an element $\theta\alpha \in L$ with $\theta \in V_4$. Since $(\theta\alpha)^2 = 1$ we have that θ commutes with α. Hence $\theta \in \langle AB, CD \rangle$ and $L = W_\alpha$ which is impossible by the previous result. Similar arguments work for $\langle \beta \rangle$ and $\langle \alpha\beta \rangle$. If $\pi(L) = \langle E \rangle$ then $\theta E \in L$ and $\theta \in \langle AD, BC \rangle$. It follows that $L \subset \langle E, \alpha\beta, AD, BC \rangle$. A similar argument works if $\pi(L) = \langle E\alpha\beta \rangle$.

Suppose $\pi(L) = \langle E, \alpha\beta \rangle$. Again it follows that $L \subset \langle E, \alpha\beta, AD, BC \rangle$, so the only case which remains is $\pi(L) = \langle \alpha, \beta \rangle$. In this case $\theta\alpha$ and $\tau\beta$ are contained in L with $\theta \in \langle AB, CD \rangle$, $\tau \in \langle AC, BD \rangle$. The element common to these groups is $ABCD$, so, since $L \cap V_4 = \mathbb{Z}/2$, it follows that $ABCD \in L$. Thus, we can assume that $\theta = AB$ or $\theta = 1$. Suppose that $\theta = 1$. Then, $\tau\beta$ and α commute so $\tau = 1$, and $L = \mathscr{B}$. If $\theta = AB$ then a similar check shows that $\tau = AC$ so $L = A\mathscr{B}A$. ∎

COROLLARY 2.8. *Each of the three extremal elementary two groups* V_4, W_4, V_3 *is weakly closed in* $\mathrm{Syl}_2(M_{22})$ *in* M_{22}.

Proof. We know that $\mathrm{Syl}_2(M_{22})$ only contains the two copies of 2^4, V_4 and W_4. Moreover, they are not conjugate in M_{22} since they have non-isomorphic normalizers there. Thus they are weakly closed in $\mathrm{Syl}_2(M_{22})$, and the result above shows that $V_3 = \mathscr{B}$ is also weakly closed in $\mathrm{Syl}_2(M_{22})$. ∎

The core of the structure of M_{22} comes from the amalgamation

$$V_4 : \mathscr{A}_6 \quad \bullet\!\!\underset{V_4 : \mathscr{S}_4}{\longrightarrow}\!\!\bullet \quad W_4 : \mathscr{S}_5$$

where the group $V_4 : \mathscr{S}_4 = G_1 \cap G_2$. Let $\mathscr{M} = W_4 : (\mathscr{S}_3 \times \mathbb{Z}/2) \subset W_4 : \mathscr{S}_5$ be the subgroup

$$\mathscr{M} = \langle W_4, \tau, \alpha, \lambda \rangle$$

where $\lambda \in W_4 : \mathscr{S}_5$ is represented by the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$\mathscr{M}$ is not conjugate to any subgroup of $V_4 : \mathscr{A}_6$ though $\mathrm{Syl}_2(\mathscr{M})$ is conjugate to the second $UT_4(2) \subset \mathrm{Syl}_2(M_{22})$, $\langle W_4, \alpha, AB \rangle$. Set $\mathscr{N} = \mathscr{M} \cap G_1 \cap G_2 \cong W_4 : \mathscr{S}_3$. Then we have the following lemma.

LEMMA 2.9. *Let* $\theta \in H^*(G_1 \cap G_2)$. *Then* $\theta \in \mathrm{im}(\mathrm{res}^*: H^*(W_4:\mathcal{S}_5) \to H^*(G_1 \cap G_2))$ *if and only if* $res^*(\theta) \in H^*(\mathcal{N})^\lambda$.

Proof. There are two double cosets of $G_1 \cap G_2$ in $W_4:\mathcal{S}_5$, corresponding to the double cosets of $\mathcal{S}_4$ in $\mathcal{S}_5$. In particular, for the non-trivial double coset $G_1 \cap G_2 \lambda G_1 \cap G_2$ we have $\lambda^2 = 1$ and $\lambda G_1 \cap G_2 \lambda \bigcap G_1 \cap G_2 = W_4:\mathcal{S}_3$, with λ commuting with $\mathcal{S}_3$. Thus, an element in $H^*(G_1 \cap G_2)$ is stable if and only if the condition of (2.9) is satisfied. ■

LEMMA 2.10. *An element* $\theta \in H^*(G_1 \cap G_2)$ *is in the image of restriction from* $H^*(V_4:\mathcal{A}_6)$ *if and only if*

(a) *the restriction of* θ *to* $H^*(V_4)$ *is contained in* $H^*(V_4)^{\mathcal{A}_6}$ *and*
(b) *the restriction of* θ *to* $H^*(UT_4(2))$ *is contained in* $H^*(UT_2(2))^{\langle T \rangle}$.

Proof. Once more we look at the double coset decomposition of $V_4:\mathcal{A}_6$ in terms of $G_1 \cap G_2$. As in (2.9) this is determined by the decomposition of $\mathcal{A}_6$ in terms of the copy of $\mathcal{S}_4 = \langle (12)(56), (123), (13)(24) \rangle \subset \mathcal{A}_6$. Thus there are three double cosets

$$V_4:\mathcal{A}_6 = G_1 \cap G_2 \sqcup G_1 \cap G_2(456)G_1 \cap G_2 \sqcup G_1 \cap G_2(35)(46)G_1 \cap G_2.$$

Moreover, $G_1 \cap G_2 \bigcap (456)G_1 \cap G_2(654) = \langle V_4, (123) \rangle$, so the constraint due to this double coset is subsumed in the assumption that the restriction of θ is contained in $H^*(V_4)^{\mathcal{A}_6}$. Finally $G_1 \cap G_2 \bigcap (35)(46)G_1 \cap G_2(35)(46) = UT_4(2)$ and $UT_4(2)$ together with $(35)(46)$ are both contained in $\langle UT_4(2), E, T \rangle$, the centralizer of $\langle ABCD \rangle$ in M_{22}. ■

We next note the following lemma.

LEMMA 2.11. *Let* $\theta \in H^*(Syl_2(M_{22}))$, *then* θ *is in the image of restriction from* $H^*(G_1 \cap G_2)$ *if and only if the restriction of* θ *to* $H^*(\mathrm{Syl}_2(L_3(4)))$ *is contained in* $H^*(\mathrm{Syl}_2(L_3(4)))^\tau$. (This is again an easy exercise with double cosets.)

Finally, we note that the results above give an effective method for determining when an element in $H^*(\mathrm{Syl}_2(M_{22}))$ is in the image of restriction from $H^*(M_{22})$: $\theta \in H^*(\mathrm{Syl}_2(M_{22}))$ is contained in the image of restriction from $H^*(M_{22})$ if and only if it is in the image from $H^*(V_4:\mathcal{A}_6)$, the image from $H^*(W_4:\mathcal{S}_5)$ and its restriction to $H^*(V_3)$ lies in $H^*(V_3)^{L_3(2)}$.

The group $H^*(\mathrm{Syl}_2(L_3(4)))^\tau$ has been studied in [1]. It has a radical but the restriction map $H^*(\mathrm{Syl}_2(L_3(4)))^\tau/\mathrm{Rad} \to H^*(V_4) \oplus H^*(W_4)$ is injective. Also, $\mathrm{Syl}_2(\mathcal{N})$ is isomorphic to the wreath product $(\mathbb{Z}/2)^2 \wr \mathbb{Z}/2$ and its cohomology is detected by W_4, $\langle ABCD, \alpha\beta, \alpha \rangle = V_3$, both of which are normalized by λ. It follows that the constraints imposed by $\mathcal{N}$ are subsumed in the requirements that the restriction to $H^*(W_4)$ be $\mathcal{S}_5$ invariant and to $H^*(V_3)$ be invariant under $L_3(2)$. Consequently, we only need to study $H^*(UT_4(2))$ before we can completely control $H^*(M_{22})$.

To conclude this section we give, as promised in Section 1, a list of the centralizers of the four 2^2's in M_{22}. From the structure of $\mathrm{Syl}_2(M_{22})$ we see easily that each 2^2 in $\mathrm{Syl}_2(M_{22})$ is contained in a 2^3, so there are no more than four conjugacy classes. Moreover, since V_3 has Weyl group $L_3(2)$ any two 2^2's contained in V_3 are conjugate. Thus the 2^2 in V_3 is $V_3 \cap W_4 = \langle \alpha\beta, ABCD \rangle = V_{2,1}$. Likewise $V_{2,0} = V_4 \cap W_4 = \langle AD, ABCD \rangle$, $V_{2,\zeta} = \langle ABCD, E \rangle$, and the remaining subgroup $V_{2,s} = \langle ABCD, ABC \rangle \subset V_4$. In particular, each of these groups contains $ABCD$ and so its centralizer is contained in $\langle \mathrm{Syl}_2(M_{22}), T \rangle$.

Consequently, we have

$$\begin{aligned} C(V_{2,0}) &= \mathrm{Syl}_2(L_3(4)) = \langle V_4, W_4 \rangle \\ C(V_{2,1}) &= \langle W_4, \alpha \rangle \\ C(V_{2,\zeta}) &= W_4 \\ C(V_{2,s}) &= \langle V_4, T \rangle \end{aligned} \tag{2.12}$$

and, as claimed in Section 1, all the centralizers are distinct so these subgroups cannot be conjugate in M_{22}.

3. THE COHOMOLOGY OF $UT_4(2)$

The ring $H^*(UT_4(2))$ plays a decisive role in determining $H^*(\mathrm{Syl}_2(M_{22}))$. In this section we determine $H^*(UT_4(2))$ and show that it injects into the sum of the cohomology rings of its elementary 2-subgroups. The procedure is to use the Stiefel–Whitney classes of its irreducible representations to construct enough elements to detect $H^*(UT_4(2))$. Consequently, we begin by determining the irreducible representations of $UT_4(2)$.

There are two irreducible four-dimensional representations of $UT_4(2)$: the first, r_1, is induced up from the one-dimensional representation of V, $A \mapsto -1$, $B, C, D \mapsto 1$, while the second is $E(r_1)$. Let

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then, on generators they are given explicitly as

$$r_1: \quad \begin{aligned} &A \mapsto \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \quad B \mapsto \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \\ &C \mapsto \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix} \quad D \mapsto \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} \\ &\alpha \mapsto \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} \quad \beta \mapsto \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \end{aligned}$$

and for r_2 the same matrices for α, β while

$$r_2: \quad \begin{aligned} &A \mapsto \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \quad B \mapsto \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} \\ &C \mapsto \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \quad D \mapsto \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix} \end{aligned}.$$

Next there are six two-dimensional representations. They form two orbits under the action of the element of order three, T, constructed in Section 2, one which we denote $(+)$ and the other (−). The ones associated to α are given as follows:

$$E_{\alpha,\pm}:\quad \alpha \mapsto \pm I, \beta \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad A, B \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad C, D \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Finally, there are the eight one-dimensional representations with generators

$$\langle A \rangle : A \mapsto -1, \alpha, \beta \mapsto 1$$

$$\langle \alpha \rangle : A \mapsto 1, \alpha \mapsto -1, \beta \mapsto 1$$

$$\langle \beta \rangle : A \mapsto 1, \alpha \mapsto 1, \beta \mapsto -1.$$

We obtain from Table 1 restrictions of the Stiefel–Whitney classes of these representations to the 5 elementary 2-subgroups of H described above, where $\Gamma = e(l+m) + (l+m)^2$, $\theta = (lm(l+e)(m+e))$, σ_i is the ith symmetric sum and $\bar{w}_3 = \sigma_3 + \sigma_2\sigma_1$. Table 1 is, of course, highly redundant. Simplifying, we obtain a table of generators (Table 2) which are in

Table 1. Restrictions for S–W classes

Rep.	S–W class	$\mathscr{B}$	W_α	$W_{\alpha\beta}$	W_β	V
r_1	w_1	0	0	0	0	σ_1
r_1	w_2	d_2	$e^2+\Gamma$	$e^2+\Gamma$	$e^2+\Gamma$	σ_2
r_1	w_3	d_3	$e\Gamma$	$e\Gamma$	$e\Gamma$	σ_3
r_1	w_4	$v^4+v_2d_2+vd_3$	θ	θ	θ	σ_4
r_2	w_1	0	0	0	0	σ_1
r_2	w_2	d_2	$e^2+\Gamma$	$e^2+\Gamma$	$e^2+\Gamma$	$\sigma_2+\sigma_1^2$
r_2	w_3	d_3	$e\Gamma$	$e\Gamma$	$e\Gamma$	$\sigma_3+\sigma_1^3$
r_2	w_4	$v^4+v^2d_2+vd_3$	θ	θ	θ	$\sigma_4+\bar{w}_3\sigma_1$
$E_{\alpha,+}$	w_1	k	0	e	e	σ_1
$E_{\alpha,+}$	w_2	0	0	Γ	Γ	$(a+b)(c+d)$
$E_{\alpha,-}$	w_1	k	0	e	e	σ_1
$E_{\alpha,-}$	w_2	$(k+h)h$	e^2	Γ	Γ	$(a+b)(c+d)$
$E_{\beta,+}$	w_1	h	e	e	0	σ_1
$E_{\beta,+}$	w_2	0	Γ	Γ	0	$(a+c)(b+d)$
$E_{\beta,-}$	w_1	h	e	e	0	σ_1
$E_{\beta,-}$	w_2	$k(k+h)$	Γ	Γ	e^2	$(a+c)(b+d)$
$E_{\alpha\beta,+}$	w_1	$h+k$	e	0	e	σ_1
	w_2	0	Γ	0	Γ	$(a+d)(b+c)$
$E_{\alpha\beta,-}$	w_1	$h+k$	e	0	e	σ_1
	w_2	hk	Γ	e^2	Γ	$(a+d)(b+c)$
$\langle\alpha\rangle$	w_1	h	e	e	0	0
$\langle\beta\rangle$	w_1	k	0	e	e	0
$\langle A\rangle$	w_1	0	0	0	0	σ_1

Table 2. Image of restrictions from $H^*(UT_4(2))$

Name	$\mathscr{B}$	W_α	$W_{\alpha\beta}$	W_β	V
w_1	0	0	0	0	σ_1
h	h	e	e	0	0
k	k	0	e	e	0
w_2	0	Γ	Γ	Γ	σ_2
w_3	0	$e\Gamma$	$e\Gamma$	$e\Gamma$	σ_3
w_4	$v^4+v^2d_2+vd_3$	θ	θ	θ	σ_4
γ_{12}	0	0	Γ	Γ	$(a+b)(c+d)$
γ_{13}	0	Γ	Γ	0	$(a+c)(b+d)$
γ_{14}	0	Γ	0	Γ	$(a+d)(b+c)$

the image of restriction from $H^*(UT_4(2))$, where the notation is the same as that in Table 1. Out of these generators we construct three elements of dimension three, $v_\alpha = h(a+d)(b+c) = h\gamma_{14}$, $v_{\alpha\beta} = h(w_2 + \gamma_{14})$ and $v_\beta = k\gamma_{14}$. They restrict as shown in Table 3.

In particular, we find that within the subring of $H^*(UT_4(2))$ generated by the elements above, the ideal given as the kernel of restriction to $H^*(V) \oplus H^*(\mathscr{B})$ has the form

$$\mathbb{F}_2[h, w_2, w_4](v_\alpha, v_{\alpha\beta}) \oplus \mathbb{F}_2[k, w_2, w_4]v_\beta.$$

Moreover, the quotient by this ideal has the form

$$\mathbb{F}_2[w_4]\{\mathbb{F}_2[h,k] \oplus \mathbb{F}_2[w_1, ab+cd, ac+bd,](1, ad+bc, w_3, w_3(ad+bc))\}.$$

It follows that the Poincaré series for $H^*(UT_4(2))$ is at least as big as

$$\frac{1}{(1-x)(1-x^4)}\left[\frac{3x^3}{1-x^2} + \frac{1}{1-x} + \frac{1+x^2+x^3+x^5}{(1-x^2)^2} - 1 + x\right]$$
$$= \frac{1+2x+2x^2+x^3-x^4-x^5}{(1-x)(1-x^2)^2(1-x^4)}.$$

We now show the following theorem.

THEOREM 3. *$H^*(UT_4(2))$ is exactly the ring above. In particular, $H^*(UT_4(2))$ is detected by restriction to the 5 elementary 2-groups $\mathscr{B}$, W_α, $W_{\alpha\beta}$, W_β and V. Moreover, $H^*(UT_4(2))$ is generated by the Stiefel–Whitney classes of its irreducible representations.*

Proof. The index 2 subgroup $L \subset UT_4(2)$ generated by α, A, B, C, D is isomorphic to the wreath product $(\mathbb{Z}/2)^2 \wr \mathbb{Z}/2$. Consequently,

$$H^*(L) = \mathbb{F}_2[a+b, c+d, ab, cd](1, ad+bc) \oplus \mathbb{F}_2[ab, cd, h]h.$$

β acts on $H^*(L)$ to interchange $a+b$ and $c+d$. It also acts to interchange ab and cd. It follows that the E_2 term of the spectral sequence for the extension from L to $UT_4(2)$ (which equals $H^*(\mathbb{Z}/2; H^*_\beta(L))$) is given explicitly as

$$\begin{aligned}\mathbb{F}_2[w_1, (a+b)(c+d), ab+cd, w_4](1, ad+bc, w_3, w_3(ad+bc))\\ \oplus \mathbb{F}_2[ab+cd, w_4, h]h\\ \oplus \mathbb{F}_2[w_4, h, k]hk\\ \oplus \mathbb{F}_2[(a+b)(c+d), w_4, k](k, k(ad+bc)).\end{aligned} \tag{3.2}$$

When we compare the Poincaré series for the E_2 term above with the Poincaré series of the subalgebra described before the statement of the theorem we see directly that they are equal. Consequently, the spectral asequence collapses and the result follows. ■

Remark 3.3. The cohomology of $UT_4(2)$ has been previously determined by Tezuka and Yagita [16]. However, the point of view here is quite different and the explicit identification

Table 3.

Name	$\mathscr{B}$	W_α	$W_{\alpha\beta}$	W_β	V
v_α	0	$e\Gamma$	0	0	0
$v_{\alpha\beta}$	0	0	$e\Gamma$	0	0
v_β	0	.0	0	$e\Gamma$	0

of the cohomology generators in our treatment is crucial in our determination of $H^*(\mathrm{Syl}_2(M_{22}))$. For a related discussion of some of the elements in $H^*(UT_4(2))$ see also [9].

We can understand (3.2) best in terms of invariants. Note that V is normal in $UT_4(2)$ but the Weyl group for $\mathscr{B}$ is $2^2 = \langle AB, AC\rangle$, and the Weyl group of W_α is $D_8 = \langle A, \beta\rangle$ so W_α, W_β and $W_{\alpha\beta}$ are normal in $UT_4(2)$ as well, while the normalizer of $\mathscr{B}$, $N_{UT_4(2)}(\mathscr{B}) = D_8 * D_8$.

LEMMA 3.4. *For notational convenience write $H = UT_4(2)$, then we have*

(1) *The ring of invariants $H^*(V)^{W_H(V)}$ is given as*

$$\mathbb{F}_2[\sigma_1, ab + cd, ac + bd, \sigma_4](1, ad + bc, \sigma_3, (ad + bc)\sigma_3).$$

where $H^*(V) = \mathbb{F}_2[a,b,c,d]$ with a dual to A, b dual to B, etc. while σ_i is the ith symmetric monomial in a, b, c, d.

(2) *The ring of invariants $H^*(W_\alpha)^{W_H(W_\alpha)}$ is given as*

$$\mathbb{F}_2[e, (e + l + m)(l + m), lm(l + e)(m + e)]$$

where e is dual to α, l is dual to AB and m is dual to CD.

(3) *The ring of invariants $H^*(\mathscr{B})^{W_H(\mathscr{B})}$ is given as*

$$\mathbb{F}_2[h, k, c(c + h)(c + k)(c + h + k)]$$

where h is dual to α, k is dual to β and c is dual to ABCD.
Thus the image of the restriction map from $H^*(H)$ lies in these invariant subrings, and indeed, except for multiple image classes surjects onto these subrings.

(This is direct.)

4. THE COHOMOLOGY OF $\mathrm{Syl}_2(M_{22})$

There are at least three ways of looking at $\mathrm{Syl}_2(M_{22})$: first as a central extension of $UT_4(2)$, second as a semi-direct product $UT_4(2)$:2, and third as the semi-direct product $\mathrm{Syl}_2(L_3(4))$: 2_2. In this section we determine the ring $H^*(\mathrm{Syl}_2(M_{22}))$ using these different decompositions to construct a sufficient number of non-zero cohomology classes so that we can show there are no possible differentials in the spectral sequence associated to the third decomposition (with E_2 term $H^*(\mathbb{Z}/2: \mathscr{H}^*(UT_4(2))))$. We initially wrote this E_2 term down in [1] and recall it in (4.1). We will construct these classes from the Stiefel–Whitney classes of the irreducible representations of $\mathrm{Syl}_2(M_{22})$ and as classes in the image of transfers. Then we will show they are non-zero by restricting to the abelian subgroups in (2.5). Thus we turn now to the structure of these representations.

As we remarked in (2.1)–(2.3) $UT_4(2)$ occurs both as a subgroup of $\mathrm{Syl}_2(M_{22})$ and as its central quotient. So far we have concentrated on the subgroup. Now we look at the central extension

$$\mathbb{Z}/2 \overset{\triangleleft}{\to} \mathrm{Syl}_2(M_{22}) \overset{\pi}{\to} UT_4(2).$$

The most basic thing is to determine the K-invariant of the extension as that determines the kernel of $\pi^*: H^*(UT_4(2)) \to H^*(\mathrm{Syl}_2(M_{22}))$.

In (2.2) we see that α and β commute with each other, $AB\alpha$ and $AC\beta$ also commute. Moreover each of these elements has order two in $\mathrm{Syl}_2(M_{22})$, as do A, E. However, the commutators $[A, E] = [\alpha, AC\beta] = [\beta, AB\alpha] = ABCD$, the central element. Consider now the 5 detecting groups corresponding to the groups in (2.5) but for the quotient $UT_4(2)$ given in (2.2) and the central extension restricted to them. We find

$$\langle \alpha, \beta, AB, AC \rangle \mapsto D_8 * D_8$$
$$\langle A, AB, AC \rangle \mapsto 2^4$$
$$\langle E, \alpha\beta, AD\alpha\beta \rangle \mapsto 2^4$$
$$\langle AE, AB\alpha\beta, BC \rangle \mapsto Q_8 \times 2$$
$$\langle A, E, BC \rangle \mapsto D_8 \times 2.$$

In particular, the K-invariant for the extension restricts to the first group as $\alpha^*(AC)^* + \beta^*(AB)^*$, trivially to the second and third groups and as

$$(AE)^*(AB\alpha\beta)^* + ((AB\alpha\beta)^* + (AE)^*)^2$$

in the fourth group. Finally, in the fifth group it restricts to A^*E^*. Consequently, the K-invariant has the form

$$(hk, 0, \Gamma + e^2, 0, ad + bc),$$

and this is the restriction of the element $(E^* + A^*)^2 + w_2(r_1) + w_2(E_{AE,+})$.

Write K for this K-invariant. Note that $(A^* + E^*)K \mapsto (d_3, 0, 0, 0, 0)$. On the other hand, $Sq^1(K) \mapsto (d_3, 0, e\Gamma, 0, (a + d)ad + (b + c)bc)$. Thus the element which restricts to $(0, 0, e\Gamma, 0, Sq^1(ad + bc))$ maps to zero in $H^*(\mathrm{Syl}_2(M_{22}))$, and, in particular h times this element, which restricts to $(0, 0, e^2\Gamma, 0, 0)$ also maps to zero in $H^*(\mathrm{Syl}_2(M_{22}))$. But the K-invariant shows that A^*E^* maps to the same element as the element which restricts to $(0, 0, \Gamma, 0, ad + bc)$. Consequently, the image of $A^*(E^*)^3$ is the same as that of the element which restricts to $(0, 0, e^2\Gamma, 0, 0)$, and is thus zero. This shows that there are nilpotent elements in the ring $H^*(\mathrm{Syl}_2(M_{22}))$ so $H^*(\mathrm{Syl}_2(M_{22}))$ cannot be detected by restriction to 2-elementary subgroups.

The projection $(\mathbb{Z}/2) \vartriangleleft \mathrm{Syl}_2(M_{22}) \to UT_4(2)$ lifts back a copy of the representation ring of $UT_4(2)$ as a direct summand of the representation ring of $\mathrm{Syl}_2(M_{22})$. There is one further representation of this group that we need, r_3, given by inducing r_1 on the copy of $UT_4(2)$ in (2.1) to $\mathrm{Syl}_2(M_{22})$. It restricts back to $UT_4(2)$ as the direct sum $r_1 + r_2$, and by Frobenius reciprocity is thus irreducible. Now, by a dimension count, we have found all the irreducible representations of $\mathrm{Syl}_2(M_{22})$. Call the representations of $\mathrm{Syl}_2(M_{22})$ obtained by pulling back the irreducible representations of the quotient $UT_4(2)$ by the same names that they had in the previous section. Then, on restricting back to the subgroup $UT_4(2) = \langle \alpha, \beta, A \rangle \subset \mathrm{Syl}_2(M_{22})$ we obtain Table 4 of Stiefel–Whitney classes where we have again left out redundant classes.

To give the restrictions of the Stiefel–Whitney classes for the representation r_3 to these elementary 2-groups we introduce some notation $\bar{w}_3 = Sq^1(w_2) = w_1w_2 + w_3$,

Table 4.

Rep	class	$\mathscr{B}$	W_α	$W_{\alpha\beta}$	W_β	V
r_1	w_1	$h + k$	e	0	e	0
r_1	w_4	0	0	Γ^2	0	$(a + b)(a + c)(b + d)(c + d)$
E_{E+}	w_2	hk	0	e^2	0	0
E_{A-}	w_1	$h + k$	e	0	e	σ_1
E_{A-}	w_2	0	Γ	0	Γ	$(a + d)(b + c)$

$\gamma_4 = w_2(w_2 + w_1^2)$, $\gamma_5 = Sq^2\bar{w}_3 = w_4 w_1 + \bar{w}_3 w_2$, $\gamma_8 = w_4(w_4 + w_1\bar{w}_3)$ and $\mu = v^4 + v^2 d_2 + v d_3$ *in* $H^*(\mathscr{B})$. Then the only Stiefel–Whitney classes to restrict non-trivially are w_4, w_6, w_7 and w_8, and we have Table 5.

There is also one other element that we should note which lies in the image of restriction from $H^*(\mathrm{Syl}_2(M_{22}))$, $K_5 = (h+k)w_4(r_1)$. Indeed, it restricts to

$$((h+k)(v^4 + v^2 d_2 + v d_3), e\theta, 0, e\theta, 0)$$

which shows that K_5 has the form $(1 + E^*)(hw_4(r_1))$ and hence is in the image of the restriction preceeded by transfer.

We now recall the partial results of [1]. We considered the spectral sequence associated to the index 2 subgroup $\mathrm{Syl}_2(L_3(4)) \subset \mathrm{Syl}_2(M_{22})$, where $\mathrm{Syl}_2(L_3(4))$ can be identified with the explicit subgroup $\langle A, B, C, D, E, \alpha\beta \rangle$. The E_2 term is given explicitly as follows, where we have modified the notation of [1] to write the result more in keeping with the structure of $H^*(UT_4(2))$.

$$\begin{aligned} &\mathbb{F}_2[\gamma_4,\gamma_8]\{\mathbb{F}_2[\sigma_1,(a+d)(b+c)](\gamma_5(1)) \oplus \mathbb{F}_2[E^*,\{hk\}](\gamma_5(2))\}(1,N_6) \\ &\oplus \mathbb{F}_2[\gamma_4,\gamma_8]\{\mathbb{F}_2[\sigma_1,(a+d)(b+c)] \oplus \mathbb{F}_2[E^*,\{hk\}]\}(\bar{w}_3,N_7) \\ &\oplus \mathbb{F}_2[\gamma_4,\gamma_8]\{\mathbb{F}_2[\sigma_1,(a+d)(b+c)](N_4,N_8) \oplus \mathbb{F}_2[E^*,\{hk\}](N_4',N_8')\} \\ &\oplus \mathbb{F}_2[\gamma_4,\gamma_8](\sigma_1 E^*, \sigma_1 E^{*2}, T_6, T_7) \\ &\oplus \{\mathbb{F}_2[\gamma_4,\gamma_8,s]/(s\gamma_4) = 0)\}(V_5, N_6') \\ &\oplus \{\mathbb{F}_2[\gamma_8,(a+d)(b+c),\{hk\},s]/((a+d)(b+c)\{hk\} = 0)\}(s, N_6 s). \end{aligned} \tag{4.1}$$

The way to read and understand what (4.1) means is

(1) In [1] it is shown that the ring of invariants

$$\mathbb{F}_2[a,b,c,d]^{D_8} = \mathbb{F}_2[\sigma_1,(a+d)(b+c),\gamma_4,\gamma_8](1,\bar{w}_3,N_4,\gamma_5,N_6,N_7,N_8,\gamma_5 N_6)$$

and the first three lines of (4.1) show that the images under restriction to $H^*(V_4)$ and $H^*(W_4)$ are these three invariant subrings.

(2) The subring of multiple image classes in $H^*(V_4) \oplus H^*(W_4)$ is

$$\mathbb{F}_2[\gamma_4,\gamma_8](1,\bar{w}_3,N_7)$$

so, in particular, there is an element which restricts to $(\bar{w}_3^2, 0)$ in the direct sum but none which restricts to $(\bar{w}_3, 0)$. (It appears from the above that N_6 is also multiple, and it is, but the class N_6' restricts to $(0, N_6)$ in the direct sum, though N_6' also has non-trivial restriction to $H^*(V_3)$.)

(3) The fourth line gives the terms in $\mathrm{Rad}(H^*(\mathrm{Syl}_2(M_{22})))$.

(4) The last two lines are the parts which cup non-trivially with s. Here s is dual to α or β, consequently restricts to $h+k$ in $UT_4(2)$, and has filtration (1,0) while every other generator θ has filtration $(0, \dim(\theta))$.

Table 5.

Class	$\mathscr{B}$	W_α	$W_{\alpha\beta}$	W_β	V
w_4	d_2^2	$e^4 + \Gamma^2$	$e^4 + \Gamma^2$	$e^4 + \Gamma^2$	$\gamma_4 + \sigma_1\bar{w}_3 + \sigma_1^4$
w_6	d_3^2	$e^2\Gamma^2$	$e^2\Gamma^2$	$e^2\Gamma^2$	$\sigma_1\gamma_5 + \bar{w}_3^2 + \sigma_1^2\gamma_4 + \sigma_1^3\bar{w}_3$
w_7	0	0	0	0	$w_1\bar{w}_3^2 + w_1^2\gamma_5$
w_8	μ^2	θ^2	θ^2	θ^2	γ_8

Table 6. Restriction images

Class	Restriction
$s^i w_2(E_{A-})$	$(0, e^i\Gamma, 0, e^i\Gamma, 0)$
$s^i w_2(E_{E+})$	$((h+k)^i hk, 0, 0, 0, 0)$
$s^i w_2(E_{A-})^2$	$(0, e^i\Gamma^2, 0, e^i\Gamma^2, 0)$
$s^i w_2(E_{E+})^2$	$((h+k)^i(hk)^2, 0, 0, 0, 0)$
$s^i K_5$	$(0, e^{i+1}\theta, 0, e^{i+1}\theta, 0)$

(5) The class $\gamma_5(1)$ was originally described in [1] as $\mathscr{A}v + \mathscr{B}w$ which, when multiplied by $\mathscr{E} + \mathscr{F}$, gives the class $\mathscr{A}\mathscr{F}v + \mathscr{B}\mathscr{E}w$. In turn, this class corresponds to the class T_6 in the description above, while T_7 corresponds to $(\mathscr{E} + \mathscr{F})^2(\mathscr{A}v + \mathscr{B}w)$. Thus we see that T_6 and T_7 are represented, respectively, as $E^*\gamma_5(1)$ and $(E^*)^2\gamma_5(1)$.

It follows that the only possible differentials occur on elements not annihilated by s. The generators of this subalgebra are $\gamma_8, (a+d)(b+c), \{hk\}, V_5, N_6$ and N'_6. γ_8 can be taken to be $w_8(r_3)$ and is thus non-zero. Moreover, the class $(a+d)(b+c)$ is represented as $w_2(E_{A-})$ while the class $\{hk\}$ is given as $w_2(E_{A+})$, and s restricts to $w_1(r_1) = (h+k, e, 0, e, 0)$. The class E^* restricts to 0 in $H^*(UT_4(2))$ and, indeed, the kernel of this restriction map is exactly the ideal (E^*). Finally, we will represent the class γ_4 above as $w_4(r_3)$.

We have already seen that the class $K_5 = ((h+k)(v^4 + v^2d_2 + vd_3), e\theta, 0, e\theta, 0)$ is in the image of the restriction map from $H^*(\mathrm{Syl}_2(M_{22}))$. On the other hand, the elements $\gamma_5(1)$ and $\gamma_5(2)$ are both of the form $(1+\alpha^*)\lambda$ in $H^*(UT_3(4))$, hence cup trivially with s. It follows that the class V_5 must be an infinite cycle in the spectral sequence and has a representative which restricts to K_5. As a consequence we have Table 6 of restriction images and these classes are all linearly independent. It follows that none of these classes, nor any linear combinations of them are hit by differentials in the spectral sequence. Thus N_6 and N'_6 must both be infinite cycles and, as was asserted in [1], the spectral sequence collapses.

5. THE INVARIANT SUBALGEBRAS FOR $H^*(M_{22})$ AND RAD$(H^*(M_{22}))$

We begin by determining the invariant subrings which occur in $H^*(V_4)$, $H^*(W_4)$ and $H^*(V_3)$. Then the main difficulty in specifying $H^*(M_{22})$ will be to determine the radical.

As we discussed in the introduction $\mathbb{F}_2[x_1, x_2, x_3]^{L_3(2)} = \mathbb{F}_2[d_4, d_6, d_7]$, the Dickson algebra, where $Sq^2(d_4) = d_6$, $Sq^1(d_7) = d_7$. From [3] we have

$$\mathbb{F}_2[a,b,c,d]^{\mathscr{A}_6} = \mathbb{F}_2[\bar{w}_3, \gamma_5, d_8, d_{12}](1, \gamma_9, b_{15}, \gamma_9 b_{15})$$

where $\bar{w}_3 = Sx_i^2 x_j$, where the x_i, x_j run over a, b, c, d, while $\gamma_5 = Sq^2(\bar{w}_3)$, $\gamma_9 = Sq^4(\gamma_5)$ and d_8, d_{12} are the Dickson elements. In [1] we show that

$$\mathbb{F}_2[a,b,c,d]^{\langle E, \alpha\beta\rangle} = \mathbb{F}_2[r,s,v,w](1, L, M, LM)$$

where

$$r = a + d$$

$$s = b + c$$

$$v = a^2d^2 + ad(rs + s^2)$$

$$w = b^2c^2 + bc(rs + r^2)$$

$$L = adr + a^2c + ac^2 + bd^2 + b^2d$$

$$M = a^2c + ac^2 + bcs + b^2d + bd^2.$$

L and M are both invariant under the action of $L_2(4) = \mathscr{A}_5$ and α acts to interchange r and s, L and M, v and w in pairs. In particular, $L + M$ and LM are $\mathscr{S}_5$ invariants. Finally we have the following lemma.

LEMMA 5.1. *Let $\mathscr{S}_5$ act on V_4 as the extension $\mathscr{S}_5 = L_2(4){:}2_2$, then*

$$\mathbb{F}_2[a,b,c,d]^{\mathscr{S}_5} = \mathbb{F}_2[\bar{w}_3, \gamma_5, d_8, d_{12}](1, n_6, n_8, \gamma_9, n_{10}, n_6^2, x_{12}, x_{14}, x_{15}, x_{16}, x_{18}, x_{24}).$$

Here $n_6 = LM$, $n_8 = Sq^2(n_6)$, $n_{10} = Sq^4(n_6)$, $x_{12} = Sq^4(n_8)$, $x_{14} = n_6 Sq^2(n_6)$ and $\bar{w}_3$ is represented by $L + M$. In particular, this invariant subalgebra is Cohen–Macaulay over the same polynomial subalgebra as occurs for $\mathscr{A}_6$.

Proof. Consider the inclusion $\langle E, \alpha\beta\rangle \subset \mathscr{S}_5$. This gives an inclusion in the reverse order of the invariant subrings. Set $\bar{w}_3 = L + M$. (If we make the change of variables $a \mapsto a + b + c$, $b \mapsto b + c$, $c \mapsto a + b$, $d \mapsto d$, then $L + M \mapsto Sx_i^2 x_j$, while d_8 and d_{12} are fixed, so we can regard $L + M$ as equal to $\bar{w}_3$.) Thus, we see that $\mathbb{F}_2[a,b,c,d]^{\mathscr{S}_5}$ contains the polynomial subalgebra $\mathscr{R} = \mathbb{F}_2[\bar{w}_3, \gamma_5, d_8, d_{12}]$. In particular, we have the explicit relations

$$L^2 = (r+s)^2 v + r^2 w + (r^2 s + rs^2)L$$

$$M^2 = s^2 v + (r+s)^2 w + (r^2 s + rs^2)M$$

$$\gamma_5 = rv + sw + (r^2 + rs + s^2)(L + M)$$

$$d_8 = v^2 + vw + w^2 + (r^2 + rs + s^2)^4$$

$$d_{12} = (r^2 + rs + s^2)d_8 + (r^2 + rs + s^2)^6 + (r^2 s + rs^2)^4 + v^2 w + vw^2. \tag{5.2}$$

It suffices to show that $\mathbb{F}_2[a,b,c,d]^{\langle E, \alpha\beta\rangle}$ is Cohen–Macaulay over $\mathscr{R}$. To do this consider the surjective map from a polynomial algebra on six formal variables (given the names of their images) to $\mathbb{F}_2[a,b,c,d]^{\langle E, \alpha\beta\rangle}$

$$\mathbb{F}_2[r, s, v, w, L, L + M] \to \mathbb{F}_2[r, s, v, w](1, L, M, LM)$$

with kernel I, the ideal generated by the relations for L^2 and $(L + M)^2$ generated by the first two relations above. It follows that $\mathbb{F}_2[r,s,v,w](1, L, M, LM)/(\mathscr{R} - \{1\})$ is exactly $\mathbb{F}_2[r,s,v,w,L]$ modulo the ideal, J, generated by the relations

$$rv + sw,\ v^2 + vw + w^2 + r^8 + r^4 s^4 + s^8$$

$$v^2 w + vw^2 + r^{12} + r^{10}s^2 + r^8 s^4 + r^6 s^6 + r^4 s^8 + r^2 s^{10} + s^{12},\ r^2 v + s^2 w$$

$$(r^2 + s^2)v + r^2 w + (r^2 s + rs^2)L + L^2.$$

To find this quotient explicitly we used Macaulay to construct a resolution of J over $\mathbb{F}_2[r,s,v,w,L]$. Table 7 shows the generators and degrees in the resolution.

Table 7. Generators and degrees in the resolution

Dim	Number of gens.	Degrees
1	5	5 6 6 8 12
2	10	11 11 12 13 14 14 17 18 18 20
3	10	17 19 19 20 23 23 24 25 26 26
4	5	25 29 31 31 32
5	1	37

It follows that the Poincaré series of the quotient $\mathbb{F}_2[r,s,v,w,L]/J$ is given as the alternating sum of the terms

$$\frac{x^{37}}{(x-1)^2(x^3-1)(x^4-1)^2} - \frac{x^{25}(1+x^4+2x^6+x^7)}{(x-1)^2(x^3-1)(x^4-1)^2}$$
$$+\frac{x^{17}(1+2x^2+x^3+2x^6+x^7+x^8+2x^9)}{(x-1)^2(x^3-1)(x^4-1)^2}$$
$$-\frac{x^{11}(2+x+x^2+2x^3+x^6+2x^7+x^9)}{(x-1)^2(x^3-1)(x^4-1)^2}$$
$$+\frac{x^5(1+2x+x^3+x^7)}{(x-1)^2(x^3-1)(x^4-1)^2} - \frac{1}{(x-1)^2(x^3-1)(x^4-1)^2}$$

This factors and simplifies to give the polynomial

$$\begin{aligned}p(x) = x^{24} &+ 2x^{23} + 3x^{22} + 5x^{21} + 9x^{20} + 12x^{19} + 14x^{18} + 18x^{17} + 23x^{16}\\ &+ 25x^{15} + 25x^{14} + 28x^{13} + 30x^{12} + 28x^{11} + 25x^{10} + 25x^9\\ &+ 23x^8 + 18x^7 + 14x^6 + 12x^5 + 9x^4 + 5x^3 + 3x^2 + 2x + 1.\end{aligned}$$

On the other hand, this quotient can be regarded as representing a generating set for $\mathbb{F}_2[r,s,v,w](1,L,M,LM)$ over $\mathcal{G}$ and we see that there are exactly 360 generators required. But a short calculation shows that $p(x)$ is also equal to the quotient

$$\frac{(1+x^3)^2(x^3-1)(x^5-1)(x^8-1)(x^{12}-1)}{(x-1)^2(x^4-1)^2}$$

which represents the minimal possible number of generating elements, and these two numbers are equal if and only if $\mathbb{F}_2[r,s,v,w](1,L,M,LM)$ is free and finitely generated, i.e. Cohen–Macaulay, over $\mathcal{G}$. But then $\mathbb{F}_2[a,b,c,d]^{\mathcal{S}_5}$ is also Cohen–Macaulay over $\mathcal{G}$.

It remains to see that the list of generators is correct. But in [1] we determine $\mathbb{F}_4[a,b,c,d]^{\mathcal{S}_5}$. Its Poincaré series is given after some simplification as

$$\frac{1+x^6+x^8+x^9+x^{10}+2x^{12}+x^{14}+x^{15}+x^{16}+x^{18}+x^{24}}{(1-x^3)(1-x^5)(1-x^8)(1-x^{12})}.$$

Moreover, we know that $n_6 = LM$ is $\mathcal{S}_5$ invariant and from this and the first two relations in the Grobner basis, it follows that $(LM)^2$ will be part of a generating set for $\mathbb{F}_2[a,b,c,d]^{\mathcal{S}_5}$ over $\mathcal{G}$. ∎

COROLLARY 5.3. *The images of the restriction maps* res: $H^*(M_{22}) \to H^*(V)$, *where* V *runs over the three extremal elementary* 2-*subgroups of* M_{22} *are given as follows*:

(1) *For* $V = F_4$ *the image of* res* *is*

$$\mathbb{F}_2[a,b,c,d]^{\mathcal{A}_6} = \mathbb{F}_2[\bar{w}_3,\gamma_5,d_8,d_{12}](1,\gamma_9,\gamma_{15},\gamma_9\gamma_{15}).$$

(2) *For* $V = W_4$ *the image of* res* *is* $H^*(W_3)^{\mathcal{S}_5}$ *determined above*,

(3) *For* $V = V_3$ *the image of* res* *is* $\mathbb{F}_2[d_6,d_7,d_4^2](1,d_4d_6,d_4d_7)$.

Proof. As we discussed in the introduction the Cardenas–Kuhn theorem shows that if $V \subset \mathrm{Syl}_2(M_{22})$ is weakly closed in G, then the image of res* is

$$\mathrm{im}(\mathrm{res}^*: H^*(\mathrm{Syl}_2(M_{22})) \to H^*(V)) \bigcap H^*(V)^{W_G(V)}.$$

But (4.1) shows that the image of $H^*(\mathrm{Syl}_2(M_{22}))$ in V_4 is $H^*(V_4)^{D_8}$ and similarly for $H^*(W_4)$ and the first two statements follow from the Cardenas–Kuhn theorem.

To prove (3) we note that $(v^4 + v^2d_2 + vd_3)d_2 + d_3^2 = d_6$, but this is the restriction of $N'_6 + K_5(h + k) + d_3^2$. Also, $d_7 = (v^4 + v^2d_2 + vd_3)d_3$, and we have seen that this element is in the image of restriction from $H^*(\mathrm{Syl}_2(M_{22}))$. The other classes are obtained similarly. ■

From (4.1) we see that the kernel of the sum of the three restriction maps above is $H^*(M_{22}) \cap \mathbb{F}_2[\gamma_4, \gamma_8](\sigma_1 E^*, \sigma_1(E^*)^2, T_6, T_7)$ and is the radical in the ring $H^*(M_{22})$. We now prove the following result.

PROPOSITION 5.4. *$H^*(M_{22}) \cap \mathbb{F}_2[\gamma_4, \gamma_8](\sigma_1 E^*, \sigma_1(E^*)^2, T_6, T_7)$ is equal to*

$$H^*(G_1 \cap G_2) \cap \mathbb{F}_2[\gamma_4, \gamma_8](\sigma_1 E^*, \sigma_1(E^*)^2, T_6, T_7)$$

and this in turn has the form $\mathbb{F}_2[d_8, d_{12}](\sigma_1 E^*, T_7, a_{11}, a_{14})$.

Proof. From the discussion of double coset decompositions at the end of Section 3 we see that the only constraint on the radical, since $H^*(UT_4(2))$ is detected by 2-elementaries, is the condition $\mathrm{res}^*(\theta) \in H^*(\mathrm{Syl}_2(L_3(4)))^\tau$. But this is clearly the same as saying the elements lie in $\mathrm{Rad}(H^*(G_1 \cap G_2))$.

We now determine $\mathrm{Rad}(H^*(G_1 \cap G_2)$. From (4.8) of [1] we see that

$$\mathrm{Rad}(H^*(\mathrm{Syl}_2(L_3(4))) \otimes \mathbb{F}_4 \cong \mathbb{F}_4[v_4, w_4](\mathcal{AF}, \mathcal{BE}, \mathcal{AF}^2, \mathcal{FA}^2)$$

and β acts to interchange the elements in the pairs (v, w), $(\mathcal{AF}, \mathcal{BE})$, $(\mathcal{AF}^2, \mathcal{FA}^2)$. Thus

$$\mathrm{res}^* : \mathrm{Rad}(H^*(\mathrm{Syl}_2(M_{22}))) \to \mathrm{Rad}(H^*(\mathrm{Syl}_2(L_3(4)))$$

is an injection with image $\mathrm{Rad}(H^*(\mathrm{Syl}_2(L_3(4)))^\beta$, and it follows that

$$\mathrm{Rad}(H^*(M_{22})) \cong \mathrm{Rad}(H^*(\mathrm{Syl}_2(L_3(4)))^{\mathcal{S}_3}$$

where $\mathcal{S}_3 = \langle \beta, T \rangle$. Now, the calculations in [1] at the beginning of Section 3 determine this invariant ideal and (5.4) follows. ■

6. THE MULTIPLE IMAGE ANALYSIS

Although it is not strictly necessary, we begin this section by listing the images of restriction to the four 2^2 subgroups of M_{22}. This will give an additional proof that n_6 is multiple image.

LEMMA 6.1. *Consider the restriction maps*

$$res_W^* : H^*(W_4) \to H^*(V_{2,1})$$
$$res_V^* : H^*(V_3) \to H^*(V_{2,1})$$
$$res_0^* : H^*(V_4) \to H^*(V_{2,0})$$
$$res_p^* : H^*(W_4) \to H^*(V_{2,0}).$$

We have that each restriction map on $\bar{w}_3$ is zero, each restriction map on d_8 is $(x^2 + xy + y^2)^4$, each restriction map on d_{12} is $(xy(x + y))^4$, $res_V^*(d_4) = (x^2 + xy + y^2)^2$ and $res_V^*(d_6) = res_W^*(n_6) = (xy(x + y))^2$.

Proof. A representative for $V_{2,1}$ is $\langle ABCD, \alpha\beta\rangle$. The embedding $V_{2,1} \subset W_4 \subset UT_4(2)$ of (2.3) shows that using the outer automorphism which switches V_4, W_4 in $\mathrm{Syl}_2(M_{22})$ that we can identify $V_{2,1}$ with $\langle ABCD, AB\rangle \subset V_4$ and use the restriction map from $H^*(V_4)^{\mathscr{S}_5}$ to $\langle ABCD, AB\rangle$ to calculate the desired restriction, using (5.1) to write down explicit generators for the invariant subring. On generators we find $x_1 \mapsto x + y$, $x_2 \mapsto x + y$, $x_3 \mapsto x$, $x_4 \mapsto x$. Consequently, both L and M mapto $x^2y + xy^2 = xy(x + y)$. The restriction map for $V_{2,0}$ is determined on generators by $x_1 \mapsto x + y$, $x_2 \mapsto x$, $x_3 \mapsto x$, $x_4 \mapsto x + y$, and from this L and M map to zero. The other assertions are similar. ∎

We now complete our determination of $H^*(M_{22})$ by showing that $(b_{15}, 0, 0)$ is in the image of restriction from $H^*(M_{22})$.

LEMMA 6.2. *There is a class h_{18} in $H^*(M_{22})$ which restricts to $(\bar{w}_3 b_{15}, 0, 0)$ in $H^*(V_4) \oplus H^*(W_4) \oplus H^*(V_3)$.*

Proof. From [6] we have that the centralizer of the involution $ABCD$ in M_{22} is the other subgroup $V_4 : \mathscr{S}_4 \subset V_4 : \mathscr{A}_6$. From Theorem (3.2) of [1] we have that the invariants under this action of $\mathscr{S}_4$ give the ring $\mathbb{F}_2[\sigma_1, \bar{w}_3, \gamma_4, \gamma_8](1, \gamma_5, b_7, \gamma_5 b_7)$ where $\gamma_4 = \sigma_2(\sigma_2 + \sigma_1^2)$, $\gamma_8 = \sigma_4(\sigma_4 + \bar{w}_3\sigma_1)$ and $b_7 = \sigma_1 b_6 + \sigma_4 \bar{w}_3$. Here the notation is that of Table 2. Also, the polynomial submodule $\mathbb{F}_2[\gamma_4, \gamma_8](1, \bar{w}_3, b_7)$ is multiple image, but the rest is not. Now $b_{15} = S_8 b_7 + S_{10}\gamma_5 + S_{12}\bar{w}_3 + S_{14}\sigma_1$, and multiplying by $\bar{w}_3$ gives the existence of a class in $H^*(\mathrm{Syl}_2(M_{22}))$ which restricts to the desired class. But such a class is manifestly stable under all the double coset conditions so it comes from $H^*(M_{22})$. ∎

At this stage the only problem is whether b_{15} is a multiple image class or not. To verify that it must be we check the structure of the Bockstein spectral sequence. First we apply the derivation Sq^1 to $H^*(M_{22})$. The resulting homology groups form a ring and the mod(4) Bockstein is a derivation on this ring. The resulting homology groups admit β_8 as a derivation, and so on. In the limit we have only a single copy of $\mathbb{Z}/2$ in degree zero.

There are three keys to this calculation. The first is the observation that

$$H_*(\mathbb{F}_2[\bar{w}_3, \gamma_5, d_8, d_{12}](1, \gamma_9) \colon Sq^1) = \mathbb{F}_2[d_8, d_{12}](1, \bar{w}_3)$$

since, as we have pointed out $Sq^1(\gamma_5) = \bar{w}_3^2$, $Sq^1(\gamma_9) = \gamma_5^2$. The second is the observation that we can write

$$\mathbb{F}_2[d_4, d_6, d_7]d_7 = \mathbb{F}_2[d_4, d_6^2](d_7, d_6 d_7, d_7^2, d_6 d_7^2, \ldots, d_7^i, d_6 d_7^i, \ldots)$$

and $Sq^1(d_6 d_7^j) = d_7^{j+1}$. Hence, the resulting Sq^1 homology of this piece is simply

$$\mathbb{F}_2[d_4^2, d_6^2](d_7, d_4 d_7).$$

On the other hand, $Sq^1(n_6) = d_7$, $Sq^1(n_{10}) = d_4 d_7 + \bar{w}_3 n_8 + \gamma_5 n_6$, and from this it follows that some Bockstein of $(\bar{w}_3 x_8 + \gamma_5 n_6) = n_{12}$, so $n_{12} d_8$ is an integral class since n_{12} is. Also, we recall, in particular the result $Sq^1(\bar{w}_3 x_{12} + \gamma_5 n_{10}) = \gamma_5^2 n_6$ from the introduction. Using these partial calculations and the other Sq^1's for $\mathbb{F}_2[a, b, c, d]^{\mathscr{S}_5}$ as listed in the introduction we reduce ourselves to x_{15}, x_{16}, x_{18} and x_{24}.

Through dimension 20 we are uncertain of whether $(b_{15}, x_{15}, 0)$ or $(b_{15}, 0, 0)$ occurs in the image of restriction. We are also uncertain of Sq^1 on x_{15}, x_{16}, x_{18}. But, modulo that uncertainty we obtain that the following classes generate the Sq^1 homology in dimensions 13–20 at most. There may be further Sq^1's among these generators which cut down the Sq^1 homology by removing further pairs of elements in successive dimensions, but there are no other possible homology generators.

dim 13	14	15	16	17	18	19	20
	a_{14}	$b_{15}?$			$\bar{w}_3 b_{15}$	$a_{11}d_8$	$d_8 d_{12}$
		$a_7 d_8$	d_8^2		$\bar{w}_3 x_{15}$	$a_7 d_{12}$	
		x_{15}	x_{16}	$\bar{w}_3 x_{14}$	x_{18}	$\bar{w}_3 x_{16}$	
					$a_2 d_8^2$	$\bar{w}_3 d_8^2$	
						$(\bar{w}_3 n_8 + \gamma_5 n_6)d_8$	$n_{12}d_8$

Here we know $d_8 d_{12}$ is integral and consequently must be in the image of some Bockstein. Also, $\beta_4(a_2 d_8^2) = \bar{w}_3 d_8^2$ and some Bockstein of $a_7 d_8 = d_8^2$. Thus the only way that the Bockstein spectral sequence can work out is if both b_{15} and x_{15} are present which can only happen if $(b_{15}, 0, 0)$ is in the image of restriction.

This completes our discussion of $H^*(M_{22})$.

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