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The Subgroup Structure and mod 2 Cohomology of O'Nan's Sporadic Simple Group

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This is the first of two papers in which we determine the mod(2) cohomology of the sporadic group O'N of order $2^9\,3^4\,5\,7^3\,11\,19\,31 = 460,815,505,920$. O'N has 2-rank three, and, from the Gorenstein–Harada Theorem [GH], together with the results of [AMM, FM, M1, We], which determine the mod(2) cohomology rings of $J_1, M_{12}, G_2(q), {}^3D_4(q)$, with q odd, there now remain only the Chevalley groups $U_3(8)$ and $Sz(8)$ among the 2-rank three simple groups for which the mod(2) cohomology ring has not been determined. In what follows we often use the *Atlas* notation [Co].

There are two maximal 2-local subgroups in O'N, $\mathrm{Alp}_2^2 = 4^3 \cdot L_3(2)$, which is the non-split extension of the form

$$1 \to 4^3 \xrightarrow{\triangleleft} \mathrm{Alp}_2^2 \to L_3(2) \to 1$$

studied by Alperin [Alp], and the centralizer of an involution, the non-split extension $4 \cdot L_3(4){:}2_1$. They can be chosen to intersect in a subgroup $H_\beta = K_\beta : \mathcal{S}_3$ where $K_\beta = 4^3 : 2^2$. In particular, their intersection contains a Sylow 2-subgroup. (The β denotes a particular element of order three in the $\mathcal{S}_3 \subset \mathrm{Alp}_2^2$ which normalizes K_β.)

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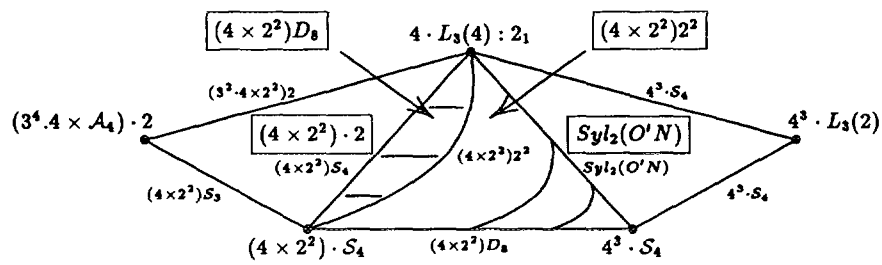
In order to obtain cohomology results we have to get very precise information about the conjugacy classes of elementary 2-subgroups and their centralizers in O'N. Here is our first main result, sharpening results of O'Nan [ON] and Yoshiara [Yo]:

THEOREM A. (a) *There are two conjugacy classes of maximal elementary 2 groups in* O'N, *both isomorphic to* 2^3 *with centralizers* $\mathrm{I} \simeq 4^3$ *and* $\mathrm{II} \simeq 4 \times 2^2$ *respectively.*

(b) *The Weyl groups of the two conjugacy classes*[1] *are the general linear group* $L_3(2)$ *for the first class and* $\mathcal{S}_4$ *for the second. In the second case there is a particular choice for the subgroups* 4 *and* 2^2 *so that the* 4 *is fixed by* $\mathcal{S}_4$ *and, at the same time, there is a subgroup* $\mathcal{S}_3 \subset \mathcal{S}_4$ *which acts as* $L_2(2)$ *on the subgroup* 2^2.

Remark. (b) is contained in [ON, Yo], but (a) requires the very detailed information about the structure of the Alperin group Alp_2^2 contained in [Gr], and much of this paper is devoted to the proof of (a). Indeed, we completely determine the poset space $\mathcal{A}_2(\mathrm{O'N})$ of 2-elementary subgroups and inclusions and show in Section 2 that the quotient of its geometric realization under the action of O'N induced by conjugation has the form of a two-dimensional complex:



The groups in boxes are the isotropy groups of faces and the labels in smaller type are the isotropy groups of edges.

In Section 4, using geometric interpretations of results of Webb [We], we derive a very useful consequence of the result above:

THEOREM B. *Let* Γ *be the amalgamated product* $\mathrm{Alp}_2^2 *_{H_B} (4 \cdot L_3(4)\colon 2_1)$ *and* $\pi\colon \Gamma \to \mathrm{O'N}$ *be the surjective homomorphism induced by sending* Alp_2^2 *and* $4 \cdot L_3(4)\colon 2_1$ *isomorphically to the subgroups of* O'N *which intersect in*

[1] Following the custom among topologists, if $H \subset G$ is an abelian subgroup of G then $N_G(H)/C_G(H)$ is called the Weyl group of H in G. Group theorists often call this the automizer of H in G.

H_β; *then the induced cohomology map*

$$\pi^*: H^*(\mathrm{O'N}; \mathbb{F}_2) \to H^*(\Gamma; \mathbb{F}_2)$$

is an isomorphism of rings.

Remark. This is very similar to the situations for M_{12}, $G_2(q)$, and ${}^3D_4(q)$ where similar surjective maps of amalgamated products induced isomorphisms in mod(2) cohomology. Similarly, a sporadic geometry enabled us to write the classifying space for M_{22} as a union of classifying spaces of three subgroups amalgamated over the classifying spaces of their respective intersections in [AM2]. However, in that case the generalized amalgamation turns out to be *isomorphic* to M_{22}.

Next we turn to the structure of $H^*(\mathrm{O'N}; \mathbb{F}_2)$. In Section 1.6 we show that $\mathrm{Syl}_2(\mathrm{O'N})$ has seven conjugacy classes of 2^3's, and in Section 3 we begin the determination of the cohomology ring by describing the exact structure of the fusions of their centralizers in O'N. Thus, we construct *explicit* fusions which fuse them into two conjugacy classes.

In Part II of [M2], it is shown that restriction to the seven centralizers of these 2^3's is injective in mod(2) cohomology for $H^*(\mathrm{Syl}_2(\mathrm{O'N}); \mathbb{F}_2)$. Thus, using the Cartan–Eilenberg double coset formula, the elements in $H^*(\mathrm{Syl}_2(\mathrm{O'N}); \mathbb{F}_2)$ which are in the image from $H^*(\mathrm{O'N}; \mathbb{F}_2)$ are precisely those elements whose images are fixed under the Weyl groups of the centralizers, as well as under the fusion identifications. In particular, we have

COROLLARY C. *The two restriction maps in cohomology*

$$\mathrm{res}^{\mathrm{I}} \oplus \mathrm{res}^{\mathrm{II}}: H^*(\mathrm{O'N}; \mathbb{F}_2) \to H^*(4^3; \mathbb{F}_2)^{L_3(2)} \oplus H^*(4 \times 2^2; \mathbb{F}_2)^{\mathcal{S}_4}$$

together are injective in cohomology where I *and* II *are the centralizers of the two conjugacy classes of* 2^3*'s in* O'N *and are discussed in Theorem A.*

The invariant subrings in Corollary C are given as follows. Recall that

$$H^*(4^3; \mathbb{F}_2) = \mathbb{F}_2[b_2(1), b_2(2), b_2(3)] \otimes E(e_1(1), e_1(2), e_1(3)),$$

the tensor product of a polynomial algebra on two-dimensional generators and an exterior algebra on one-dimensional generators[2] and that the $e_1(i)$'s and the $b_2(i)$'s are related by the $\mathbb{Z}/4$-Bockstein, $\beta_4(e_1(i)) = b_2(i)$, $i = 1, 2, 3$, which extends as a derivation to products. The action of $L_3(2)$ is the usual one induced from the linear actions on the one- and two-dimen-

[2] A subscript on a generator in this paper will always denote its dimension.

sional generators, so from the results of [Mui] we have that the ring of invariants has the form

$$\mathbb{F}_2[d_8, d_{12}, d_{14}]\left(1, E_3, \tilde{M}_4, \tilde{M}_6, \tilde{X}_7, \tilde{M}_{10}, \tilde{X}_{11}, \tilde{X}_{13}\right)$$

where the d_i are the Dickson elements in $\mathbb{F}_2[b_2(1), b_2(2), b_2(3)]$,

$$\begin{aligned} d_8 = {} & b_2(1)^4 + b_2(2)^4 + b_2(3)^4 \\ & + b_2(1)^2 b_2(2)^2 + b_2(1)^2 b_2(3)^2 + b_2(2)^2 b_2(3)^2 \\ & + b_2(1) b_2(2) b_2(3)(b_2(1) + b_2(2) + b_2(3)), \end{aligned}$$

while

$$d_{12} = Sq^4(d_8), \qquad d_{14} = Sq^2(d_{12}), \qquad E_3 = e_1(1)e_1(2)e_1(3),$$

$$\tilde{M}_4 = \beta_4(E_3) = b_2(1)e_1(2)e_1(3) + e_1(1)b_2(2)e_1(3) + e_1(1)e_1(2)b_2(3),$$

$$\begin{aligned} \tilde{M}_6 = Sq^2\left(\tilde{M}_4\right) = {} & b_2(1)^2 e_1(2)e_1(3) + e_1(1)b_2(2)^2 e_1(3) \\ & + e_1(1)e_1(2)b_2(3)^2, \end{aligned}$$

$$\begin{aligned} \tilde{X}_7 = \beta_4\left(\tilde{M}_6\right) = {} & \left(b_2(1)^2 b_2(2) + b_2(1)b_2(2)^2\right)e_1(3) \\ & + \left(b_2(1)^2 b_2(3) + b_2(1)b_2(3)^2\right)e_1(2) \\ & + \left(b_2(2)^2 b_2(3) \right. \\ & \quad \left. + b_2(2)b_2(3)^2\right)e_1(1), \end{aligned}$$

$$\tilde{M}_{10} = Sq^4\left(\tilde{M}_6\right), \qquad \tilde{X}_{11} = Sq^4\left(\tilde{X}_7\right), \qquad \tilde{X}_{13} = Sq^2\left(\tilde{X}_{11}\right).$$

We now describe the structure of the invariants for II. Recall that $H^*(G_1 \times G_1; \mathbb{F}_p) = H^*(G_1; \mathbb{F}_p) \otimes_{\mathbb{F}_p} H^*(G_2; \mathbb{F}_p)$, $H^1(G; \mathbb{F}_p) = \mathrm{Hom}(G, \mathbb{F}_p)$, and

$$H^*(2^i; \mathbb{F}_2) = \begin{cases} \mathbb{F}_2[e_1] & \text{if } i = 1, \\ E(\lambda_1) \otimes \mathbb{F}_2[b_2(\lambda)] & \text{if } i \geq 2. \end{cases}$$

Thus we have

$$H^*(4 \times 2^2; \mathbb{F}_2) = \mathbb{F}_2[b_2(\lambda), b_1, h_1] \otimes E(\lambda_1)$$

where $\lambda \in \mathrm{Hom}(4 \times 2^2, \mathbb{F}_2)$ is the unique non-trivial homomorphism with kernel $2^3 \subset 4 \times 2^2$. The $\mathcal{S}_4$ action is described in Theorem A, so if v is the generator of order four fixed by $\mathcal{S}_4$ and x, y are generators of order two, then the Klein group $K = \langle a, b\rangle \lhd \mathcal{S}_4$ acts by $a(x) = x + 2v$, $a(y) = y$, $b(x) = x$, $b(y) = y + 2v$, while an $\mathcal{S}_3$ subgroup acts as $L_2(2)$ on $\langle x, y\rangle$. Consequently, the $\mathcal{S}_4$ invariants have the form

$$(\mathbb{F}_2[b_2(\lambda), b_1, h_1] \otimes E(\lambda_1))^{\mathcal{S}_4} = \mathbb{F}_2[D_8, d_2, d_3] \otimes E(\lambda_1)$$

where $d_2 = b_1^2 + b_1h_1 + h_1^2$, $d_3 = b_1^2h_1 + b_1h_1^2$, and $D_8 = b_2(\lambda)^4 + d_2^2b_2(\lambda)^2 + d_3^2b_2(\lambda)$.

As a final step at this stage, one of the irreducible real representations of O'N is used to determine the existence of a polynomial subalgebra $\mathbb{F}_2[d_8, d_{12}, \bar{d}_{14}] \subset H^*(\mathrm{O'N}; \mathbb{F}_2)$ and its image under restriction in $H^*(4^3; \mathbb{F}_2) \oplus H^*(4 \times 2^2; \mathbb{F}_2)$.

Next one applies the specific results of Part II of [M2] to determine the exact image of $H^*(\mathrm{O'N}; \mathbb{F}_2)$ in $H^*(\mathrm{Syl}_2(\mathrm{O'N}); \mathbb{F}_2)$. We summarize the table of generators and their restrictions to the seven conjugacy classes of centralizers of 2^3's in Section 3, and from this we can read off the generators of $H^*(\mathrm{O'N}; \mathbb{F}_2)$.

The description of the ring $H^*(\mathrm{O'N}; \mathbb{F}_2)$ is given as follows. It has 12 generators, one in each dimension from 3 to 14: d_3, M_4, Y_5, M_6, Y_7, d_8, Y_9, M_{10}, X_{11}, d_{12}, X_{13}, and d_{14} with M_4, M_6, M_{10} as the generators of the radical. The action of the Steenrod algebra on these generators is given by

$$\begin{aligned} Sq^2 d_3 &= Y_5, & Sq^4(Y_5) &= Y_9, \\ Sq^2 M_4 &= M_6, & Sq^4(M_6) &= M_{10}, \\ Sq^4(X_7) &= X_{11}, & Sq^2(X_{11}) &= X_{13}, \\ Sq^4(d_8) &= d_{12}, & Sq^2(d_{12}) &= \bar{d}_{14} = d_{14} + d_3(d_3d_8 + X_{11}), \end{aligned}$$

and the action of Sq^1 on the generators is given by the formulae

$$Sq^1(Y_5) = d_3^2, \qquad Sq^1(M_6) = d_3M_4, \qquad Sq^1(Y_9) = d_3X_7,$$
$$Sq^1(M_{10}) = M_4X_7, \qquad Sq^1(X_{13}) = d_3X_{11}.$$

Sq^1 is zero on the generators which are not listed. Also, there are a number of basic relations such as

$$d_3d_{14} = Y_5d_{14} = Y_9d_{14} = 0,$$
$$Y_5^2 = d_3X_7, \qquad Y_9^2 = d_3(X_7d_8 + d_3d_{12} + d_3^5),$$
$$X_7^2 = d_3X_{11},\ X_{11}^2 = d_3(X_{11}d_8 + X_7d_{12} + X_7d_3^4).$$

The best way to understand the structure of this algebra is to explain its embedding into $H^*(4^3; \mathbb{F}_2)^{L_3(2)} \oplus H^*(4 \times 2^2; \mathbb{F}_2)^{\mathcal{S}_4}$. The map on generators is given by

$$d_3 \mapsto (0, d_3), \qquad X_7 \mapsto (\tilde{X}_7, d_2^2 d_3), \qquad X_{11} \mapsto (\tilde{X}_{11}, d_2^4 d_3),$$
$$M_4 \mapsto (\tilde{M}_4, \lambda d_3), \qquad d_8 \mapsto (d_8, D_8 + d_2^4), \qquad d_{12} \mapsto (d_{12}, d_2^2 D_8 + d_3^4),$$
$$Y_5 \mapsto (0, d_2 d_3), \qquad Y_9 \mapsto (0, d_2^3 d_3), \qquad X_{13} \mapsto (\tilde{X}_{13}, d_2^5 d_3),$$
$$M_6 \mapsto (\tilde{M}_6, \lambda d_2 d_3), \qquad M_{10} \mapsto (\tilde{M}_{10}, \lambda d_2^3 d_3), \qquad d_{14} \mapsto (d_{14}, 0).$$

Using this embedding of the algebra it is easy to verify the relations and squaring operations above, as well as to write down the remaining relations. One can, for example, verify that the image of the subalgebra $\mathbb{F}_2[d_3, d_8, d_{12}](1, Y_5, X_7, Y_9, X_{11}, X_{13})$ in the second factor is an injection.[3] In particular, the image is free over the polynomial algebra on d_3, d_8, and d_{12}. Moreover, if we look at the part generated by M_4, M_6, and M_{10} over this algebra, which we denote $\mathscr{A}$, we get a short exact sequence of modules

$$0 \to \mathbb{F}_2[d_3, d_8, d_{12}](1, Y_5, X_7, Y_9, X_{11}, X_{13})M_4 \to \mathscr{A}$$
$$\to \mathbb{F}_2[d_8, d_{12}](M_6, M_{10}) \to 0.$$

But the extension is non-trivial, $d_3 M_6 = Y_5 M_4$, $d_3 M_{10} = Y_9 M_4$, as is quickly verified by checking restrictions—for example, the restrictions of both $d_3 M_{10}$ and $Y_9 M_4$ are $(0, \lambda_1 d_2^3 d_3^2)$—so we can rewrite $\mathscr{A}$ in the form

$$\mathscr{A} = \mathbb{F}_2[d_3, d_8, d_{12}](M_4, M_6, M_{10}, X_7 M_4, X_{11} M_4, X_{13} M_4).$$

Similarly, if we look at the image in the first factor, it has the form

$$\mathbb{F}_2[d_8, d_{12}, d_{14}]\left(1, \tilde{M}_4, \tilde{M}_6, \tilde{X}_7, \tilde{M}_{10}, \tilde{X}_{11}, \tilde{X}_{13}, d_{14} E_3\right)$$

where the last factor, $d_{14} E_3$, is the image of $X_7 M_{10} = X_{11} M_6 = X_{13} M_4$.

Remark. Quillen proved that the minimal primes in $H^*(G; \mathbb{F}_p)$ are in one-to-one correspondence with the conjugacy classes of maximal elementary p-groups in G, the correspondence being given by associating to each such group the kernel of the restriction map in cohomology to the polynomial subring of $H^*(E_p; \mathbb{F}_p)$ [AM, Chap. IV]. In the case of O'N there are exactly two minimal primes, corresponding to the two 2^3's, $\mathscr{P}_{\mathrm{I}'}$ and $\mathscr{P}_{\mathrm{II}'}$. The quotient by the first is simply $\mathbb{F}_2[d_8, d_{12}, d_{14}]$, but the quotient by the second is

$$\mathbb{F}_2[d_3, d_8, d_{12}](1, Y_5, X_7, Y_9, X_{11}, X_{13}),$$

[3] When we write for example $\mathscr{P}(A, B, C)$, this denotes an algebra which is free as a module over $\mathscr{P}$ on the three generators A, B, C.

and this integral domain is *not closed in its quotient field* since d_2 is integral over it, being a root of the monic polynomial $x^6 + d_8x^2 + (d_{12} + d_3^4) = 0$, but is not contained in it.

$H^*(\mathrm{O'N}; \mathbb{F}_2)/\mathscr{P}_{\mathrm{II'}}$ is the first example with this property that is known to us.

Remark. The Sq^1 structure described above, following from the map to $H^*(4 \times 2^2; \mathbb{F}_2)^{\mathscr{S}_4}$, gives the E_2 term of the Bockstein spectral sequence which explores the higher 2-torsion in $H^*(\mathrm{O'N}; \mathbb{Z})$ as the cohomology of $H^*(\mathrm{O'N}; \mathbb{F}_2)$ with respect to Sq^1. (This makes sense since $(Sq^1)^2 = 0$.) The result is

$$E_2 = \mathbb{F}_2[d_8, d_{12}](1, d_3, M_4, X_7, X_{11}, X_{11}M_4)$$
$$\oplus\ \mathbb{F}_2[d_8, d_{12}, d_{14}]d_{14}(1, M_4, M_6, X_7, M_{10}, X_{11}, X_{13}, M_4X_{13}),$$

where the second line is the contribution from the group $H^*(4^3; \mathbb{F}_2)$ and the first line comes from $H^*(4 \times 2^2; \mathbb{F}_2)$. Next, using our knowledge of the $\mathbb{Z}/4$ Bockstein in $H^*(4^3; \mathbb{F}_2)$ we see that E_3, which is given as the homology of E_2 with respect to this $\mathbb{Z}/4$-Bockstein, comes from the following differentials: $\beta_4(d_{14}^2E_3) = d_{14}^2M_4$, $\beta_4(d_{14}M_6) = d_{14}X_7$, $\beta_4(d_{14}M_{10}) = d_{14}X_{11}$, and $\beta_4(d_{14}X_{13}) = d_{14}^2$, while d_{14} either goes to zero or $X_{11}M_4$. It follows that the E_3 term has the form

$$\mathbb{F}_2[d_8, d_{12}](1, d_3, M_4, X_7, X_{11}, d_{14}, X_{11}M_4, d_{14}M_4),$$

or

$$\mathbb{F}_2[d_8, d_{12}](1, d_3, M_4, X_7, X_{11}, d_{14}M_4).$$

We do not know the structure of the higher differentials, but it is clear how they must work: the role of X_7 is to hit d_8, that of X_{11} is to hit d_{12}, d_3 must hit M_4, and $d_{14}M_4$ must hit $X_{11}d_8 + X_7d_{12}$. The only question is at what level these differentials appear.

The Poincaré series for $H^*(\mathrm{O'N}; \mathbb{F}_2)$ is given as

$$\frac{(1 + x^4)(1 + x^5 + x^7 + x^9 + x^{11} + x^{13})}{(1 - x^3)(1 - x^8)(1 - x^{12})} + \frac{x^6 + x^{10}}{(1 - x^8)(1 - x^{12})}$$
$$+ \frac{x^{14}(1 + x^4 + x^6 + x^{10} + x^7 + x^{11} + x^{13} + x^{17})}{(1 - x^8)(1 - x^{12})(1 - x^{14})}$$

where the third term corresponds to the elements in the minimal prime $\mathscr{P}_{\mathrm{II'}} = (d_{14})$. The rational function above expands out as

$$\frac{\begin{array}{r} x^{24} + x^{21} + x^{20} + x^{19} + 2x^{18} + x^{17} + x^{16} + 3x^{15} + 2x^{14} + 2x^{13} + 2x^{12} \\ + 2x^{11} + 2x^{10} + 3x^9 + x^8 + x^7 + 2x^6 + x^5 + x^4 + x^3 + 1 \end{array}}{(1 - x^7)(1 - x^8)(1 - x^{12})}$$

and, with denominator $(1-x^8)(1-x^{12})(1-x^{14})$, as

$$\frac{f(x)}{(1-x^8)(1-x^{12})(1-x^{14})}$$

where $f(x)$ is the symmetric polynomial of degree 31,

$$\begin{aligned} x^{31} &+ x^{28} + x^{27} + x^{26} + 2x^{25} + 2x^{24} + x^{23} + 3x^{22} + 3x^{21} + 3x^{20} + 3x^{19} \\ &+ 4x^{18} + 3x^{17} + 4x^{16} + 4x^{15} + 3x^{14} + 4x^{13} + 3x^{12} + 3x^{11} \\ &+ 3x^{10} + 3x^9 + x^8 + 2x^7 + 2x^6 + x^5 + x^4 + x^3 + 1. \qquad (*) \end{aligned}$$

The reason for the expression $(*)$ above is that we believe $H^*(O'\mathrm{N};\mathbb{F}_2)$ is freely and finitely generated over $\mathbb{F}_2[d_8, d_{12}, d_{14} + w]$, for some w contained in the ideal $(d_3, Y_5, Y_9, \dots)$ of elements which project trivially into $H^*(4^3;\mathbb{F}_2)$, so $(*)$ gives a specification of the dimensions in which the generators should occur. Moreover, $(*)$ indicates the strong likelihood of the existence of a Poincaré duality space or manifold of dimension 31 which reflects many of the properties of O'N if, indeed, $H^*(\mathrm{O'N};\mathbb{F}_2)$ is freely and finitely generated over the polynomial ring above.

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1. THE 2-LOCAL STRUCTURE OF O'N

Denote by Alp_n^1 the split extension

$$1 \to (2^n)^3 \to \mathrm{Alp}_n^1 \to L_3(2) \to 1 \tag{1.1}$$

and by Alp_n^2 the non-split extension

$$1 \to (2^n)^3 \to \mathrm{Alp}_n^2 \to L_3(2) \to 1$$

where in both cases $L_3(2) = GL_3(\mathbb{F}_2)$ acts on $2^3 \subset (2^n)^3$ in the usual way. These are the Alperin groups first introduced in [Alp], which are uniquely determined and play a critical role in the classification theorem for sporadic simple groups. Specifically, Alp_1^1 is a maximal subgroup of $\mathcal{A}_8 \cong L_4(2)$ of odd index. Also, Alp_1^2 is a maximal subgroup of $G_2(q), {}^3D_4(q)$ for q odd, and is of odd index if $q \equiv 3, 5 \bmod(8)$. Let HS denote the Higman–Sims sporadic group. Then Alp_2^1 is maximal and of odd index in HS. Additionally, the Sylow 2-subgroup of Alp_1^2 is the Sylow 2-subgroup of the Mathieu group M_{12}.

At this point the obvious question arises: does there exist a finite simple group G with $\mathrm{Syl}_2(G) = \mathrm{Syl}_2(\mathrm{Alp}_2^2)$? O'Nan [O N] gave a positive answer to this question, characterizing the sporadic group O'N as the unique simple group having this Sylow 2-subgroup. Its existence and uniqueness were later verified by Andrilli and Sims [A]. O'N has order $2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31 = 460{,}815{,}505{,}920$.

In this section, we describe the 2-local structure of O'N, as this is what we need for the cohomology calculations. The following proposition combines results of [ON, Yo].

PROPOSITION 1.1. (1) O'N *has only one conjugacy class of involutions* $\langle j \rangle$. *The centralizer of* j *is the group* $4 \cdot L_3(4).2$.

(2) O'N *has exactly two conjugacy classes of maximal subgroups of odd index* $\mathrm{Alp}_2^2 = 4^3 \cdot L_3(2)$ *and* $4 \cdot L_3(4).2$.

We will show shortly that the two subgroups above, Alp_2^2 and $4 \cdot L_3(4).2$, intersect in a common subgroup H_β of order $2^9 \cdot 3$, so O'N contains the configuration

$$4 \cdot L_3(4) : 2_1 \underset{H_\beta}{\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet} \mathrm{Alp}_2^2 \qquad (1.2)$$

which completely controls the cohomology of O'N.

The Sylow 2-Subgroup of O'N

The group $\mathrm{Syl}_2(\mathrm{Alp}_2^i)$ ($i = 1$ or 2) has a presentation with five generators v_1, v_2, v_3, s, and t, so that $\langle v_1, v_2, v_3 \rangle = (\mathbb{Z}/4)^3$, $t^2 = 1$, and the remaining relations are given as

$$\begin{gathered} sv_1s^{-1} = v_2, \qquad sv_2s^{-1} = v_3, \qquad sv_3s^{-1} = v_1v_2^{-1}v_3, \\ tv_1t = v_3^{-1}, \qquad tv_2t = v_2^{-1}, \qquad tv_3t = v_1^{-1}, \qquad tst = s^{-1} \\ s^4 = \begin{cases} v_1v_3 & \text{in the case of } \mathrm{Syl}_2(\mathrm{Alp}_2^2), \\ 1 & \text{in the case of } \mathrm{Alp}_2^1. \end{cases} \end{gathered} \qquad (1.3)$$

It is pointed out in [Alp] that $\mathrm{Syl}_2(\mathrm{Alp}_2^2)$ is a rank 3 group. To obtain the structure of the Sylow 2-subgroup of Alp_n^i for $n > 2$, $i = 1, 2$, nothing changes in the presentation above except that v_1, v_2, and v_3 now generate a group $(2^n)^3$.

We also need the structure of the conjugacy classes of elements of order two, and the elementary two groups in $\mathrm{Syl}_2(\mathrm{Alp}_2^2)$. The following two results are from [ON].

LEMMA 1.4. *There are seven conjugacy classes of involutions in* $\mathrm{Syl}_2(\mathrm{Alp}_2^2)$. *Representatives and the centralizers for these representatives are given in the following table.*

Class	*Representative*	*Number in class*	*Centralizer*
A	$v_1^2v_3^2$	1	$\langle v_1, v_2, v_3, s, t\rangle$
B	$v_1^2v_2^2$	2	$\langle v_1, v_2, v_3, s^2, st\rangle$
C	v_2^2	4	$\langle v_1, v_2, v_3, t\rangle$
D	$s^2v_1^{-1}$	16	$\langle v_1v_3, v_1^2v_2^2, t, s^2v_1^{-1}\rangle$
E	st	32	$\langle v_3v_2^{-1}, v_1^2v_3^2, st\rangle$
F	t	16	$\langle v_1v_3^{-1}, v_2^2, s^2v_1^{-1}, t\rangle$
G	tv_2	16	$\langle v_1v_3^{-1}, v_2^2, s^2v_1v_2v_3, tv_2\rangle$

LEMMA 1.5. (i) *The centralizer of* $s^2v_1^{-1} = \langle v_1v_3, v_1^2v_2^2, s^2v_1^{-1}, t\rangle$ *is isomorphic to* $2 \times (4 * D_8)$ *where* $4 * D_8$ *is the central product.*

(ii) *The centralizer of* $t = \langle v_1v_3^{-1}, v_2^2, s^2v_1^{-1}, t\rangle$ *is also isomorphic to* $2 \times (4 * D_8)$.

(iii) *The centralizer of* $tv_2 = \langle v_1v_3^{-1}, v_2^2, s^2v_1v_2v_3, tv_2\rangle$ *is again isomorphic to* $2 \times (4 * D_8)$.

(iv) *The centralizer of* $st = \langle v_1v_2^{-1}, v_1^2v_3^2, st\rangle$ *is isomorphic to* $4 \times (2)^2$.

From these lemmas it is direct to enumerate the conjugacy classes of elementary 2-groups $2^3 \subset \mathrm{Syl}_2(\mathrm{Alp}_2^2)$. We have

COROLLARY 1.6. *There are seven conjugacy classes of elementary 2-groups*

$$2^3 \subset \mathrm{Syl}_2(\mathrm{Alp}_2^2).$$

They have representatives as follows: from the centralizer of v_2^2,

$$\mathrm{I}' = \langle v_1^2, v_2^2, v_3^2\rangle, \qquad \mathrm{II}' = \left\langle (v_1v_3)^2, v_2^2, t\right\rangle, \qquad \mathrm{III}' = \left\langle (v_1v_3)^2, v_2^2, v_2t\right\rangle;$$

from the centralizer of st,

$$\mathrm{IV}' = \left\langle (v_1v_3)^2, (v_1v_2)^2, st\right\rangle;$$

and from the centralizer of $s^2v_1^{-1}$,

$$\mathrm{V}' = \left\langle (v_1v_3)^2, (v_1v_2)^2, s^2v_1^{-1}\right\rangle, \qquad \mathrm{VI}' = \left\langle (v_1v_3)^2, v_1v_3t, s^2v_1^{-1}\right\rangle,$$

$$\mathrm{VII}' = \left\langle (v_1v_3)^2, t, s^2v_1^{-1}\right\rangle.$$

Proof. The listing of the possible conjugacy classes of 2^3-subgroups is routine when we note that each subgroup will contain $(v_1v_3)^2$, and if it also contains $(v_1v_2)^2$ then it must contain an element in one of the remaining five conjugacy classes. Consequently we need only list the 2^3's in the centralizers of these five elements and cancel those that are conjugate. This is aided when we note that $ts^2v_1^{-1} = stv_2s^{-1}$, which directly implies that the three 2^3's which arise from the centralizer of tv_2 are listed in the groups for v_2^2 and $s^2v_1^{-1}$. It is also helpful to note that the group $\langle (v_1v_3)^2, s^2v_3^{-1}, t\rangle$ which occurs in the centralizer of t is conjugate to $\langle (v_1v_3)^2, v_1v_3t, s^2v_1^{-1}\rangle$ under the action of s^2. ∎

Remark. We use the primes in our labeling of the conjugacy classes of elementary 2-subgroups above, since we reserve the unprimed groups I, II, ..., VII to denote the centralizers of the groups above. Thus I $= \langle v_1, v_2, v_3\rangle$, II $= \langle v_1v_3^{-1}, v_2^2, t\rangle$, and so on. Except for I, each of the centralizers is a copy of 4×2^2.

The following result is very helpful in analyzing $H^*(\mathrm{Syl}_2(\mathrm{O'N}); \mathbb{F}_2)$ since the cohomology rings of the dihedral groups, $D_{2^{n+2}}$, and wreath products are well understood [AM, Chap. IV].

PROPOSITION 1.7. *The subgroup* $\langle v_1v_3\rangle = 4$ *is normal in* $\mathrm{Syl}_2(\mathrm{Alp}_2^2)$ *with quotient*

$$\Gamma_2 = \mathrm{Syl}_2(\mathrm{Alp}_2^2)/\langle v_1v_3\rangle \cong D_8 \wr 2$$

where D_8 *is the Dihedral group of order* 8.

Proof. $\langle v_1v_3\rangle \lhd \mathrm{Syl}_2(\mathrm{Alp}_2^2)$ since $sv_1v_3s^{-1} = v_1v_3$, $tv_1v_3t = (v_1v_3)^{-1}$. The quotient is obtained by adjoining the relation $v_3 = v_1^{-1}$. In particular, $s^4 = 1$ so the projection $\Gamma_2 \to D_8$ sending v_i to 1 for $i = 1, 2$ is split, and we can write Γ_2 as the semi-direct product

$$\Gamma_2 = 4^2 : D_8.$$

We can write $D_8 = 2 \wr 2$ with $\langle t, s^2t\rangle$ being the $2^2 \subset 2^2 : 2 = 2 \wr 2$ while the element acting to interchange them is st. Also, t commutes with v_1 in Γ_2 while $tv_2t = v_2^{-1}$, $s^2tv_1ts^{-2} = s^2v_1s^{-2} = v_3 = v_1^{-1}$, and $s^2tv_2ts^{-2} = s^2v_2^{-1}s^{-2} = v_2$, so

$$\begin{aligned} 4^2 : \langle t, s^2t\rangle &= D_8 \times D_8 \\ &= \langle v_2, t\rangle \times \langle v_1, s^2t\rangle. \end{aligned}$$

Finally, $(st)^2 = 1$ and

$$stv_2ts^{-1} = sv_2^{-1}s^{-1} = v_3^{-1} = v_1,$$

so st acts to interchange the two copies of D_8 above and the proposition follows. ∎

Remark. The group Alp_2^2 is not the centralizer of an involution in O'N; that is the extension $4 \cdot L_3(4){:}\ 2_1$. In particular, from [ON] we have that the quotient group $L_3(4){:}\ 2$ of the involution centralizer has the group $\mathrm{Syl}_2(\mathrm{Alp}_2^2)/\langle v_1 v_2^2 v_3^{-1}\rangle$ as its Sylow 2-subgroup, and this group is *not* the quotient analyzed above nor is it isomorphic to it. Indeed, [ON] shows that there are precisely two conjugacy classes of elements of order 4 in O'N, the first represented by $v_1 v_3$ and the second by $v_1 v_2^2 v_3^{-1}$, and it is this second which represents the 4 in $4 \cdot L_3(4){:}\ 2_1$.

Remark. In the table in Section 3 where the generators of $H^*(\mathrm{Syl}_2(\mathrm{O'N}); \mathbb{F}_2)$ are listed, the elements e_t, e_v, e_s, $\Gamma(x)$, σ_w, V_3, and $\Gamma(w)$ all come from $H^*(D_8 \wr 2; \mathbb{F}_2)$ under the induced cohomology map for the projection above.

The Explicit Structure of Alp_2^2

Partly in order to understand the fusion in O'N, but also to understand the configuration 1.2, we now give an explicit set of generators for Alp_2^2 following an article by Griess [Gr], where he constructs Alp_2^2 as a subgroup of $\mathit{Aff}_3(\mathbb{Z}/8) \subset \mathrm{GL}_4(\mathbb{Z}/8)$ consisting of matrices of the form $\begin{pmatrix} 1 & 0 \\ V & W \end{pmatrix}$ where $W \in \mathrm{GL}_3(\mathbb{Z}/8)$ and V is a 3×1-column vector. In particular, the subgroup $\langle v_1, v_2, v_3 \rangle$ is given as the subgroup of $\mathrm{GL}_4(\mathbb{Z}/8)$ of all matrices of the form $\begin{pmatrix} 1 & 0 \\ V & I_3 \end{pmatrix}$ where I_3 is the 3×3 identity matrix and V is any 3×1 column vector with even entries.

The upper-triangular matrices in $\mathrm{GL}_3(2)$ give a copy of the 2-Sylow subgroup so we look in those tables for elements W which when reduced mod(2) give upper triangular matrices. Explicit generators for the 2-Sylow subgroup corresponding to the generators we have been studying can be given as

$$t \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 7 & 1 & 2 \\ 2 & 0 & 1 & 4 \\ 7 & 0 & 0 & 7 \end{pmatrix}, \qquad s \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 7 & 6 \\ 2 & 6 & 1 & 7 \\ 5 & 4 & 4 & 7 \end{pmatrix}.$$

With respect to these choices for s, t, we find that a consistent choice for v_1 is

$$v_1 \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 6 & 0 & 1 & 0 \\ 6 & 0 & 0 & 1 \end{pmatrix},$$

so, since $v_2 = sv_1s^{-1}$, $v_3 = s^2v_1s^{-2}$, we find that representatives for the generators v_2, v_3 are given by[4]

$$v_2 \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 4 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}, \qquad v_3 \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 6 & 0 & 0 & 1 \end{pmatrix}.$$

We record the representations of the elements ts, s^2, and ts^2:

$$ts \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 5 & 2 & 7 \\ 0 & 6 & 1 & 3 \\ 2 & 4 & 4 & 1 \end{pmatrix}, \qquad s^2 \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 6 & 1 \\ 3 & 0 & 7 & 4 \\ 0 & 0 & 4 & 5 \end{pmatrix},$$

$$ts^2 \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & 5 & 1 & 5 \\ 5 & 0 & 7 & 0 \\ 7 & 0 & 4 & 3 \end{pmatrix}.$$

Next we need two elements, α, β, of order 3, for which the W terms restrict mod(2) to

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

respectively. These matrices α, β, together with the elements above, must generate the entire group Alp_2^3. Suitable choices are

$$\alpha \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 7 & 4 & 3 & 0 \\ 4 & 1 & 3 & 2 \\ 0 & 4 & 0 & 1 \end{pmatrix}, \qquad \beta \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 \\ 7 & 0 & 4 & 3 \\ 6 & 2 & 1 & 3 \end{pmatrix}. \tag{1.8}$$

Then

$$\alpha^{-1} \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 7 & 3 & 5 & 6 \\ 7 & 7 & 4 & 0 \\ 4 & 4 & 4 & 1 \end{pmatrix}, \qquad \beta^{-1} \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 4 & 4 \\ 5 & 6 & 3 & 5 \\ 7 & 0 & 7 & 4 \end{pmatrix}.$$

[4] This model for Alp_2^3 is not quite the model in [ON], as it reflects the topologists' preference for left actions instead of right actions. In particular, in this paper conjugation of x by a is axa^{-1} and not, as is more usual with group theorists, $a^{-1}xa$. We also use the notation $\alpha(x)$ for $\alpha x \alpha^{-1}$.

LEMMA. 1. *Let* $K_\alpha = \langle v_1, v_2, v_3, ts, s^2\rangle$. *Then* α *normalizes* K_α *with action* $\alpha(s^2) = (v_1v_2)^{-1}v_3ts^3$, $\alpha(ts) = (v_1v_3)^{-1}v_2s^2$, $\alpha(v_1) = v_1(v_2v_3)^{-1}$, $\alpha(v_2) = v_1^2v_2^{-1}$, *and* $\alpha(v_3) = v_1^{-1}v_2^2$. *Also* $t\alpha t = \alpha^{-1}$.

2. *Let* $K_\beta = \langle v_1, v_2, v_3, t, s^2\rangle$. *Then* β *normalizes* K_β *with action* $\beta(t) = v_3^{-1}ts^2$, $\beta(s^2) = (v_2v_3)^{-1}t$, $\beta(v_1) = v_1v_2^{-1}$, $\beta(v_2) = v_3^{-1}$, *and* $\beta(v_3) = v_2v_3^{-1}$, *while* $ts^2\beta ts^3 = \beta^{-1}$.

Proof. This is a direct calculation. For example, we have

$$\alpha s^2\alpha^{-1} \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 7 & 4 & 3 & 0 \\ 4 & 1 & 3 & 2 \\ 0 & 4 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 6 & 1 \\ 3 & 0 & 7 & 4 \\ 0 & 0 & 4 & 5 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 7 & 3 & 5 & 6 \\ 7 & 7 & 4 & 0 \\ 4 & 4 & 4 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 7 & 7 & 0 & 0 \\ 3 & 2 & 7 & 1 \\ 4 & 4 & 0 & 1 \end{pmatrix}$$

and, in turn, this is seen to represent $v_1^3v_2^3v_3ts^3$. Next we find

$$\alpha ts\alpha^{-1} \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 6 & 3 & 6 & 1 \\ 7 & 0 & 7 & 4 \\ 6 & 0 & 4 & 5 \end{pmatrix}$$

which is the image of $(v_1v_3)^3v_2s^2$. Also, it is direct to check that $t\alpha t = \alpha^{-1}$, and we have

$$\alpha v_1\alpha^{-1} \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 6 & 0 & 0 & 1 \end{pmatrix}, \qquad \alpha v_2\alpha^{-1} \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix},$$
$$\alpha v_3\alpha^{-1} \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 6 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 6 & 0 & 0 & 1 \end{pmatrix}.$$

Consequently, $\alpha v_1\alpha^{-1} = v_1v_2^3v_3^3$, $\alpha v_2\alpha^{-1} = v_1^2v_2^3$, and $\alpha v_3\alpha^{-1} = v_1^3v_2^2$. Thus, the action of α on the subgroup of $\mathrm{Syl}_2(\mathrm{Alp}_2^2)$ which it normalizes is completely determined.

We also check directly that $ts^3\beta ts^3 = \beta^{-1}$. It remains to determine the action of β on the subgroup it normalizes. We have

$$\beta t\beta^{-1} \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & 5 & 1 & 5 \\ 3 & 0 & 7 & 0 \\ 1 & 0 & 4 & 3 \end{pmatrix}, \qquad \beta s^2\beta^{-1} \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 7 & 1 & 2 \\ 4 & 0 & 1 & 4 \\ 7 & 0 & 0 & 7 \end{pmatrix}$$

while

$$\beta v_1 \beta^{-1} \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 4 & 0 & 0 & 1 \end{pmatrix}, \qquad \beta v_2 \beta^{-1} \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 6 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix},$$

$$\beta v_3 \beta^{-1} \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 4 & 0 & 0 & 1 \end{pmatrix}.$$

Thus we have the action of β on $\langle t, s^2, v_1, v_2, v_3\rangle$: first, $\beta v_1 \beta^{-1} = v_1 v_2^{-1}$, and $\beta v_2 \beta^{-1} = v_3^{-1}$, $\beta v_3 \beta^{-1} = v_2 v_3^{-1}$; second, $\beta t\beta^{-1} = v_3^{-1}ts^2$; and third, $\beta s^2\beta^{-1} = (v_2 v_3)^{-1}t$. ∎

Remark. The action of α on $\langle v_1, v_2, v_3\rangle$ splits as a trivial action on $4 = \langle v_1^2 v_2 v_3^2\rangle$, and an action on $4^2 = \langle v_1 v_2^{-1} v_3^2, v_1 v_3^{-1}\rangle$ where

$$\alpha v_1 v_2^{-1} v_3^2 \alpha^{-1} = v_1 v_3^{-1}, \qquad \alpha v_1 v_3^{-1}\alpha^{-1} = \left((v_1 v_2^{-1} v_3^2)(v_1 v_3^{-1})\right)^{-1}.$$

We have $\langle t, \alpha\rangle \cong \mathscr{S}_3$ and

$$H_\alpha = \langle t, \alpha, ts, s^2, v_1, v_2, v_3\rangle \cong K_\alpha : \mathscr{S}_3. \tag{1.9}$$

Similarly, $\langle ts^3, \beta\rangle \cong \mathscr{S}_3$ so

$$H_\beta = \langle ts^3, \beta, t, s^2, v_1, v_2, v_3\rangle \cong K_\beta : \mathscr{S}_3. \tag{1.10}$$

Also, for both H_α, H_β, the quotients by $\langle v_1, v_2, v_3\rangle$ are copies of $\mathscr{S}_4$. Note in particular that $\beta v_1 v_3 \beta^{-1} = v_1 v_3^{-1}$ so $\beta(v_1 v_3)^2\beta^{-1} = (v_1 v_3)^2$ and, since $4 \cdot L_3(4) \cdot 2$ is the centralizer of $(v_1 v_3)^2$ in O'N, we have

$$\mathrm{Alp}_2^2 \cap 4 \cdot L_3(4) \cdot 2 = H_\beta \cong 4^3 \cdot \mathscr{S}_4.$$

Remark 1.11. When we project K_α to $\mathrm{Syl}_2(\mathrm{O'N})/\langle v_1 v_3\rangle$, we get that s^2 acts to invert every element of $\langle v_1, v_2, v_3\rangle/\langle v_1 v_3\rangle \cong 4^2$, while ts acts to exchange the two generators. But this is the description of $\mathrm{Syl}_2(M_{12})$ as given, for example, in [FM]. This observation will be crucial in the calculation of $H^*(\mathrm{Syl}_2(\mathrm{O'N}); \mathbb{F}_2)$ given in [M2].

A Key Fusion in Alp_2^2

In Section 2 we will see that there are exactly two conjugacy classes of rank 3 elementary abelian 2-subgroups in O'N. For this we will need the following lemma describing a specific fusion in Alp_2^2.

LEMMA 1.12. *In* Alp_2^2 *the groups* $\langle (v_1v_3)^2, (v_1v_2)^2, s^2v_1^{-1}\rangle$ *and* $\langle (v_1v_3)^2, (v_1v_2)^2, st\rangle$, *both isomorphic to* 2^3, *are conjugate.*

Proof. Take the element $\alpha \in \mathrm{Alp}_2^2$ of order 3 described in (1.8). Then

$$\alpha(v_1v_3)^2\alpha^{-1} = (v_1^2v_2^2v_3^2)(v_1^2) = (v_2v_3)^2$$
$$\alpha(v_1v_2)^2\alpha^{-1} = (v_1^2v_2^2v_3^2)(v_2^2) = (v_1v_3)^2$$
$$\alpha(s^2v_1^{-1})\alpha^{-1} = v_1^{-1}v_2^{-1}v_3ts^3(v_1v_2^{-1}v_3^{-1})^{-1}.$$

Now $v_1^{-1}v_2^{-1}(v_3t) = v_1^{-1}(v_2^{-1}t)v_1^{-1} = (v_1^{-1}t)v_2v_1^{-1} = tv_3v_2v_1^{-1}$, and

$$s^3(v_1^{-1}v_2v_3) = (v_2(v_1v_3)^{-1}v_1v_2)s^3 = (v_1v_2^2)(v_1v_3)^{-1}s^3 = (v_1v_2^2)s^{-1}$$

so

$$\begin{aligned}\alpha(s^2v_1^{-1})\alpha^{-1} &= v_1^{-1}v_2^{-1}v_3ts^3v_1^{-1}v_2v_3\\ &= t(v_2v_3v_1^{-1})v_1v_2^2s^{-1} = tv_3v_2^{-1}s^{-1}\\ &= ts^{-1}(v_2^{-1}v_1) = stv_1v_2^{-1}\\ &= v_2v_3ts^3.\end{aligned}$$

Now we have

$$(v_1v_2v_3)s^2(\alpha(s^2v_1^{-1}\alpha^{-1}))s^{-2}(v_1v_2v_3)^{-1} = st,$$

and (1.12) follows. ∎

Remark. We will exploit this fusion further in Section 3 when we describe the necessary and sufficient conditions that an element in $H^*(\mathrm{Syl}_2(\mathrm{O'N}); \mathbb{F}_2)$ be in $H^*(\mathrm{O'N}); \mathbb{F}_2)$.

2. THE LATTICE OF ELEMENTARY ABELIAN SUBGROUPS IN O'N

In this section we describe the lattice of elementary abelian 2-subgroups in $G = \text{O'N}$. Our main sources are the Atlas [Co, ON, Yo] and a private communication from Lyons [Ly]. To begin, we have

THEOREM 2.1. *There are exactly five distinct conjugacy classes of elementary abelian 2-groups in* $G = \text{O'N}$ *described as follows*:

Group	*Normalizer*	*Order of normalizer*
$Z \cong 2$	$4 \cdot L_3(4){:}\, 2_1$	$161{,}280 = 2^9 \cdot 3^2 \cdot 5 \cdot 7$
$V_1 \cong 2^2$	$4^3 \cdot \mathcal{S}_4$	$1536 = 2^9 \cdot 3$
$V_2 \cong 2^2$	$(3^2 4 \times \mathcal{A}_4) \cdot 2$	$864 = 2^5 \cdot 3^3$
$E_1 \cong 2^3$	$4^3 \cdot L_3(2)$	$10{,}752 = 2^9 \cdot 3 \cdot 7$
$E_2 \cong 2^3$	$(4 \times 2^2){:}\, \mathcal{S}_4$	$384 = 2^7 \cdot 3$

Proof. The fact that there is only one conjugacy class of involutions in G with centralizer $4 \cdot L_3(4) \cdot 2$ appears in O'Nan's original paper [ON]. The presence of exactly two distinct conjugacy classes of Klein groups is Lemma 3.1 in [Yo]; the normalizers of V_1, V_2 follow directly from the discussion there.

Finally, in Lemma 3.2 in [Yo] it is proved that there are at most three distinct conjugacy classes of 2^3 in G, and that the 2-rank of G is three. The representatives for these subgroups can be taken to be (using the notation from Section 1):

$$\mathrm{I}' = \langle v_1^2, v_2^2, v_3^2 \rangle, \qquad \mathrm{V}' = \langle v_1^2 v_3^2, v_1^2 v_2^2, s^2 v_1^{-1} \rangle,$$
$$\mathrm{IV}' = \langle v_2^2 v_3^2, v_1^2 v_2^2, st \rangle.$$

(Note that in [Yo] it is shown that I$'$ and V$'$ are not conjugate.) However, from our analysis of 2-fusion, (1.14), it follows that, in fact, IV$'$ is conjugate to V$'$. The structure of the normalizers appears in [ON] as well as [Yo]. ∎

From this information we will now construct a simplicial complex associated to this lattice. We recall its definition: consider the partially ordered set of 2-elementary abelian subgroups in a finite group G, denoted by $\mathcal{A}_2(G)$. Denote by $|\mathcal{A}_2(G)|$ the simplicial complex associated to it; its vertices are the elements of $\mathcal{A}_2(G)$ and its simplices are the non-empty finite chains (under inclusion). These complexes were first introduced by Brown [B] and then studied by Quillen [Q]. They have a G-action induced by conjugation.

Remark 2.2. In previous cases, such as the analysis of M_{12} in [AMM] and the analysis of $G_2(q), {}^3D_4(q)$ in [FM], these poset spaces have allowed us to reduce the determination of $H^*(G; \mathbb{F}_2)$ to the analysis of a fairly small number of subgroups and their common intersections. As we will see in Section 5, this is also the case here. However, it turns out that in this case it is at least as hard to determine the cohomology of the subgroups as it is to study $H^*(\mathrm{O'N}; \mathbb{F}_2)$ directly. This is similar to what occurred in our study of M_{22} [AM2]. But, regardless, as we see in Section 4, the structure of $|\mathscr{A}_2(\mathrm{O'N})|/\mathrm{O'N}$ will give us an important connection between the classifying space for O'N and a union of two classifying spaces over a common intersection which holds out the hope of understanding some of the implications of the existence of O'N in homotopy theory.

We now describe $|\mathscr{A}_2(\mathrm{O'N})|/\mathrm{O'N}$.

THEOREM 2.3. *$|\mathscr{A}_2(\mathrm{O'N})|$ is a two-dimensional complex which, up to conjugacy (the action of* O'N*), has five vertices, eight* 1*-cells, and four* 2*-cells. The following is a list of conjugacy classes of flags with their normalizers*:

Flag	*Normalizer*	*Order of normalizer*
(Z, V_1)	4^3D_8	$512 = 2^9$
(Z, V_2)	$(3^2 \cdot 4 \times V_2)2$	$288 = 2^5 \cdot 3^2$
(Z, E_1)	$4^3 \cdot \mathscr{S}_4$	$1536 = 2^9 \cdot 3$
(Z, E_2)	$(4 \times 2^2) \cdot \mathscr{S}_4$	$384 = 2^7 \cdot 3$
$(Z, E_2)'$	$(4 \times 2^2)2^2$	$64 = 2^6$
(V_1, E_1)	$4^3 \cdot \mathscr{S}_4$	$1536 = 2^9 \cdot 3$
(V_1, E_2)	$(4 \times 2^2) \cdot D_8$	$128 = 2^7$
(V_2, E_2)	$(4 \times V_2) \cdot \mathscr{S}_3$	$96 = 2^5 \cdot 3$
(Z, V_1, E_1)	$4^3 \cdot D_8$	$512 = 2^9$
(Z, V_1, E_2)	$(4 \times 2^2) \cdot D_8$	$128 = 2^7$
$(Z, V_1, E_2)'$	$(4 \times 2^2) \cdot 2^2$	$64 = 2^6$
(Z, V_2, E_2)	$(4 \times V_2)2$	$32 = 2^5$

Proof. The first four on the list are clear. The fifth one occurs because $N_G(E_2)$ does not act transitively on the seven subgroups of order 2 in E_2; there is an orbit with six elements and a fixed point. In [Yo] it is proved that no conjugate of V_2 lies in E_1, explaining the absence of (V_2, E_1) and (Z, V_2, E_1). The rest are then clear from these arguments. ∎

$|\mathscr{A}_2(\mathrm{O'N})|$ is a two-dimensional CW complex with 2,079,117,579 vertices, 19,900,669,640 edges, and 26,100,878,270 faces. Hence $\chi(|\mathscr{A}_2(\mathrm{O'N})|) = 8{,}279{,}326{,}209 \equiv 1 \bmod(2^9)$. The pictorial representation of the orbit space $|\mathscr{A}_2(\mathrm{O'N})|/\mathrm{O'N}$ can be found in the Introduction.

3. THE DETAILED FUSION STRUCTURE FOR THE MAXIMAL ELEMENTARY SUBGROUPS

Each of the seven centralizers of maximal elementaries of (1.6) except for VI and VII is in a known conjugacy class in O'N: it is either that of I or II. We now show that one of the remaining two classes is conjugate to a subgroup of I and the other is conjugate to II.

In [AM1] a homomorphism $\phi\colon \langle v_1, v_2, v_3, t, s^2\rangle \to \mathrm{Syl}_2(L_3(4))$ was given with kernel $\langle v_1 v_3^{-1} v_2^2\rangle$. This homomorphism cannot extend to an injection $H_\beta \to 4\cdot L_3(4)\colon 2_1$, so we begin our analysis by giving a homomorphism $\phi\colon K_\beta \to \mathrm{Syl}_2(L_3(4))$, with kernel $\langle v_1 v_3^{-1} v_2^2\rangle$ which does extend to a homomorphism from H_β to $L_3(4)\colon 2_1$, the group $L_3(4)$ extended by the "unitary involution" (see below). Set $\beta' = v_2\,\beta$. β' is again of order three. We begin the construction by setting

$$\phi(\beta') = \begin{pmatrix} \zeta & 0 & 0 \\ 0 & \zeta^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $\zeta \in \mathbb{F}_4$ is a primitive third root of unity and we are in the quotient by the center $L_3(4)$, and not in $SL_3(4)$. We have $H_\beta = K_\beta\colon 2_1 = \langle s^2, t, v_1, v_2, v_3\rangle\colon \mathscr{S}_3$, and a convenient choice for the extending $\mathscr{S}_3$ is

$$\mathscr{S}_3 = \langle v_1^{-1} v_2 t s^3, \beta'\rangle,$$

which makes H_β explicit. Now remember that the element $v_1^{-1} v_2 t s^3$ is supposed to act on $L_3(4)$ via the unitary involution $A \mapsto M(A^{-1})^t M^{-1}$ where M is the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

so that when we specify the image of t, the images of all the remaining generators in $\langle v_1, v_2, v_3, t, s^2, \beta'\rangle$ are determined. Set

$$\phi(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \zeta \\ 0 & 0 & 1 \end{pmatrix},$$

so we obtain

$$v_2^2 \mapsto \begin{pmatrix} 1 & 0 & \zeta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad v_1^2 \mapsto \begin{pmatrix} 1 & 0 & \zeta^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$t \leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \zeta \\ 0 & 0 & 1 \end{pmatrix}, \qquad ts^2 v_1^{-1} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.1}$$

$$ts^2 v_1^2 \mapsto \begin{pmatrix} 1 & \xi^2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad v_1 v_3 v_2^{-1} \mapsto \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Remark. The choice for β' was made because it was required that β' should normalize $\langle v_1 v_3^{-1} v_2^2, t, s^2 v_1^{-1}\rangle$, the group VII associated to (1.6), so the image of this group in $L_3(4)$ is the subgroup of matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}.$$

A second, inequivalent choice is given by choosing β' to normalize the other group, VI $= \langle v_1 v_3^{-1} v_2^2, v_1 v_3 t, s^2 v_1^{-1}\rangle$, and modifying ts^3 to an element xts^3 with $x \in \langle v_1, v_2, v_3\rangle$, so $\langle xts^3, \beta'\rangle = \mathscr{S}_3$.

Anyway, with the embedding in (3.1), the image of $\langle v_1 v_3^{-1} v_2^2, t, s^2 v_1^{-1}\rangle$ is conjugate in $4 \cdot L_3(4)$ to the group

$$\phi^{-1}\left\{\begin{pmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\} = \langle v_1 v_3^{-1} v_2^2, v_2^2, v_1^2\rangle.$$

On the other hand, when we explicitly embed the remaining group $\langle v_1 v_3^{-1} v_2^2, v_1 v_3 t, s^2 v_1^{-1}\rangle$ we find

$$v_1 v_3 t \mapsto \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & \zeta \\ 0 & 0 & 1 \end{pmatrix}, \qquad v_1 v_3 ts^2 v_1^{-1} \mapsto \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

which has determinant $\det\begin{pmatrix} 1 & 1 \\ \zeta & 1 \end{pmatrix} = \zeta^2$.

In particular, both groups are contained in the subgroup

$$4 * D_8 * D_8 = \langle v_1 v_3^{-1} v_1^2, v_2^2, t, ts^2 v_1^{-1} \rangle \sim \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \subset 4 \cdot L_3(4),$$

which has the Weyl group $\mathscr{A}_5$. Since this action fixes $v_1 v_3^{-1} v_2^2$, it is determined on the centralizers of the maximal elementaries by what it does on their images in the quotient $2^4 \subset L_3(4)$, and each such image is a 2^2. It was pointed out in [AM1] that there are 4 conjugacy classes of $2^2 \subset 2^4 \cong \mathbb{F}_4^2$ under this action of $\mathscr{A}_5 \cong SL_2(4)$, and they are distinguished by the determinant of any two non-zero elements in 2^2. Thus, $\langle v_1^2, v_2^2, v_1 v_3^{-1} v_2^2 \rangle$ has determinant 0 as does $\langle v_1 v_3^{-1} v_2^2, t, s^2 v_1^{-1} \rangle$, and they are conjugate. On the other hand, the determinant of $\langle v_1 v_3^{-1} v_2^2, v_2^2, t \rangle$ is also ζ^2 so this group is conjugate to $\langle v_1 v_3^{-1} v_2^2, v_1 v_3 t, s^2 v_1^{-1} \rangle$ as asserted.

This completes our analysis of the fusion in O'N. There are five subgroups 2^3 in one class, with centralizers all equal to 4×2^2 while there are two in the other class.

It remains to determine the explicit identifications of the six 4×2^2's, modulo the actions of their normalizers in order to make cohomology calculations.

We note that $(4 * D_8 * D_8)$: $\mathscr{A}_5$ contains the normalizers in O'N of the maximal 2-elementaries in II, III, V, and VI. The Weyl group $\mathscr{S}_4$ of each centralizes $v_1 v_3^{-1} v_2^2$ and is generated by elements d, e, f, g with $\langle d, e \rangle \cong 2^2$, $\langle e, f \rangle \cong \mathscr{S}_3$ inducing on $\mathrm{II}' = \langle a^2, b, c \rangle$ the following actions—where we are writing $a = v_1 v_3^{-1} v_2^2$, $b = v_2^2$, $c = t$; d: $a^2 \mapsto a^2$, $b \mapsto ba^2$, $c \mapsto c$; e: $a^2 \mapsto a^2$, $b \mapsto b$, $c \mapsto a^2 c$; f: $a^2 \mapsto a^2$, $b \mapsto c$, $c \mapsto b$; g: $a^2 \mapsto a^2$, $b \mapsto c$, $c \mapsto bc$. (Since the normalizer of II' modulo II acts faithfully on the 2^3-subgroup II' fixing $(v_1 v_3)^2$, it can be identified with a maximal parabolic subgroup 2^3: $L_2(2) \cong \mathscr{S}_4 \subset L_3(2)$ and the claim follows.)

Thus the action is uniquely determined on II, III, V, and VI. Moreover, the fusions of these groups in $4 \cdot L_3(4)$: 2_1 are also clear, since $v_1 v_3^{-1} v_2^2$ is fixed by every element in $4 * D_8 * D_8$: $\mathscr{A}_5$. It remains to determine explicitly the fusion between IV and V. We have, referring to the calculations in the proof of Lemma 1.12,

$$\alpha(v_1 v_3^{-1} v_2^2) = v_1^2 (v_2 v_3)^{-1},$$

$$\alpha((v_1 v_2)^2) = (v_1 v_3)^2,$$

$$\alpha(s^2 v_1^{-1}) = v_2 v_3 t s^3,$$

and conjugating by $v_1v_2v_3s^2$ gives the explicit map

$$v_1v_3^{-1}v_2^2 \mapsto v_1^2v_2v_3,$$
$$(v_1v_2)^2 \mapsto (v_1v_3)^2,$$
$$s^2v_1^{-1} \mapsto st.$$

Thus, the subgroup 4 fixed under the action of the normalizer of IV is $\langle v_2v_3^{-1}(v_1v_3)^2\rangle$.

Finally, VII is identified with $\langle v_1v_3^{-1}v_2^2, v_1^2, v_2^2\rangle$ in $4 * D_8 * D_8: \mathscr{A}_5$ by an element which fixes $v_1v_3^{-1}v_2^2$, and this completes the explicit descriptions of the fusions.

Now we recall one of the results of [M2] where the cohomology ring of $\mathrm{Syl}_2(\mathrm{O'N})$ is determined.

THEOREM 3.2. *$H^*(\mathrm{Syl}_2(\mathrm{O'N}); \mathbb{F}_2)$ is detected by restriction to the seven centralizers of the seven distinct conjugacy classes of elementary two subgroups.*

Consequently, we have

COROLLARY 3.3. *Identify the five subgroups* II, III, IV, V, VI *and identify* VII *with $\langle v_1v_3^{-1}v_2^2, v_1^2, v_2^2\rangle$ as discussed above. Then, $\nu \in H^*(\mathrm{Syl}_2(\mathrm{O'N}); \mathbb{F}_2)$ is in the restriction image from $H^*(\mathrm{O'N}; \mathbb{F}_2)$ if and only if its restrictions to these seven centralizers are all the same under these identifications and they belong to the invariant subrings in cohomology.*

Remark 3.4. Of course the same techniques may be applied to the groups Alp_2^2 and $4 \cdot L_3(4)$: 2_1. For example, with Alp_2^2 the fusions are as follows: β and α fuse II, III, IV, and V, but the remaining groups I, VI, and VII are not fused. Here the Weyl groups are D_8 for II, and $L_3(2)$, $\mathscr{A}_4$, and $\mathscr{A}_4$ for I, VI, and VII respectively.

Thus we get

COROLLARY 3.5. *$a \in H^*(\mathrm{Syl}_2(\mathrm{O'N}; \mathbb{F}_2)$ is in the image from $H^*(\mathrm{Alp}_2^2; \mathbb{F}_2)$ if and only if*

1. $\mathrm{Res}^{\mathrm{I}}(a) \in H^*(\mathrm{I}; \mathbb{F}_2)^{L_3(2)}$,
2. $\mathrm{Res}^{\mathrm{II}}(a) = \mathrm{Res}^{\mathrm{III}}(a) = \mathrm{Res}^{\mathrm{IV}}(a) = \mathrm{Res}^{\mathrm{V}}(a)$,
3. $\mathrm{Res}^{\mathrm{VI}}(a) \in H^*(\mathrm{VI}; \mathbb{F}_2)^{\mathscr{A}_4}$,
4. $\mathrm{Res}^{\mathrm{VII}}(a) \in H^*(\mathrm{VII}; \mathbb{F}_2)^{\mathscr{A}_4}$.

Remark 3.6. The situation for $4 \cdot L_3(4)$: 2_1 is that I is fused with VII, while II, III, V, and VI fuse with Weyl group $\mathscr{S}_4$. But IV is not fused with any other subgroup. The Weyl group of I is also $\mathscr{S}_4$.

Further discussion of the actual cohomology of O'N is deferred to [M2]. However, we now record the generators of $H^*(\mathrm{Syl}_2(\mathrm{O'N}); \mathbb{F}_2)$ and their restrictions to the seven conjugacy classes above from [M2].

First, we have three polynomial generators d_8, d_{12}, and $\tilde{d}_{14}$ which come from the Stiefel–Whitney classes of a representation of O'N. (Note that the first is indecomposable in $H^*(\mathrm{Syl}_2(\mathrm{O'N}); \mathbb{F}_2)$ but the second and third are sums of decomposables.)

The image of $H^*(4^3; \mathbb{F}_2)$ in $H^*(E_1; \mathbb{F}_2)$ is $\mathbb{F}_2[x_1(1)^2, x_1(2)^2, x_1(3)^2]$, and clearly

$$\begin{aligned}\mathrm{im}(\mathrm{res}^*: H^*(\mathrm{O'N}; \mathbb{F}_2) \to H^*(E_1; \mathbb{F}_2)) &\subseteq \mathrm{im}\big(\mathrm{res}^*: H^*(4^3 \cdot L_3(2); \mathbb{F}_2) \\ &\qquad \to H^*(E_1; \mathbb{F}_2)\big) \\ &\subseteq \mathbb{F}_2\big[x_1(1)^2, x_1(2)^2, x_1(3)^2\big]^{L_3(2)}.\end{aligned}$$

By a well known result of Dickson,

$$\mathbb{F}_2\big[x_1(1)^2, x_1(2)^2, x_1(3)^2\big]^{L_3(2)} = \mathbb{F}_2\big[d_4^2, d_6^2, d_7^2\big],$$

where

$$\begin{aligned}d_4 = {} & x_1(1)^4 + x_1(2)^4 + x_1(3)^4 + x_1(1)^2 x_1(2) x_1(3) + x_1(1) x_1(2)^2 x_1(3) \\ & + x_1(1) x_1(2) x_1(3)^2,\end{aligned}$$

$d_6 = Sq^2(d_4)$, $d_7 = Sq^3(d_4)$, and we have

THEOREM 3.7. *The image under restriction of $H^*(\mathrm{O'N}; \mathbb{F}_2)$ in $H^*(E_1; \mathbb{F}_2)$ is exactly the algebra $\mathbb{F}_2[d_4^2, d_6^2, d_7^2]$. Moreover, the image under restriction in $H^*(E_2; \mathbb{F}_2)$ contains a copy of this same algebra.*

Proof. To obtain this result, we use the Stiefel–Whitney classes of a representation. We recall a few basic facts. If $O(n)$ is the $n \times n$ orthogonal group over the real numbers, and $\rho: G \to O(n)$ is a group homomorphism, then the Stiefel–Whitney classes of ρ, denoted $w_i(\rho) \in H^i(BO(n), \mathbb{F}_2)$, are by definition $\rho^*(w_i)$, where $w_1, w_2, \ldots, w_n$ are the well-known symmetric polynomial generators for the cohomology of $BO(n)$. The total Stiefel–Whitney class is the formal sum $W(\rho) = w_1(\rho) + \cdots + w_n(\rho)$. Note that $W(\rho_1 \oplus \rho_2) = W(\rho_1)W(\rho_2)$. We refer the reader to [MS] and [AM] for more on this.

In [Co] the irreducible representations of O'N and their characters are given. We concentrate on χ_{16} which has dimension 64790. Any element of order 2 in O'N has character 70 under this representation. Moreover, the representation is real. Hence, when restricted to E_1 or E_2, the representation has the form $70\epsilon + (8090)R$ where ϵ is the trivial representation and R is the regular representation. The total Stiefel–Whitney class of R is $1 + d_4 + d_6 + d_7$, and consequently the total Stiefel–Whitney class of χ_{16}, when restricted to E_1 or E_2, is

$$(1 + d_4 + d_6 + d_7)^{2+8+\cdots} = 1 + d_4^2 + d_6^2 + d_7^2 + d_4^8 + \cdots .$$

In particular, in both $H^*(E_1; \mathbb{F}_2)$ and $H^*(E_2; \mathbb{F}_2)$ the classes d_4^2, d_6^2, and d_7^2 are in the image of restriction from $H^*(\mathrm{O'N}; \mathbb{F}_2)$. Equation (3.7) follows. ∎

We have included Table I describing the restrictions (on the generators) from the cohomology of the 2-Sylow subgroup of O'N to the cohomology of the centralizers of the seven distinct conjugacy classes of elementary abelian 2-subgroups.

To illustrate the techniques, note that $e_t\sigma_w + e_s\Gamma(x)_2 + L_3 + S_3 + V_3$ restricts to 0 at I, VII, while it restricts to d_3 at each of II–VI. Consequently, it is in the image of $H^*(\mathrm{O'N}; \mathbb{F}_2)$. Also, M_4 is directly seen from the table to lie in the image of $H^*(\mathrm{O'N}; \mathbb{F}_2)$. Since $Sq^2(d_3) = d_2 d_3$, this is also an element in the image, representing Y_5. It thus remains only to construct X_7. But this is represented by

$$d_3\left(e_t^4 + \left(e_s^2 + e_t^2\right)e_{t'}^2 + e_{t'}^4 + \sigma_w^2\right) + S_7 + L_3\Gamma(w)_4 + \Gamma(x)_2 S_5.$$

It follows that $H^*(\mathrm{O'N}; \mathbb{F}_2)$ is at least as large as asserted in the Introduction. To see that it can be no larger requires a close look at the possible invariant classes in the restriction image in $H^*(\mathrm{IV}; \mathbb{F}_2)$. The only possibility which is not already in the image from $H^*(\mathrm{O'N}; \mathbb{F}_2)$ is $\lambda d_2^5 d_3$, but to build this class requires $\Gamma(w)$, and by looking simultaneously in $H^*(V; \mathbb{F}_2)$ we see that this class cannot occur in the restriction image from $H^*(\mathrm{O'N}; \mathbb{F}_2)$. Further details will be found in [M2].

Remark. In a similar way we can construct classes in the image of $H^*(\mathrm{Alp}_2^2; \mathbb{F}_2)$. For example, $L_3 + e_s\Gamma(x)$ is in the image, as is $e_t^2 + e_t e_{t'} + e_{t'}^2$.

4. APPLICATIONS OF A FORMULA OF WEBB

We will now apply a formula due to Webb [We] to the poset complex of O'N discussed in Section 2. Let G be any finite group, $|\mathscr{A}_2(G)|$ be the poset complex as above, and denote the rank of G at 2 by $r(G)$. Then Webb's result can be restated as follows (see [AM, Chap. V] for details). Let C^i

TABLE I

The Restriction Images of the Generators of $H^*(\mathrm{Syl}_2(\mathrm{O'N}); \mathbb{F}_2)$ in the Seven Conjugacy Classes

Name: Group	I $\langle v_1, v_2, v_3 \rangle$	II $\langle j, v_2^2, t \rangle$	III $\langle j, v_2^2, v_2 t \rangle$	IV $\langle l, (v_1 v_3)^2, st \rangle$	V $\langle j, (v_1 v_2)^2, s^2 v_1^{-1} \rangle$	VI $\langle j, v_1 v_3 t, s^2 v_1^{-1} \rangle$	VII $\langle j, t, s^2 v_1^{-1} \rangle$
e_t	0	h	h	h	0	b	b
e_v	$e(1) + e(2) + e(3)$	0	h	0	h	h	h
e_s	0	0	0	h	0	0	0
$\Gamma(x)_2$	$(e(1) + e(3))e(2)$	0	0	λh	0	0	0
σ_w	$b(1) + b(2) + b(3)$	$S + \lambda h$	$S + \lambda h$	0	0	λb	λb
V_3	$\mathscr{R}$	0	0	0	$d_3 + h^2\lambda$	$\lambda h(b + h)$	$\lambda h(b + h)$
S_3	$\mathscr{R}$	0	0	$d_3 + h^2\lambda$	0	0	0
L_3	0	λh^2	λh^2	0	λh^2	$\lambda d_2 + d_3$	λd_2
$\Gamma(w)_4$	$(b(1) + b(3))b(2)$	0	0	$b(\lambda)^2 + b(\lambda)h^2$	S^2	0	0
M_4	$\tilde{M}_4$	λd_3	λd_3	λd_3	λd_3	λd_3	0
S_5	$\mathscr{T}$	0	0	hS^2	0	0	0
S_6	$\mathscr{N}$	0	0	$\lambda h S^2$	$\lambda h S^2$	0	0
S_7	$\tilde{X}_7$	0	0	N_1	N_1	0	0

Note. We have not included d_8, d_{12}, and $\bar{d}_{14}$ in this table. Their restrictions have already been described in (3.7). As regards the elements in the table we have used the following abbreviations: $\mathscr{R} = (b(1) + b(3))e(2) + (e(1) + e(3))b(2)$, $S = b(b + h)$, $N_1 = (d_3 + \lambda h^2)S^2$, $\mathscr{T} = (e(1) + e(2) + e(3))b(2)(b(1) + b(3))$, and $\mathscr{N} = (b(1) + b(2) + b(3))\tilde{M}_4 + Sq^2\tilde{M}_4 = b(3)(b(1) + b(2))e(1)e(2) + b(2)(b(1) + b(3))e(1)e(3) + b(1)(b(2) + b(3))e(2)e(3)$. For the groups II–VII, the element b is dual to the middle generator, e.g., v_2^2 in II, III; h is dual to the third generator, e.g., t in II, $v_2 t$ in III; λ is dual to the first generator; and $b(\lambda) = \beta_4(\lambda)$. Also, $j = v_1 v_3^{-1} v_2^2$ and $l = v_2 v_3^{-1}(v_1 v_3)^2$. This table is set up so that the fusion of the various subgroups preserves the form of expressions $b \mapsto b$, $h \mapsto h$, and $\lambda \mapsto \lambda$. Finally, as usual, $d_2 = h^2 + bh + b^2$, $d_3 = b^2 h + bh^2$.

denote the ith cellular chain group over $\mathbb{F}_2$ of the G-CW complex $|\mathscr{A}_2(G)|$; then the coboundary map defines a cochain complex $\mathscr{D}^*(q) = H^q(G, C^*)$:

$$H^q(G, C^0) \to H^q(G, C^1) \to \cdots \to H^q(G, C^{r(G)-1})$$

with $H^r(\mathscr{D}(q)) \cong 0$ if $r > 0$ and $H^0(\mathscr{D}(q)) \cong H^q(G)$.

In particular, taking Euler characteristics gives the formula

$$H^*(G;\mathbb{F}_2) \oplus \coprod_{\substack{\sigma_i \in |\mathscr{A}_2(G)|/G \\ i \text{ odd}}} H^*(G_{\sigma_i};\mathbb{F}_2) \cong \coprod_{\substack{\sigma_i \in |\mathscr{A}_2(G)|/G \\ i \text{ even}}} H^*(G_{\sigma_i};\mathbb{F}_2)$$

where σ_i is a representative for an orbit of i-cells in $|\mathscr{A}_2(G)|$. For the group O'N this formula simplifies drastically after cancellation and we have the following

THEOREM 4.1. *There is an isomorphism*

$$H^*(\mathrm{O'N};\mathbb{F}_2) \oplus H^*(4^3 \cdot \mathscr{S}_4;\mathbb{F}_2) \cong H^*(4 \cdot L_3(4):2;\mathbb{F}_2) \oplus H^*(4^3 \cdot L_3(2);\mathbb{F}_2).$$

We can draw several conclusions from the structure of the poset space and its quotient.

THEOREM 4.2. *The* 2*-local structure of* O'N *defines a surjective homomorphism*, π, *from an amalgamated product to* O'N,

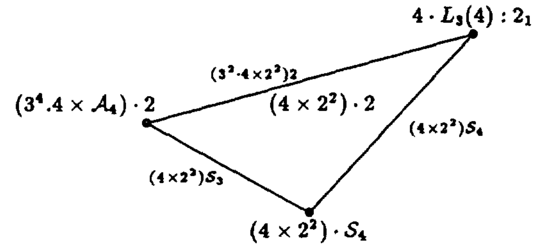
$$\pi : 4 \cdot L_3(4):2_1 *_{4^3 \cdot \mathscr{S}_4} 4^3 \cdot L_3(2) \to \mathrm{O'N},$$

and π *induces isomorphisms in* mod(2) *homology and cohomology.*

Proof. By a theorem of Brown [B] and Webb's formula we have an isomorphism $H^*(|\mathscr{A}_2(\mathrm{O'N})| \times_{\mathrm{O'N}} E_{\mathrm{O'N}};\mathbb{F}_2) \cong H^*(\mathrm{O'N};\mathbb{F}_2)$ where $E_{\mathrm{O'N}}$ is the universal cover of the classifying space of O'N. Consider the projection

$$q : |\mathscr{A}_2(\mathrm{O'N})| \times_{\mathrm{O'N}} E_{\mathrm{O'N}} \to |\mathscr{A}_2(\mathrm{O'N})|/\mathrm{O'N}.$$

The inverse image of every open simplex in $|\mathscr{A}_2(\mathrm{O'N})|/\mathrm{O'N}$ is the product of that open simplex with the classifying space of the isotropy group of that simplex, and this gives a decomposition of a space having the same homology as O'N. Now, consider the leftmost triangle.



Note that the top edge and the entire simplex have isotropy groups with the same mod(2) cohomology. Hence, they cancel out geometrically as well as algebraically, and we can remove the edge and the simplex. Note also that the leftmost vertex and the lower edge also have isotropy groups with the same mod(2) cohomology, and so we are left with only the right edge. A similar consideration holds for the curved edge with isotropy group $(4 \times 2^2)2^2$ which is exactly the same isotropy group as that of the "curved" triangle, so the two cancel out. We may now cancel the lower edge of the central triangle (with isotropy group $(4 \times 2^2)D_8$) and the interior of the triangle. That leaves us with the edge with isotropy group $(4 \times 2^2)\mathscr{S}_4$ and the rightmost triangle with isotropy group $\mathrm{Syl}_2(\mathrm{O'N})$. This triangle cancels with its lower edge. The edge cancels with its lower vertex, and finally the lower right-hand edge cancels with the vertex with isotropy group $4^3\mathscr{S}_4$. This leaves only the diagram

$$\overset{4 \cdot L_3(4): 2_1}{\bullet} \underset{4^3 : \mathscr{S}_4}{\text{———}} \overset{4^3 \cdot L_3(2).}{\bullet} \tag{4.3}$$

Algebraically what we have is that the chain complex $\mathscr{D}^*(q)$ is homologous to the one-dimensional sub-cochain complex

$$H^q(4 \cdot L_3(4): 2_1) \oplus H^q(4^3 \cdot L_3(2)) \to H^q(4^3 \cdot \mathscr{S}_4),$$

hence from Webb's theorem we conclude that the sum of the restriction maps is onto, and that its kernel is precisely $H^q(\mathrm{O'N})$. On the other hand, if we denote by Γ the corresponding amalgamated product, the associated Mayer–Vietoris sequence also implies that the kernel is isomorphic to $H^q(\Gamma)$. Consequently, by rank considerations and the fact that the finite subgroups in Γ are mapped isomorphically into O'N by the projection π, we obtain the desired result. ▮

Identifying the kernel of the sum of the restrictions, we obtain

COROLLARY 4.4. *$H^*(\mathrm{O'N}; \mathbb{F}_2)$ is given explicitly as the intersection in $H^*(H_\beta; \mathbb{F}_2)$ of the images,* $\mathrm{im}(H^*(4 \cdot L_3(4)\colon 2_1; \mathbb{F}_2))$ *and* $\mathrm{im}(H^*(4^3 \cdot L_3(2); \mathbb{F}_2))$,

$$\begin{aligned} H^*(\mathrm{O'N}; \mathbb{F}_2) &= \mathrm{im}\big(H^*(4 \cdot L_3(4): 2_1; \mathbb{F}_2)\big) \cap \mathrm{im}\big(H^*(4^3 \cdot L_3(2); \mathbb{F}_2)\big) \\ &\subset H^*(H_\beta; \mathbb{F}_2). \end{aligned}$$

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