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# Automorphisms and Cohomology of Discrete Groups

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## INTRODUCTION

Let $\Gamma$ denote a discrete group of finite cohomological dimension. Calculating the cohomology of such groups is notoriously difficult, involving complicated geometric information attached to the group. For example, the cohomology of torsion-free arithmetic groups involves delicate questions about symmetric spaces and number theory. Perhaps the key difficulty lies in that there is no practical method for building up the cohomology of $\Gamma$ from that of its subgroups (such as one can do for finite groups).

In this paper we outline a method for constructing non-trivial classes in $H^*(\Gamma, \mathbb{F}_p)$ (where $\mathbb{F}_p$ denotes a field with $p$ elements) based on the use of finite automorphism groups of $\Gamma$. Given an explicit presentation for $\Gamma$, $\mathrm{Aut}(\Gamma)$ can often be approached, and in particular its finite subgroups are sometimes accessible. Furthermore, it is an elementary observation that many interesting classes of groups admit numerous finite symmetries. An obvious but very important example is given by a *normal* torsion-free subgroup $\Gamma$ in an arithmetic group $U$; *any* finite subgroup $K \subseteq U$ will act on it via conjugation.

Given $G \subseteq \mathrm{Aut}(\Gamma)$, we can form the semidirect product $\overline{\Gamma} = \Gamma \times_\varphi G$. Let $\overline{H} \subseteq \overline{\Gamma}$ denote a finite subgroup mapping onto $G$ under the natural projection map. Then $C_\Gamma(\overline{H}) = C_{\overline{\Gamma}}(\overline{H}) \cap \Gamma \subseteq \Gamma$ and in the particular case when $\overline{H} = \{(1, x) | x \in G)\}$, we have that

$$C_\Gamma(\overline{H}) = \Gamma^G = \{\gamma \in \Gamma | g \cdot \gamma = \gamma \text{ for all } g \in G\}.$$

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After recalling that there is a $1-1$ correspondence between the $\Gamma$-conjugacy classes of such subgroups $\overline{H} \subseteq \overline{\Gamma}$ and the non-abelian cohomology $H^1(G,\Gamma)$ we prove the following

THEOREM 3.3. *If $P \subseteq \mathrm{Aut}(\Gamma)$ is a finite $p$-group, then*

$$\dim_{\mathbb{F}_p} H^*(\Gamma, \mathbb{F}_p) \geq \sum_{\overline{H} \in H^1(P,\Gamma)} \dim_{\mathbb{F}_p} H^*\left(C_\Gamma(\overline{H}), \mathbb{F}_p\right).$$

Here $\dim_{\mathbb{F}_p} H^*(\ ,\mathbb{F}_p)$ denotes the total dimension of the mod $p$ cohomology. If $M$ is a set, we denote its cardinality by $\#[M]$; then as a corollary we obtain that

$$\dim_{\mathbb{F}_p} H^*(\Gamma, \mathbb{F}_p) \geq \max\left\{\#\left[H^1(P,\Gamma)\right], \dim_{\mathbb{F}_p} H^*\left(\Gamma^P, \mathbb{F}_p\right)\right\}.$$

The results above clearly indicate that each of the subgroups $C_\Gamma(\overline{H})$ will contribute to the mod $p$ cohomology of $\Gamma$, in particular the cohomology of the fixed subgroup $\Gamma^P$ will produce cohomology for $\Gamma$, in what can be thought of as a group-theoretic version of a classical result due to P. Smith. This has interesting consequences, some of which we describe in Section 3.

To illustrate this, we apply it to the case of a level $q$ congruence subgroup ($q$ an odd prime) $\Gamma_n(q) \subseteq \mathrm{SL}_n(\mathbb{Z})$ and $P = \langle A \rangle$, where $A$ is the involution defined by

$$A_{ij} = \begin{cases} 0, & i \neq j \\ -1, & i = j, 1 \leq i \leq n-1 \\ 1, & i = j = n. \end{cases}$$

Given $B \in \mathrm{SL}_n(\mathbb{Z})$, define an action via conjugation,

$$\hat{B}_{ij} = \begin{cases} -B_{ij} & i = n \text{ or } j = n \text{ but not both} \\ B_{ij} & \text{otherwise}. \end{cases}$$

Then the classes in $H^1(P, \Gamma_n(q))$ are equal to

$$\left\{B \in \Gamma_n(q) | \hat{B} = B^{-1}\right\} / \sim\ ,$$

where $B_1 \sim B_2$ if there exists $C \in \Gamma(q)$ with $C^{-1} B_1 \hat{C} = B_2$. For each class $[B] \in H^1(P, \Gamma_n(q))$ we have that the corresponding centralizer is the subgroup

$$C([B]) = \left\{D \in \Gamma_n(q) | B\hat{D} = DB\right\}.$$

Then our result indicates that

$$\dim_{\mathbb{F}_2} H^*\left(\Gamma_n(q), \mathbb{F}_2\right) \geq \sum_{[B]} \dim_{\mathbb{F}_2} H^*\left(C[B], \mathbb{F}_2\right).$$

In particular $C([I]) \cong \Gamma_{n-1}(q)$ from which we derive the rather interesting fact that

$$\dim_{\mathbb{F}_2} H^*\left(\Gamma_n(q), \mathbb{F}_2\right) \geq \dim_{\mathbb{F}_2} H^*\left(\Gamma_{n-1}(q), \mathbb{F}_2\right).$$

Using an automorphism of order $p$, an odd prime, we show in Application 3.8 that if $n = k(p-1) + t$, with $0 \leq t < p-1$, then

$$\dim_{\mathbb{F}_p} H^*\left(\Gamma_n(q), \mathbb{F}_p\right) \geq 2^{k \cdot ((p-3)/2)} \cdot \dim_{\mathbb{F}_p} H^*\left(\Gamma_t(q), \mathbb{F}_p\right).$$

Our theorem seems to be a basic result for demonstrating the existence of non-trivial cohomology for $\Gamma$. However, it can also be used conversely to prove the finiteness of $H^1(P, \Gamma)$ and $H^*(C_\Gamma(\overline{H}), \mathbb{F}_p)$.

In Section 5 we describe a formula for computing the number of conjugacy classes of elements of finite order in a semidirect product. More precisely, we prove

THEOREM 5.1. *If $\Gamma$ is a torsion-free discrete group and $G \subseteq \operatorname{Aut}(\Gamma)$ is a finite automorphism group, then $\overline{\Gamma} = \Gamma \times_\varphi G$ has the following number of conjugacy classes of elements of finite order*:

$$\sum_{\substack{(g) \\ \text{conjugacy} \\ \text{classes in } G}} \#\left[H^1(\langle g \rangle, \Gamma)/C_G(g)\right].$$

Here $C_G(g) =$ centralizer of $g$ in $G$, which acts on $H^1(\langle g \rangle, \Gamma)$ (see Section 5 for details).

The proof of Theorem 3.3 is an application of Smith theory to a construction of Serre for the group $\overline{\Gamma}$. More precisely, we construct an *admissible* $\overline{\Gamma}$-complex $X$ and analyze the $G$-action on $X/\Gamma$ (see Section 1 for definitions). The sets $H^1(K, \Gamma)$ arise naturally from fixed-point data, as do the subgroups $C_\Gamma(\overline{H})$. Although Theorem 5.1 is purely algebraic, it is closely related to the calculation of $K_G^*(X/\Gamma)$, as described in [A].

This work is motivated by the paper of Rohlfs and Schwermer [RS], where they use finite automorphisms to construct non-trivial cohomology via intersection theory. Our main contribution is the introduction of cohomological ideas which produce non-zero classes in a more general context, without using products. From our point of view the ring structure of the $H^*(C_\Gamma(\overline{H}))$ and possible intersections are best expressed homologically, as we do in Section 4. We think that the results in this paper may help put their results in perspective, showing how specific information about arithmetic groups is required in cohomology calculations. The subgroups $C_\Gamma(\overline{H})$ yield "special cycles" which are in fact simply arising from $BC_\Gamma(\overline{H}) \subseteq B\Gamma$, and they contribute non-trivial cohomology without the need of any intersection arguments.

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## 1. PRELIMINARIES

In this section we provide the necessary algebraic and topological background.

To begin we will assume that $\Gamma$ is a discrete group of finite cohomological dimension and that $G \underset{\varphi}{\hookrightarrow} \mathrm{Aut}(\Gamma)$ is a finite automorphism group. We will need the following algebraic concept.

DEFINITION 1.1. The non-abelian cohomology $H^1(G, \Gamma)$ is the set of equivalence classes of functions $\theta$: $G \to \Gamma$ satisfying $\theta(g_1 g_2) = \theta(g_1)\varphi(g_1)[\theta(g_2)]$ for all $g_1, g_2 \in G$, where $\theta$ is said to be equivalent to $\theta'$ if there exists $\gamma \in \Gamma$ with

$$\theta(g) = \gamma\theta'(g)\varphi(g)[\gamma^{-1}]$$

for all $g \in G$.

Note in particular the distinguished element $1 \in H^1(G, \Gamma)$, corresponding to the trivial homomorphism.

Using $\Gamma$ and $G$, we can construct the semidirect product $\Gamma \times_\varphi G$, consisting of the set of pairs $(\gamma, g) \in \Gamma \times G$ with product

$$(\gamma_1, g_1)(\gamma_2, g_2) = (\gamma_1\varphi(g_1)[\gamma_2], g_1 g_2).$$

By construction $\Gamma \lhd \Gamma \times_\varphi G$ with quotient $G$, hence $\Gamma \times_\varphi G$ has finite *virtual cohomological dimension* (i.e., it has a subgroup of finite index which has finite cohomological dimension).

Next we recall a construction due to Serre (see [B, p. 190] for details).

THEOREM 1.2. *If $Q$ is a discrete group of finite v.c.d., then there exists a finite dimensional $\Gamma$-complex $X$ with the following properties*

(1) $X^H \neq \varnothing \Leftrightarrow H \subseteq \Gamma$ *is finite.*

(2) *For all $H \subseteq \Gamma$ finite, $X^H$ is contractible.*

From now on we denote by $X$ the complex associated to $\Gamma \times_\varphi G$. Note that $\Gamma$ acts freely on $X$, a contractible space and hence $X/\Gamma \simeq B\Gamma$, the classifying space of $\Gamma$. We recall a basic fixed-point formula due to K. Brown [B] applied to this particular case: if $K \subseteq G$ is any subgroup, then

$$(X/\Gamma)^K = \coprod_{H \in \mathscr{C}} X^H/\Gamma \cap N_{\Gamma \times_\varphi G}(H), \tag{1.3}$$

where $\mathscr{C}$ is a set of representatives for the $\Gamma$-conjugacy classes of finite subgroups in $\Gamma \times_\varphi G$ whose image in $G$ is $K$. Note that by the defining properties of $X$, we have

$$(X/\Gamma)^K \simeq \coprod_{H \in \mathscr{C}} B\big(\Gamma \cap N_{\Gamma \times_\varphi G}(H)\big). \tag{1.4}$$

Finally, we recall a basic result from Smith theory (see [AP, p. 210] for details):

THEOREM 1.5. *Let $Y$ be a finite dimensional complex with an action of a finite $p$-group $P$; then*

$$\sum_{i=0}^{\dim Y} \dim_{\mathbb{F}_p} H^i\big(Y, \mathbb{F}_p\big) \geq \sum_{i=0}^{\dim Y^P} \dim_{\mathbb{F}_p} H^i\big(Y^P, \mathbb{F}_p\big).$$

Note in particular that if $H^*(Y, \mathbb{F}_p)$ has finite total dimension, then $Y^P$ has a finite number of components, each of which has finite total dimension. The proof is based on using a central subgroup in $P$ of order $p$ and applying induction on $|P|$ to reduce it to the case of $\mathbb{Z}/p$. In that case equality occurs only under rather restrictive conditions.

## 2. SEMI-DIRECT PRODUCTS AND NON-ABELIAN COHOMOLOGY

To simplify notation, let $\overline{\Gamma} = \Gamma \times_\varphi G$. We now state the main result in this section, which is a basic fact (see [S]) which we include for completeness.

THEOREM 2.1. *Let $\pi\colon \overline{\Gamma} \to G$ be the natural projection map. There is a one-to-one correspondence between $\Gamma$-conjugacy classes of finite subgroups of $\overline{\Gamma}$ mapping onto $K \subseteq G$ and $H^1(K, \Gamma)$.*

*Proof.* Let $\overline{K} \subseteq \overline{\Gamma}$ be a finite subgroup such that $\pi(\overline{K}) = K$; note that $\pi$ maps $\overline{K}$ isomorphically onto $K$, as $\ker \pi = \Gamma$ is torsion-free. We can describe $\overline{K}$ as

$$\overline{K} = \left\{\left(\gamma_x^{\overline{K}}, x\right) \middle| x \in K,\ \gamma_x^{\overline{K}} \in \Gamma\right\}$$

for some function $\gamma^{\bar K}\colon K \to \Gamma$. Note that

$$\left(\gamma_x^{\bar K}, x\right)\left(\gamma_y^{\bar K}, y\right) = \left(\gamma_x^{\bar K}\varphi(x)\left[\gamma_y^{\bar K}\right], xy\right)$$

and hence

$$\gamma_{xy}^{\bar K} = \gamma_x^{\bar K}\varphi(x)\left[\gamma_y^{\bar K}\right],$$

which simply means that $\gamma^{\bar K}$ is a *cocycle*.

Hence we can define a function

$$\rho\colon \left\{\begin{array}{c}\bar K \subseteq \Gamma \text{ finite}\\ \text{with}\\ \pi(\bar K) = K\end{array}\right\} \to \left\{\begin{array}{c}\text{cocycles}\\ K \to \Gamma\end{array}\right\}$$

via $\rho(\bar K) = \gamma^{\bar K}$. Conversely, given any cocycle $\xi\colon K \to \Gamma$, we consider

$$K(\xi) = \left\{(\xi_x, x) \middle| x \in K\right\}.$$

We claim that $K(\xi)$ is a finite subgroup in $\bar\Gamma$, isomorphic to $K$. Indeed, if $(\xi_x, x), (\xi_y, y) \in K(\xi)$, then

$$\begin{aligned}(\xi_x, x)^{-1} &= \left(\varphi(x^{-1})\left[\xi_x^{-1}\right], x^{-1}\right)\\ &= \left(\xi_{x^{-1}}, x^{-1}\right)\end{aligned}$$

(here we use the fact that $1 = \xi_1 = \xi_{xx^{-1}} = \xi_x\varphi(x)[\xi_{x^{-1}}]$, from which $\xi_x^{-1} = \varphi(x)[\xi_{x^{-1}}]$ and $\varphi(x^{-1})[\xi_x^{-1}] = \xi_{x^{-1}}$) and

$$\begin{aligned}(\xi_x, x)\cdot(\xi_y, y) &= \left(\xi_x\varphi(x)\left[\xi_y\right], xy\right)\\ &= \left(\xi_{xy}, xy\right)\end{aligned}$$

as $\xi$ is a cocycle. It remains to show that this association induces a bijection modulo $\Gamma$-conjugacy and equivalence of cocycles, respectively.

Assume first that $\bar K, \bar K'$ are $\Gamma$-conjugate, i.e., there exists $(\xi, 1) \in \Gamma$ such that $(\xi, 1)\bar K(\xi, 1)^{-1} = \bar K'$. Hence we have

$$\begin{aligned}&(\xi, 1)\left(\gamma_x^{\bar K}, x\right)(\xi, 1)^{-1} = \left(\gamma_x^{\bar K'}, x\right) \qquad \text{for all } \left(\gamma_x^{\bar K}, x\right) \in \bar K.\\ &\Rightarrow \left(\xi\gamma_x^{\bar K}\varphi(x)\left[\xi^{-1}\right], x\right) = \left(\gamma_x^{\bar K'}, x\right)\\ &\Rightarrow \gamma_x^{\bar K'} = \xi\gamma_x^{\bar K}\varphi(x)\left[\xi^{-1}\right] \qquad \text{for all } x \in K.\end{aligned}$$

This means precisely that $\gamma^{\bar K} = \gamma^{\bar K'}$ as elements in $H^1(K, \Gamma)$. The converse can be proved similarly and our proof is complete. ∎

*Remark*. Under this correspondence, the trivial cocycle $1 \in H^1(K, \Gamma)$ corresponds to the subgroup $\overline{K} = \{(1, x) | x \in K\} \subseteq \overline{\Gamma}$. This will be a distinguished class in our considerations.

Given $\overline{K} \subseteq \overline{\Gamma}$ mapping onto $K$, we define its centralizer in $\overline{\Gamma}$

$$C_{\overline{\Gamma}}(\overline{K}) = \left\{(\gamma, y) \in \overline{\Gamma} \,|(\gamma, y)\mu(\gamma, y)^{-1} = \mu \ \forall \mu \in \overline{K}\right\}.$$

Note that this simplifies to yield

$$(\gamma, y)\left(\gamma_x^{\overline{K}}, x\right)(\gamma, y)^{-1} = \left(\gamma_x^{\overline{K}}, x\right)$$

$$\left(\gamma\varphi(y)\left[\gamma_x^{\overline{K}}\right]\varphi(yxy^{-1})\left[\gamma^{-1}\right], yxy^{-1}\right) = \left(\gamma_x^{\overline{K}}, x\right)$$

$$\gamma\varphi(y)\left[\gamma_x^{\overline{K}}\right]\varphi(x)\left[\gamma^{-1}\right] = \gamma_x^{\overline{K}}, \qquad yxy^{-1} = x$$

for all $(\gamma_x^{\overline{k}}, x) \in \overline{K}$. Note the special case when $y = 1$, i.e., $(\gamma, y) \in C_{\overline{\Gamma}}(\overline{K}) \cap \Gamma = C_{\Gamma}(\overline{K})$, then the second condition is superfluous and we have

$$\gamma\gamma_x^{\overline{K}}\varphi(x)\left[\gamma^{-1}\right] = \gamma_x^{\overline{K}}.$$

For later use we note that if $N_{\overline{\Gamma}}(\overline{K})$ denotes the normalizer of $\overline{K}$ in $\overline{\Gamma}$, then $N_{\overline{\Gamma}}(\overline{K}) \cap \Gamma = C_{\Gamma}(\overline{K})$; indeed if $(\gamma\gamma_x^{\overline{K}}\varphi(x)[\gamma^{-1}], x) \in \overline{K}$, then necessarily we recover the identity above.

Note that if $\xi^{\overline{K}} = 1$, then

$$C_{\Gamma}(\overline{K}) = \{\gamma \in \Gamma | \varphi(x)[\gamma] = \gamma \ \forall x \in K\},$$

the subgroup of "fixed points" in $\Gamma$ under the action of $K$. We will denote this group by $\Gamma^K$.

## 3. FIXED POINTS, CENTRALIZERS, AND COHOMOLOGY

Let $X$ be the finite dimensional $\overline{\Gamma}$-complex described in Section 1. Combining (1.4) with the observation at the end of Section 2 and Theorem 2.1, we obtain

PROPOSITION 3.1. *Let $K \subseteq G$ be any subgroup, then*

$$(X/\Gamma)^K \simeq \coprod_{\overline{K} \in H^1(K, \Gamma)} BC_{\Gamma}(\overline{K}).$$

Note that we have established a $1-1$ correspondence

$$\pi_0\big((X/\Gamma)^K\big) \leftrightarrow H^1(K,\Gamma).$$

This result appears in a very special context in [RS]. The explanation for this is that in fact there is an underlying action of a semidirect product on the symmetric spaces which they consider.

In addition we have

COROLLARY 3.2. *If $X/\Gamma$ is compact, then for every $G \subseteq \mathrm{Aut}(\Gamma)$ finite, $H^1(G,\Gamma)$ is a finite set and $BC_\Gamma(\overline{G})$ admits a compact model for all $\overline{G} \in H^1(G,\Gamma)$.*

We can now prove one of our main results. For a space $X$, let

$$\dim_{\mathbb{F}_p} H^*(X,\mathbb{F}_p) = \sum_{i=0}^{\dim X} \dim_{\mathbb{F}_p} H^i(X,\mathbb{F}_p).$$

THEOREM 3.3. *Let $\Gamma$ be a discrete group of finite cohomological dimension and $P \subset \mathrm{Aut}(\Gamma)$ a finite $p$-group. Then we have that*

$$\dim_{\mathbb{F}_p} H^*(\Gamma,\mathbb{F}_p) \geq \sum_{\overline{H}\in H^1(P,\Gamma)} \dim_{\mathbb{F}_p} H^*\big(C_\Gamma(\overline{H}),\mathbb{F}_p\big),$$

*and in particular*

$$\dim_{\mathbb{F}_p} H^*(\Gamma,\mathbb{F}_p) \geq \max\Big\{\#H^1(P,\Gamma), \dim_{\mathbb{F}_p} H^*\big(\Gamma^P,\mathbb{F}_p\big)\Big\}.$$

*Proof.* The proof is a straightforward application of Theorem 1.5 and Proposition 3.1. ∎

To make this meaningful, we assume from now on that $H^*(\Gamma,\mathbb{F}_p)$ is totally finite. This will imply that $H^1(P,\Gamma)$ is finite for all $P \subseteq \mathrm{Aut}(\Gamma)$ and that each $C_\Gamma(\overline{H})$ is homologically finite mod $p$. Of course the right hand side of Theorem 3.3 achieves its maximum possible value (as we vary the automorphism group) when $|P| = p$. We now give a few applications.

*Application* 3.4. $\Gamma = F(n)$, free group on $n$ generators. In this case we obtain that for any $P \subseteq \mathrm{Aut}(\Gamma)$,

$$n+1 \geq \sum_{\overline{H}\in H^1(P,\Gamma)} \big(\mathrm{rank}\, C_\Gamma(\overline{H}) + 1\big)$$

and in particular

$$n + 1 \geq \max\left\{ \# H^1(P, \Gamma), \operatorname{rank} \Gamma^P + 1 \right\}.$$

*Application* 3.5. Suppose that $\Gamma$ is a discrete group with the mod $p$ homology of a point. Then $H^1(P, \Gamma) = \{1\}$, and $\Gamma^P$ is also mod $p$ homologous to a point. There are examples of groups $\Gamma$ (of finite c.d.) satisfying such a condition. Perhaps the simplest one is Higman's group, for which a presentation can be given as

$$\Gamma = \left\langle x_0, x_1, x_2, x_2 | x_{i-1} x_i x_{i-1}^{-1} = x_i^2, i = 0, 1, 2, 3 \bmod (4) \right\rangle.$$

This group evidently has an automorphism of order 4 (simply rotating the generators) and we deduce that its fixed point group must be mod 2 acyclic.

*Application* 3.6. Suppose that $\Gamma$ has the same mod $p$ cohomology as a sphere. Then, for any $P \subseteq \operatorname{Aut}(\Gamma)$, we have either $H^1(P, \Gamma) = \{1\}$ and $\Gamma^P$ has the mod $p$ homology of a sphere, or $\#[H^1(P, \Gamma)] = 2$ and for both classes $[\overline{H}] \in H^1(P, \Gamma)$ we have that $C_\Gamma(\overline{H})$ is mod $p$ acyclic. Examples of such groups $\Gamma$ can be easily provided. Let $r, s, t$ be positive, pairwise relatively prime integers satisfying $1/r + 1/s + 1/t < 1$. Denote by $T$ the group generated by elements $\gamma_1, \gamma_2, \gamma_3$ subject to the relations

$$\gamma_1^r = \gamma_2^s = \gamma_3^t = \gamma_1 \gamma_2 \gamma_3.$$

Then if $\Gamma = [T, T] \subseteq T$, it is a well-known fact [Mi] that $\Gamma$ has the integral homology of a 3-sphere and in fact has cohomological dimension three.

*Application* 3.7. Let $\Gamma_n(p) \subseteq SL_n(\mathbb{Z})$ denote the level $p$ congruence subgroup, for $p$ an odd prime. It is known that the $\Gamma_n(p)$ have finite cohomological dimension. Now if $A \in GL_n(\mathbb{Z})$ is a finite subgroup then it acts on $\Gamma_n(p)$ via conjugation.

Let

$$A = \begin{pmatrix} -1 & & & & \\ & -1 & & & \\ & & \ddots & & \\ & & & -1 & \\ & & & & 1 \end{pmatrix},$$

an element of order two which clearly acts non-trivially on $\Gamma_n(p)$. It is direct to verify that $\Gamma_n(p)^{\langle A\rangle} \cong \Gamma_{n-1}(p)$ and so Theorem 3.3 yields

$$\dim_{\mathbb{F}_2} H^*\big(\Gamma_n(p), \mathbb{F}_2\big) \geq \dim_{\mathbb{F}_2} H^*\big(\Gamma_{n-1}(p), \mathbb{F}_2\big)$$

for all $n \geq 2$.

One should note however that $H^1(\langle A\rangle, \Gamma_n(p))$ may have more than one element. The cocycles can be described as $\theta$: $\langle A\rangle \to \Gamma_n(p)$ such that

$$A\theta(A)A = \theta(A)^{-1}$$

or more precisely if for $B \in \mathrm{GL}_n(\mathbb{Z})$ we denote

$$\hat{B}_{ij} = \begin{cases} -B_{ij}, & i = n \text{ or } j = n \text{ but not both} \\ B_{ij}, & \text{otherwise}, \end{cases}$$

then

$$Z^1(\langle A\rangle, \Gamma_n(p)) = \big\{B \in \Gamma_n(p) | \hat{B} = B^{-1}\big\}.$$

Two elements $B_1, B_2$ are equivalent if there exists $C \in \Gamma_n(p)$ with

$$C^{-1}B_1\hat{C} = B_2.$$

The cocycle $B$ corresponds to the subgroup $\{(B, A), (1, 1)\} \subseteq \overline{\Gamma}$, with

$$C_\Gamma(B, A) = \big\{D \in \Gamma_n(p) | B\hat{D} = DB\big\}.$$

It should be possible to use the above to obtain explicit numerical lower bounds on $\dim H^*(\Gamma_n(p), \mathbb{F}_2)$.

*Application* 3.8. Consider $\Gamma \in \mathrm{GL}_{p-1}(\mathbb{Z})$ an element of order $p$. Then, for $p$ odd $\geq 5$,

$$C_{\mathrm{GL}}(T) \cong \mathbb{Z}^{((p-3)/2)} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/p.$$

Let $\Gamma = \Gamma_{p-1}(q)$, $q$ any odd prime. Then, from the extension

$$1 \to \Gamma \to \mathrm{GL}_{p-1}(\mathbb{Z}) \xrightarrow{\pi} \mathrm{GL}_{p-1}(\mathbb{F}_q) \to 1$$

and the action of $H = \langle T\rangle$ by conjugation, we infer that there is an exact sequence

$$1 \to \Gamma^H \to \mathbb{Z}^{((p-3)/2)} \oplus \mathbb{Z}/p \oplus \mathbb{Z}/2 \to C_G(\pi(H)),$$

where $G = \mathrm{GL}_{p-1}(\mathbb{F}_q)$. We deduce that

$$\Gamma^H \cong \mathbb{Z}^{((p-3)/2)}$$

and hence that, for all odd $q$,

$$\dim_{\mathbb{F}_p} H^*\big(\Gamma_{p-1}(q), \mathbb{F}_p\big) \geq 2^{((p-3)/2)}.$$

Now let $n$ be any integer larger than $p-1$, and write $n = k(p-1)+t$, where $0 \leq t < p-1$. Consider the $n \times n$ block matrix $S$ with $T$ in the upper left hand corner, and then $n-p+1 \times n-p+1$ identity matrix in the bottom right. It is not hard to see that the centralizer of this matrix in $\mathrm{GL}_n(\mathbb{Z})$ is precisely the diagonal product $C_{\mathrm{GL}_{p-1}(\mathbb{Z})}(T) \times \mathrm{GL}_{n-p+1}(\mathbb{Z})$. Using the previous result we can deduce that

$$\dim_{\mathbb{F}_p} H^*\big(\Gamma_n(q), \mathbb{F}_2\big) \geq 2^{k\cdot((p-3)/2)} \cdot \dim_{\mathbb{F}_p} H^*\big(\Gamma_t(q), \mathbb{F}_p\big).$$

Given the cohomological applications in this section, it is quite natural to expect some consequences from the well-known localization methods (see [AP]). For example, we have:

THEOREM 3.9. *Let* $P = (\mathbb{Z}/p)^r \underset{\varphi}{\hookrightarrow} \mathrm{Aut}(\Gamma)$, *where* $\Gamma$ *is a discrete group of finite cohomological dimension. Then there exists a class* $\mathscr{C} \in H^*(P, \mathbb{F}_p)$ *such that the localized map induced by inclusion*

$$\mathscr{C}^{-1} H^*\big(\Gamma \times_T P, \mathbb{F}_p\big) \to \mathscr{C}^{-1}\Bigg[\bigoplus_{\overline{H} \in H^1(P,\Gamma)} H^*\Big(C_\Gamma(\overline{H}), \mathbb{F}_p\Big)\Bigg] \otimes H^*\big(P, \mathbb{F}_p\big)$$

*is an isomorphism. In particular if* $H^1(P, \Gamma) = \{1\}$, *the inclusion* $\Gamma^P \hookrightarrow \Gamma$ *induces an isomorphism* $H^*(\Gamma \times_\varphi P, \mathbb{F}_p) \to H^*(\Gamma^P \times P, \mathbb{F}_p)$ *after localizing.*

It is evident that in many interesting situations this formula can be used to relate the structure of $H^*(\Gamma, \mathbb{F}_p)$ to that of the $H^*(C_\Gamma(\overline{H}), \mathbb{F}_p)$ (and in particular $H^*(\Gamma^P, \mathbb{F}_p)$). For example, if $B\Gamma$ is a Poincaré Duality space, it follows from Proposition 4.1 that if $H^*(\Gamma, \mathbb{F}_p)$ is totally non-homologous to zero in $H^*(\Gamma \times_T P, \mathbb{F}_p)$, then each $BC_\Gamma(\overline{H})$ will be a Poincaré Duality space. One may conjecture more generally that $\Gamma^P$ must be a Poincaré Duality group if $\Gamma$ is.

More delicate comparisons between the cohomology of $\Gamma$ and that of $\Gamma^P$ (as well as the other components) are possible in several cases, in addition one can consider the action of the Steenrod algebra.

## 4. INTERSECTIONS

In this section we will summarize how our methods can be used together with intersection theory to produce non-trivial classes in the cohomology of a discrete group. This is based on and motivated by the work of Rohlfs and Schwermer [RS]. We present a simplified account of this in a purely topological setting, which has the advantage of wider applicability, although the arithmetic case is of preponderant interest.

To begin we need a little background in homology of manifolds. Let us assume that $Y^N$ is a connected $N$-manifold, $X^{m_1} \hookrightarrow Y^N$, $Z^{m_2} \hookrightarrow Y^N$ compact submanifolds such that $\emptyset \neq X \cap Z$ is a compact $L$-manifold, and $m_1 + m_2 = N$. Assume in addition that they are all $R$-oriented ($R$ a coefficient field, which we now suppress). We have Gysin−Thom isomorphisms (see [Sp])

$$\theta_X \colon H^*(X) \to H^{m_2+*}(Y, Y - X)$$
$$\theta_Z \colon H^*(Z) \to H^{m_1+*}(Y, Y - Z).$$

In particular we have distinguished classes $\theta_X(1) \in H^{m_2}(Y, Y - X)$, $\theta_Z(1) \in H^{m_1}(Y, Y - Z)$ and their product

$$\theta_X(1) \cup \theta_Z(1) \in H^N(Y, Y - X \cup Y - Z) = H^N(Y, Y - X \cap Z).$$

Hence $\theta^{-1}_{X \cap Z}(\theta_X(1) \cup \theta_Z(1)) \in H^L(X \cap Z)$. Let $(X \cap Z)_i$ denote a component in $X \cap Z$ with fundamental class $\mu_i \in H_L((X \cap Z)_i)$. Hence we can evaluate, to obtain $\rho_i = \theta^{-1}_{X \cap Z}(\theta_X(1) \cup \theta_Z(1))[\mu_i] \in R$; then we have that the inclusions $(Y, \emptyset) \to (Y, Y - X)$ and $(Y, \emptyset) \to (Y, Y - Z)$ induce homomorphisms

$$j_X \colon H^*(Y, Y - X) \to H^*_{\text{comp}}(Y),$$
$$j_Y \colon H^*(Y, Y - Z) \to H^*_{\text{comp}}(Y),$$

and that from the commutativity of

$$\begin{array}{ccc} H^*(Y, Y - X) \otimes H^*(Y, Y - Z) & \xrightarrow{j_X \otimes j_Z} & H^*_{\text{comp}}(Y) \otimes H^*_{\text{comp}}(Y) \\ \cup\downarrow & & \downarrow\cup \\ H^*(Y, Y - X \cap Z) & \longrightarrow & H^*_{\text{comp}}(Y) \end{array}$$

we can conclude that

$$(j_X \theta_X(1) \cup j_Z \theta_Z(1))[\mu_Y] = \sum_i \rho_i.$$

We denote this number by $\rho(X, Z)$; this can evidently be applied to show

PROPOSITION 4.1. *Under the above conditions, if* $\rho(X, Z) \neq 0$ *then there exist classes* $v_X \in H^{m_2}_{\text{comp}}(Y)$, $v_Z \in H^{m_1}_{\text{comp}}(Y)$ *such that* $v_X \cup v_Z \neq 0$.

*Remarks*. (1) Note that by Poincaré Duality we obtain non-trivial dual classes in $H_{m_1}(Y)$, $H_{m_2}(Y)$.

(2) In case $X, Z$ intersect transversally, the invariant above is simply the elementary intersection number $X \cdot Z$.

(3) More generally assume $Y$ is a $C^\infty$-manifold, $X, Z$ closed immersed submanifolds. Then the intersection is said to be clean if the components of $X \cap Z$ are immersed submanifolds of $V$ and if for all such components $W$ of $X \cap Z$ one has $TW = TX|_W \cap TZ|_W$. In this case the number above can be interpreted as an intersection number $X \cdot Z$, which in turn can be computed from the *Euler number* of the *excess bundle* $\mathcal{N}$ of the intersection.

We now specialize to our type of situation. We assume $G = H \times K$ is a finite automorphism group of $\Gamma$, construct $X$ as before, and consider the intersection diagram

$$\begin{array}{ccccc} & & X/\Gamma^H & & \\ & \nearrow & & \searrow & \\ X/\Gamma^G & & & & X/\Gamma \\ & \searrow & & \nearrow & \\ & & X/\Gamma^K & & \end{array}$$

assuming the relevant additional hypotheses. First we observe that if $\overline{H}_0, \overline{K}_0$ correspond to the trivial elements in $H^1(H, \Gamma), H^1(K, \Gamma)$, respectively, then

$$X^{\overline{H}_0}/C_\Gamma\left(\overline{H}_0\right) \cap X^{\overline{K}_0}/C_\Gamma\left(\overline{K}_0\right) = \coprod_{\overline{G} \in \Lambda} X^{\overline{G}}/C_\Gamma(\overline{G}),$$

where $\Lambda \subseteq H^1(G, \Gamma)$ is defined as $\ker \operatorname{res}_H \times \operatorname{res}_K$, and

$$\operatorname{res}_H \colon H^1(G, \Gamma) \to H^1(H, \Gamma)$$

$$\operatorname{res}_K \colon H^1(G, \Gamma) \to H^1(K, \Gamma)$$

are the restriction maps. This condition simply arises from the fact that $\overline{G}$ must restrict trivially on each factor (given our choice of $\overline{H}_0$ and $\overline{K}_0$). Using a subscript to denote the distinguished components corresponding to 1, we have that

$$(X/\Gamma)_0^H \cap (X/\Gamma)_0^K \simeq \coprod_{\overline{G} \in \Lambda} BC_\Gamma(\overline{G}).$$

It is now possible to use Proposition 4.1 to produce non-trivial classes in $H^*(\Gamma, R)$, which will be arising from the subgroups $C_\Gamma(\overline{H}), C_\Gamma(\overline{K})$. Specifically, we have that if

$$\rho\left(X^{\overline{H}_0}/C_\Gamma\left(\overline{H}_0\right), X^{\overline{K}_0}/C_\Gamma\left(\overline{K}_0\right)\right) \neq 0$$

(an intersection of classifying spaces) then there exist classes $x_H, x_K \in H^*(\Gamma, R)$ such that

$$\operatorname{res}^\Gamma_{C_\Gamma(\overline{H})} x_H = \left[X^{\overline{H}_0}/C_\Gamma\left(\overline{H}_0\right)\right]^* \neq 0$$
$$\operatorname{res}^\Gamma_{C_\Gamma(\overline{K})} x_K = \left[X^{\overline{K}_0}/C_\Gamma\left(\overline{K}_0\right)\right]^* \neq 0.$$

In certain situations the number $\rho(\ ,\ )$ can be computed in terms of intrinsic information associated to the group $\Gamma$. In [RS] the important case of an invariant arithmetic subgroup in an algebraic group having two commuting automorphisms of finite order is discussed. The associated symmetric space and its quotient will inherit an action of the finite group they generate, and the respective fixed-point sets are called ‘‘special cycles.’’ Understanding their intersection is a critical element in their method for producing non-trivial classes in $H^*(\Gamma, \mathbb{C})$. In fact they obtain an impressive general formula for $\rho$ in purely arithmetic terms, using Euler characteristics. The key technical point is that the ‘‘clean intersection formula’’ can be applied under certain rather general assumptions. This important result illustrates how the cohomology of certain discrete groups arises from underlying arithmetic data. However, from our point of view it is simply an example of how the presence of symmetries on a topological space forces the existence of non-trivial cohomology. This is of course one of the guiding principles in fixed-point theory.

By using a cohomological approach we are able to extend this method to a much broader context. In fact the specific geometric conditions can be sufficiently weakened so that we can see how the *group theoretic* nature of $\Gamma$ produces non-trivial cohomology—the new consideration here is the nature of its finite automorphisms. One would expect our more general format to have additional specific geometric applications to other classes of groups, and that other more sophisticated methods from equivariant topology can be usefully applied in analyzing the cohomology of discrete groups.

## 5. CONJUGACY CLASSES IN A SEMI-DIRECT PRODUCT

In this section we will apply our methods to study a purely algebraic problem, namely, how many conjugacy classes of elements of finite order are there in $\bar{\Gamma} = \Gamma \times_\varphi G$? This has some relevance to the calculation of the complex $K$-theory of the classifying space of this group (see [A]), but we will not elaborate on this here.

Let $g \in G$, and denote its centralizer in $G$ by $C(g)$. Consider an element $\theta \in Z^1(\langle g \rangle, \Gamma)$; if $h \in C(g)$, define $h(\theta)(x) = \varphi(h)[\theta(x)]$. Then

$$\begin{aligned} h(\theta)(x_1x_2) &= \varphi(h)[\theta(x_1x_2)] = \varphi(h)[\theta(x_1)\varphi(x_1)[\theta(x_2)]] \\ &= \varphi(h)[\theta(x_1)]\varphi(hx_1)[\theta(x_2)] \\ &= \varphi(h)[\theta(x_1)]\varphi(x_1)[\varphi(h)[\theta(x_2)]] \\ &= h(\theta)(x_1)\varphi(x_1)[h(\theta)(x_2)]. \end{aligned}$$

This means that $h(\theta)$ is also a cocycle, and this clearly induces a $C(g)$-action on the set $H^1(\langle g \rangle, \Gamma)$. We can now state

THEOREM 5.1. *Let $\Gamma$ denote a torsion-free discrete group and $G \underset{\varphi}{\hookrightarrow} \mathrm{Aut}(\Gamma)$ a finite automorphism group. Then the number of conjugacy classes of elements of finite order in $\Gamma \times_\varphi G$ is precisely equal to*

$$\sum_{(g)} \#[H^1(\langle g \rangle, \Gamma)/C(g)],$$

*where the sum ranges over all conjugacy classes of elements in $G$.*

*Proof.* Consider $(\gamma, g) \in \bar{\Gamma} = \Gamma \times_\varphi G$ of finite order. If $\pi: \bar{\Gamma} \to G$ is the natural projection then $\pi(\langle(\gamma, g)\rangle) = \langle g \rangle$. Also $\langle(\gamma, g)\rangle$ is a finite subgroup mapping isomorphically onto $\langle g \rangle \subseteq G$. If $\mathrm{Tors}(\bar{\Gamma})$ is the set of all elements of finite order in $\bar{\Gamma}$, define

$$\theta: \mathrm{Tors}(\bar{\Gamma}) \to \coprod_{g \in G} \{g\} \times H^1(\langle g \rangle, \Gamma)$$

by $\theta((\gamma, g)) = (g, [\langle(\gamma, g)\rangle]) \in \{g\} \times H^1(\langle g \rangle, \Gamma)$. If $\langle(\gamma_1, g_1)\rangle \underset{\Gamma}{\sim} \langle(\gamma_2, g_2)\rangle$, then $g_1 = g_2$, and we have that $[\langle(\gamma_1, g_1)\rangle] = [\langle(\gamma_2, g_2)\rangle]$ in $H^1(\langle g \rangle, \Gamma)$. Furthermore, if $\theta(\gamma_1, g_1) = \theta(\gamma_2, g_2)$, then $g_1 = g_2$ and there exists an $n > 0$ such that $(\gamma_1, g_1)$ is $\Gamma$ conjugate to $(\gamma_2, g_2)^n$, but then

$g_1 = g_2^n = g_2$, and so $(\gamma_1, g_1) \underset{\Gamma}{\sim} (\gamma_2, g_2)$. It is also clear that $\theta$ is onto, hence it establishes a bijection

$$\operatorname{Tors}(\bar{\Gamma})/\Gamma \xleftarrow{\bar{\theta}} \coprod_{g \in G} \{g\} \times H^1(\langle g \rangle, \Gamma).$$

Assume now that $\xi: \langle g \rangle \to \Gamma$ is a cocycle. We define $h\xi: \langle hgh^{-1} \rangle \to \Gamma$ by $h\xi(hxh^{-1}) = \varphi(h)[\xi(x)]$, this will again be a cocycle and in fact $h$ induces a bijection $H^1(\langle g \rangle, \Gamma) \to H^1(\langle hgh^{-1} \rangle, \Gamma)$. Hence $(g, \xi) \mapsto (hgh^{-1}, h\xi)$ defines a $G$-action on the set $\coprod_{g \in G} \{g\} \times H^1(\langle g \rangle, \Gamma)$. Using the natural $G$-action on $\operatorname{Tors}(\bar{\Gamma})/\Gamma$ induced by conjugation, it is direct to verify that $\bar{\theta}$ is $G$-equivariant, as is its inverse. We deduce that there is a bijection

$$\begin{aligned} \operatorname{Tors}(\bar{\Gamma})/\bar{\Gamma} &\cong \left[ \coprod_{g \in G} \{g\} \times H^1(\langle g \rangle, \Gamma) \right] \Big/ G \\ &\cong \coprod_{(g)} \{g\} \times H^1(\langle g \rangle, \Gamma)/C(g). \end{aligned}$$

From this we infer that the number of conjugacy classes of elements of finite order in $\bar{\Gamma}$ is precisely

$$\sum_{(g)} \#\left[ H^1(\langle g \rangle, \Gamma)/C(g) \right]. \quad \blacksquare$$

The following is a simple application of this formula.

EXAMPLE 5.2. Let $A$ be a free abelian group, and $G \subseteq \mathrm{GL}(A)$ a finite subgroup. Then for $\bar{\Gamma} = A \times_{\varphi} G$ the number of conjugacy classes of elements of finite order can be computed from the standard cohomology invariants $H^1(\langle g \rangle, A)^{C(g)}$. For example, let $A = \mathbb{Z}^{p-1}$, with an action of $\mathbb{Z}/p$ represented by the $(p-1) \times (p-1)$ matrix

$$M = \begin{pmatrix} 0 & 0 & & 0 & -1 \\ 1 & 0 & & 0 & -1 \\ 0 & 1 & & \cdot & \cdot \\ \cdot & 0 & \cdots & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & 0 & \cdot \\ 0 & 0 & & 1 & -1 \end{pmatrix}$$

Note that this matrix could be taken as a representative for the element $T$ discussed in Application 3.8. In this case the number of conjugacy classes is precisely

$$\sum_{1}^{p-1} \#H^1(\mathbb{Z}/p, A) + 1 = p^2 - p + 1.$$

More generally, it is well known [CR] that any integral representation of $L$ of $\mathbb{Z}/p$ decomposes as

$$L \cong \mathbb{Z}^r \oplus \bigoplus_{1}^{s} P_i \oplus \bigoplus_{1}^{t} A_i,$$

where

$$P_i \otimes \mathbb{Z}_p \cong \mathbb{Z}_p[\mathbb{Z}/p], \qquad A_i \otimes \mathbb{Z}_p \cong A \otimes \mathbb{Z}_p$$

($A$ as above). Furthermore $H^1(\mathbb{Z}/p, L) = (\mathbb{Z}/p)^t$. Hence we obtain that $\overline{\Gamma}$ has exactly $(p-1)p^t + 1$ conjugacy classes of elements of finite order.

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