1990 Barcelona Conference
on Algebraic Topology.

ON THE GEOMETRY AND COHOMOLOGY OF FINITE SIMPLE GROUPS

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§0. INTRODUCTION

Let G be a finite group and k a field of characteristic $p \mid |G|$. The cohomology of G with k-coefficients, $H^*(G,k)$, can be defined from two different points of view:

(1) $H^*(G,k) = H^*(BG;k)$, the singular cohomology of the classifying space BG

(2) $H^*(G,k) = Ext^*_{kG}(k,k)$.

From (1) it is clear that group cohomology contains important information about G-bundles and the rôle of G as a transformation group. From (2) we can deduce that $H^*(G,k)$ contains characteristic classes for representations over k, and indeed is the key tool to understanding modular representations via the method of cohomological varieties introduced by Quillen.

The importance of classifying spaces in topology is perhaps most evident in the method of "finite models". Roughly speaking, the classifying spaces of certain finite groups can be assembled to yield geometric objects of fundamental importance. For example, let Σ_m denote the symmetric group on m letters; then the natural pairing $\Sigma_n \times \Sigma_m \to \Sigma_{n+m}$ induces an associative monoid structure on $C(S^\circ) = \coprod_{n\geq 0} B\Sigma_n$. We have the result due to Dyer and Lashof [D-L]:

THEOREM

$Q(S^\circ) = \Omega^\infty S^\infty$ is the homotopy group completion of $C(S^\circ)$ (using the loop sum), and hence

$$H_*(Q(S^\circ), \mathbf{F}_2) \cong \varinjlim H_*(\Sigma_m, \mathbf{F}_2) \otimes \mathbf{F}_2[\mathbf{Z}]\,. \qquad \blacksquare$$

Given that $\pi_i(Q(S^\circ)) = \pi_i^S(S^\circ)$ (the stable homotopy groups of spheres), $Q(S^\circ)$ is the "ground ring" for stable homotopy; and the result above indicates that it can be constructed from classifying spaces of finite groups.

A similar approach (due to Quillen [Q1]) uses the $\mathrm{GL}_n(\mathbf{F}_q)$ to construct the space Im J, which essentially splits off $Q(S^\circ)$ in the natural way. From this space we obtain the algebraic K-theory of the field $\mathbf{F}_q$.

Now from the point of view of cohomology of groups, very few noteworthy examples have been calculated. Most strikingly, the important computations have been for groups involved in the geometric scheme outlined above. The understanding of the cohomology

* Partially supported by an NSF grant.

of the symmetric and alternating groups (only additively) is linked to the geometry of $Q(S^\circ)$; indeed this space can be identified with $B\Sigma_\infty^+$ (the plus construction), and the homology of the finite symmetric groups can be expressed in a precise manner using Dyer-Lashof operations. Analogously, Quillen constructed $\mathrm{BGL}(\mathbf{F}_q)^+ = F\psi^q$ by comparing its cohomology to what he calculated for $\mathrm{BGL}_n(\mathbf{F}_q)$ away from q, for all n.

From a purely calculational point of view, the cases above are very special: the cohomology is detected on abelian subgroups. Hence these results cannot truly reflect the general difficulties of the subject and may not be good examples for relating $H^*(G,k)$ to geometric and algebraic properties of G.

At this point we introduce an ingredient usually ignored by topologists — the classification of finite simple groups. From the topological point of view, only the alternating and groups associated to those of Lie type have been considered (as outlined above). The classification theorem indicates that there are 26 additional groups which are as essential as the two infinite families — the so-called sporadic simple groups. A natural question arises: what rôle do the sporadic simple groups play in this framework?

In this paper I will discuss some recent work on this problem which I have done jointly with R. J. Milgram. By now it is clear that the first step in any such project should be to understand the cohomology first as a calculation and then as a source of algebraic and geometric data. In §1 I will describe the combination of techniques required to calculate the cohomology of groups of this complexity. Then, in §2 I will present some calculations. The particular example which will be dealt with in detail is the Mathieu group M_{12}, a group of order 95,040. The cohomology ring $H^*(M_{12}, \mathbf{F}_2)$ will be fully described. In the following section, a geometric analysis of the results in §2 will be provided. This example will show how well cohomology carries the local structure of the group, as well as plentiful topological information. Finally in §4 I will discuss current work on M_{22}, M_{23}, M^cL and $O'N$, which are even larger groups.

The aim of this paper is to bring the reader up to date on recent progress, hence he will be spared the technical details of the calculations, which are either available ([AMM], [AM]) or will appear soon. All coefficients are taken to be $\mathbf{F}_2$, so they will be suppressed.

A note of caution: it is not the author's viewpoint to encourage mindless calculation, which is seen so often. Rather, the main point is that with some work we can extend our knowledge of group cohomology by understanding the key models on which the whole framework of finite group theory is now based. The sporadic simple groups seen to be the natural objects we should now turn our attention to, if there is any possibility of understanding behaviour in algebraic topology which is modelled on finite group theory.

The results in this paper were presented at a talk given by the author during the 1990 Barcelona Conference on Algebraic Topology. The author is very grateful to the organizers for being invited and in particular to M. Castellet for his kind hospitality.

§1. CALCULATIONAL TECHNIQUES

From the landmark work of Quillen [Q2] it is known that $H^*(G)$ is detected up to nilpotence on its elementary abelian subgroups. Hence a clear understanding of these subgroups is necessary before embarking on a calculation. The relevant data can all be assembled into a finite complex (first introduced by K. Brown) as follows [Q3]. Let

$A_p(G)$ denote the partially ordered set of p-elementary abelian subgroups of G under inclusion. In the usual way $A_p(G)$ can be given the structure of a simplicial complex; the n-simplices correspond to flags of the form $O \neq (\mathbf{Z}/2)^{j_1} \underset{\neq}{\subseteq} \cdots \underset{\neq}{\subseteq} (\mathbf{Z}/2)^{j_n}$. Then G acts on this simplicial set by conjugation, and its realization $|A_p(G)|$ has the structure of a finite G-CW complex, in fact of dimension $\mathrm{rank}_p(G) - 1$. If $P \leq G$ is a p-subgroup, then this complex satisfies one key property: $|A_p(G)|^P$ is contractible. We can then apply the following theorem, which is a reformulation of a result due to P. Webb [We]:

THEOREM 1.1

If X is a finite G-CW complex such that $X^p \simeq *$ for each p-subgroup $P \leq G$, then the mod p Leray spectral sequence associated to the map $X \underset{G}{\times} EG \to X/G$ satisfies

$$E_2^{p,q} \cong \begin{cases} 0 & \text{for } p > 0 \\ H^q(G, \mathbf{F}_p) & \text{for } p = 0. \end{cases} \qquad \blacksquare$$

Using the usual combinatorial description of the E_1-term of this spectral sequence yields:

COROLLARY 1.2

$$\left(\bigoplus_{\substack{\sigma_i \in (X/G)^{(i)} \\ i \text{ odd}}} H^*(G_{\sigma_i}) \right) \oplus H^*(G) \cong \bigoplus_{\substack{\sigma_i \in (X/G)^{(i)} \\ i \text{ even}}} H^*(G_{\sigma_i})$$

where X may be taken to be $|A_p(G)|$, and $\mathbf{F}_p$ coefficients are used. $\blacksquare$

The isotropy subgroups for the poset space are intersections of normalizers of p-tori. This formula in general can be of no use either because of its size or the fact that G may normalize a p-torus. However, in case G *is simple*, this reduces the calculation of $H^*(G)$ to calculations for proper, local subgroups.

With this reduction in mind, it seems likely that a cohomology calculation will involve analyzing a local subgroup which contains the 2-Sylow subgroup (recall that we take only $p = 2$ in this paper). To understand how to use the cohomology of such a subgroup, the classical double coset formula of Cartan-Eilenberg is often extremely useful [C-E]. We recall how that goes.

Assume $K \subseteq G$ is a subgroup of odd index. Take a double coset decomposition $G = \coprod_{i=1}^{n} K g_i K$. Using a simple transfer-restriction argument, it is easy to see that $H^*(G) \overset{\mathrm{res}_H^G}{\hookrightarrow} H^*(K)$ is injective. Let $c_x^* : H^*(k) \to H^*(xKx^{-1})$ denote the usual isomorphism induced by conjugation. Recall that $\alpha \in H^*(k)$ is said to be ***stable*** if $\mathrm{res}^H_{H \cap xHx^{-1}}(\alpha) = \mathrm{res}^{xHx^{-1}}_{H \cap xHx^{-1}} \circ c_x^*(\alpha)$ for $x = g_1, \ldots, g_n$. Then we have, for [G:K] odd.

THEOREM 1.3 (Cartan-Eilenberg)

$\alpha \in H^*(K)$ is in $\mathrm{im}\ \mathrm{res}_H^G$ *if and only if α is stable.* $\blacksquare$

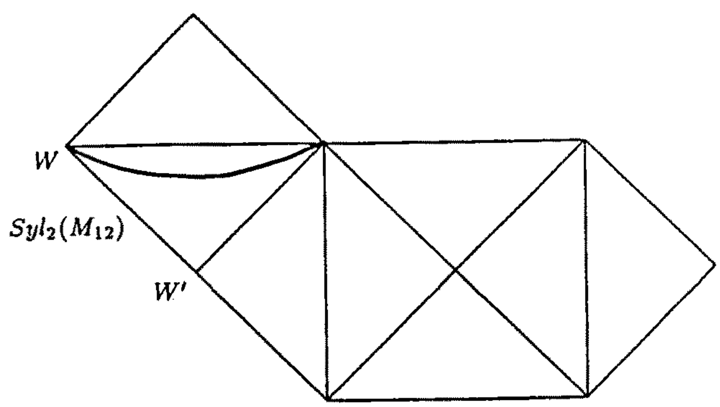
Applying the techniques described above it is in principle possible to reduce the calculation of $H^*(G)$ to understanding the cohomology of its local subgroups, together with the stability conditions. At this point one must dispense with formal reductions and calculate the cohomology of the key local subgroups using the spectral sequences associated to the different extensions they may appear in. Aside from the usual difficulties of understanding the differentials, often some very delicate modular invariant theory must be used to understand the E_2-terms or images of different restriction maps.

A combination of these methods will be used to calculate $H^*(M_{12})$, the main example in §2.

§2. COHOMOLOGY CALCULATIONS FOR M_{12}

M_{12} is the Mathieu group of order 95,040. We will give a complete analysis of its cohomology using the methods outlined in §1.

First, we have that $|A_2(M_{12})|/M_{12}$ is the following 2-dimensional cell-complex:



We have only labelled the isotropy on one edge, because 1.2 leads to

Proposition 2.1

Let $H = Syl_2(M_{12})$; then there exist two non-isomorphic subgroups $W, W' \subseteq M_{12}$ of order 192, such that

$$H^*(M_{12}) \oplus H^*(H) \cong H^*(W) \oplus H^*(W'). \qquad \blacksquare$$

Next, we use 1.3 to prove

Proposition 2.2

The image of $\mathrm{res}_H^{M_{12}}$ in $H^*(H)$ is the intersection of the two subalgebras,

$$H^*(M_{12}) = H^*(W) \cap H^*(W'). \qquad \blacksquare$$

In other words, the configuration

$$\begin{array}{rccc} W \cap W' = & H & \hookrightarrow & W' \\ & \downarrow & & \downarrow \\ & W & \hookrightarrow & M_{12} \end{array}$$

completely controls the cohomology of M_{12}.

We now give explicit descriptions of W, W'.

$$W: \qquad 1 \to Q_8 \to W \to \Sigma_4 \to 1$$

a split semidirect product (the holomorph of Q_8).

$$W': \qquad 1 \to (\mathbf{Z}/4 \times \mathbf{Z}/4) \underset{T}{\times} \mathbf{Z}/2 \to W' \to \Sigma_3 \to 1$$

also a split semidirect product.

Although $|H| = 64$, its cohomology is very complicated. It is obtained by using two index two subgroups with accessible cohomology. We have

THEOREM 2.3

$$H^*(Syl_2(M_{12})) \cong \mathbf{F}_2[e_1, s_1, t_1, s_2, t_2, L_3, k_4]/R$$

where R is the set of relations

$$\begin{aligned} &s_1 e_1 = s_2 e_1 = t_1 e_1 = 0,\ s_1^3 = s_1^2 t_1 + s_1 t_2\,, \\ &s_2^2 = s_1 s_2 t_1 + s_2 t_1^2,\ s_1^2 s_2 = t_1 L_3 + s_2 t_1^2 + s_2 t_2\,, \\ &s_1 L_3 = t_1 L_3 + s_1 s_2 t_1 + s_2 t_1^2 + s_2 t_2\,, \\ &s_2 L_3 = s_1 s_2 t_1^2 + s_2 t_1 t_2 + s_2 t_1^3\,, \\ &L_3^2 = t_1 t_2 L_3 + s_1 s_2 t_1^3 + s_2 t_1^2 t_2 + s_2 t_1^4 + e_1^2 k_4 + e_1 t_2 L_3\,. \end{aligned}$$

The Poincaré Series for $H^*(H)$ is given by $p_H(t) = 1/(1-t)^3$. ∎

Similarly, we determine $H^*(W)$, $H^*(W')$ and their intersection in $H^*(H)$. We now describe $H^*(M_{12})$:

DIMENSION	GENERATOR
2	α
3	x, y, z
4	β
5	γ
6	δ
7	ϵ

RELATIONS

$\alpha(x+y+z)=0$	$x^3=\alpha^3x+\alpha\beta x+x\delta$
$xy=\alpha^3+x^2+y^2$	$xz=\alpha^3+y^2$
$x^2y=\alpha^3z+\alpha\beta z+y\delta+\alpha\epsilon$	$yz=\alpha^3+x^2$
$\epsilon x=\beta x^2+\alpha^2x^2$	$\alpha\gamma=\alpha^2y$
$\epsilon y=\alpha^2\delta+\alpha^2y^2+\beta x^2+\beta y^2$	$y\gamma=\alpha y^2$
$\epsilon z=\gamma^2+\alpha^2\delta+\alpha^2x^2+\beta x^2+\beta z^2$	$x\gamma=\alpha^4+\alpha z^2$
$z^4=\gamma\epsilon+x^4+\alpha^4\beta+z^2\delta$	$\epsilon^2=z^3\gamma+\alpha^2\beta\delta+\alpha^5\beta+z\beta\epsilon$
	$+z\delta(\gamma+\alpha z)+\beta^2(\alpha^3+xz+yz)$

The Poincaré Series for $H^*(M_{12})$ is given by

$$P_{M_{12}}(t)=\frac{1+t^2+3t^3+t^4+3t^5+4t^6+2t^7+4t^8+3t^9+t^{10}+3t^{11}+t^{12}+t^{14}}{(1-t^4)(1-t^6)(1-t^7)}. \tag{2.4}$$

§3. AN INTERPRETATION OF THE RESULTS

We now provide a geometric analysis of our calculation of $H^*(M_{12})$. First, note that $\mathbf{F}_2[\beta_4,\delta_6,\epsilon_7]\subseteq H^*(M_{12})$ is a polynomial subalgebra of maximal rank, and in fact the cohomology ring is free and finitely generated over it (this implies that it is Cohen-Macaulay). One can also verify that $Sq^2\beta=\delta$, $Sq^1\delta=\epsilon$, $Sq^4\epsilon=\beta\epsilon$, $Sq^6\epsilon=\delta\epsilon$. On the other hand, let $\mathbf{G_2}$ denote the usual exceptional compact Lie group of automorphisms of the Cayley octaves. A result due to Borel is that

$$H^*(B\mathbf{G_2},\mathbf{F}_2)\cong\mathbf{F}_2[\beta_4,\delta_6,\epsilon_7]$$

with Steenrod operations as above. $\mathbf{G_2}$ is also a 14-dimensional closed manifold. From (2.4) we see that the numerator of $p_{M_{12}}(t)$ looks like the Poincaré Series of a 14-dimensional Poincaré Duality Complex. How does $\mathbf{G_2}$ relate to the group M_{12}?

At this point it is pertinent to mention a technique used by Borel to compute the cohomology of $B\mathbf{G_2}$. Although $\mathbf{G_2}$ has rank 2, it contains a 2-torus of rank 3, denoted by V. Then Borel [Bo] analyzed the spectral sequence for the fibration

$$\begin{array}{ccc}\mathbf{G_2}/V & \hookrightarrow & BV\\ & & \downarrow\\ & & B\mathbf{G_2}\end{array},$$

showing that in fact it collapses, and so

$$p_V(t) = p_{B\mathbf{G_2}}(t) \cdot (P.S.[\mathbf{G_2}/V]).$$

He computes $P.S.[G_2/V] = C_3(t) \cdot C_5(t) \cdot C_6(t)$, where $C_i(t)$ denotes the i-th cyclotomic polynomial.

Now consider, if $V \leq M_{12} \leq \mathbf{G_2}$, it would be reasonable to expect

$$p_{M_{12}}(t) = p_{B\mathbf{G_2}}(t) \cdot (P.S.[\mathbf{G_2}/M_{12}]).$$

Unfortunately, M_{12} is not a subgroup of $\mathbf{G_2}$. However, one does find that $\mathbf{G_2}$ contains a subgroup E of order 1344 which normalizes V, and fits into an extension

$$1 \to V \to E \to \mathrm{GL}_3(\mathbf{F}_2) \to 1 \tag{3.0}$$

which is *non-split.*

By analogues of Borel's arguments, we have that

$$p_E(t) = p_{B\mathbf{G_2}}(t) \cdot (P.S.[\mathbf{G_2}/E]).$$

Our interest is in M_{12} however, so why does it matter? It matters because of the following result:

THEOREM 3.1

$$H^*(E) \cong H^*(M_{12}) \oplus (H^*(V) \otimes \mathrm{St})^{GL_3(\mathbf{F}_2)},$$

where St *denotes the Steinberg module associated to $GL_3(\mathbf{F}_2)$.* ∎

This result is proved by splitting (3.0) via the Tits Building for $GL_3(\mathbf{F}_2)$, and then reassembling the pieces in the form above. The factor $(H^*(V) \otimes St)^{GL_3(\mathbf{F}_2)}$ is some kind of "error term". What this shows is that the geometry pertaining to $E \subsetneq \mathbf{G_2}$ is transported via cohomology to M_{12}, a rather unexpected occurrence.

The obvious next step is to compare E and M_{12} as groups. They turn out to be close cousins, as manifested by the following result due to Wong [Wo].

THEOREM 3.2

Let G be a finite group with precisely two distinct conjugacy classes of involutions, and such that the centralizer of one of them is isomorphic to $W =$ $Hol(Q_8)$. Then either $G \cong E$ or $G \cong M_{12}$, E as before. ∎

This is a very positive development, as it shows that cohomology measures in a very precise way how close E and M_{12} are from the 2-local point of view. Combined with the previous remarks, we see that group cohomology will inherit geometric properties through a local similarity, with difference measured by a very standard algebraic object. From the geometric point of view, there arises an obvious question: does there exist a 14-dimensional Poincaré Duality complex X with $\pi_1(X) = M_{12}$, and whose mod 2 cohomology has Poincaré Series equal to the numerator of $p_{M_{12}}(t)$? Just as $\mathbf{G_2}$ has important homotopy-theoretic data, so should X.

We now turn to another very unexpected geometric aspect of our calculation. Recall that the cubic tree is defined as the tree whose vertices have valence three — it is clearly infinite. We have

THEOREM 3.3

There exists a group Γ of automorphism of the cubic tree such that

(i) there is an extension $1 \to F^{496} \to \Gamma \xrightarrow{\pi} M_{12} \to 1$, where F^{496} = free group on 496 generators, and

(ii) $\pi^* : H^*(M_{12}, \mathbf{F}_2) \to H^*(\Gamma, \mathbf{F}_2)$ is an isomorphism. ∎

Part (i) is a result due to Goldschmidt [G], in fact if we take H, W, W' as in §3, $\Gamma = W \underset{H}{*} W'$. We see then that at the prime $p = 2$, BM_{12} can be modelled using the classifying space of an infinite, virtually-free group. Aside from the relationship to trivalent graphs, it allows us to make the previous geometric considerations for Γ instead of M_{12}. One can then pose the problem of constructing a 14-dim manifold with Γ as its fundamental group and having the required Poincaré Series. The fact that we are now dealing with an amalgamated product may make this considerably easier than for M_{12}.

§4. OTHER SPORADIC SIMPLE GROUPS

The other groups which we are currently considering are M_{22}, M_{23} (Mathieu groups), M^{c}L (the McLaughlin group), J_2, J_3 (the Janko groups) and $O'N$ (the O'Nan group). At this point we have calculated the cohomology of the 2-Sylow subgroups of M_{22} (same for M_{23}) and M^{c}L. For M_{22} we have an odd index subgroup with a three term double coset decomposition, and the stability conditions are clear. We only require some detailed understanding of $\mathbf{F}_2[x_1, x_2, x_3, x_4]^{A_6}$, where $A_6 \subseteq S_{p_4}(\mathbf{F}_2) \cong \Sigma_6$.

For $O'N$, even though the group has order larger than 460 billion, we have calculated its poset space $|A_2(O'N)|$, and used this to show

$$H^*(O'N) \oplus H^*(4^3 \cdot \Sigma_4) \cong H^*(4^3 \cdot GL_3(\mathbf{F}_2)) \oplus H^*(4 \cdot PSL_3(\mathbf{F}_4) \cdot 2)$$

a considerable reduction. Here we are using the Atlas notation for extensions and cyclic groups of order n. Using the Tits Buildings for the Chevalley groups in this formula we have made considerable progress towards a complete calculation (we already have $H^*(PSL_3(\mathbf{F}_4))$ and the cohomology of related extensions [AM]).

For the group $O'N$, the algebra seems to dictate that a 30-dimensional Poincaré Duality complex will play the same rôle as the 14-dimensional one for M_{12}. From the homotopy point of view these are indeed interesting dimensions.

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