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Topological models and cohomology of Galois groups

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Abstract. Let F denote a field of characteristic different from 2. In this Note we describe the mod 2 cohomology of a Galois group $\mathcal{G}_F$ which is determined by the Witt ring $\mathrm{W}F$. © Académie des Sciences/Elsevier, Paris

Modèles topologiques et cohomologie des groupes de Galois

Résumé. *Soit F un corps de caractéristique différente de 2. Dans cette Note, nous décrivons la cohomologie* mod 2 *d'un groupe de Galois $\mathcal{G}_F$ qui est determiné par l'anneau de Witt* $\mathrm{W}F$. © Académie des Sciences/Elsevier, Paris

Version française abrégée

Soit F un corps de caractéristique différente de 2. Récemment le rapport étroit entre l'anneau de Witt $\mathrm{W}F$ et la théorie de Galois de F a été explicité par Minac et Spira, qui, dans [11], ont montré que $\mathrm{W}F$ détermine le groupe de Galois $\mathcal{G}_F$ d'une certaine extension $F^{(3)}$ de F. Si F_q dénote la clôture quadratique de F, alors en fait $F^{(3)} \subset F_q$, et $\mathcal{G}_F$ est un quotient de $G_F = \mathrm{Gal}(F_q/F)$.

Dans cette Note, nous étudions la structure cohomologique de $\mathcal{G}_F$ relativement au nombre premier 2. Il est naturel de prévoir que $\mathrm{H}^*(\mathcal{G}_F, \mathbb{F}_2)$ reflètera les propriétés principales du corps F. C'est en effet le cas ; en particulier, nous montrons qu'il contient la cohomologie galoisienne mod 2 de F (2.1) ainsi que des renseignements cohomologiques supplémentaires liés à la théorie des corps (voir 2.2, 2.3 et 2.4). Quand $|\dot{F}/\dot{F}^2| < \infty$, le groupe $\mathcal{G}_F$ est un 2-groupe fini et nous pouvons alors utiliser des idées et des techniques de la cohomologie des groupes finis où des méthodes de calculs effectifs sont disponibles (*voir* [1]). Ces calculs s'appliquent aussi aux W-groupes infinis car souvent nous pouvons réduire les calculs au cas fini (comme dans 4.3, par exemple).

Étant donné un corps F, comme ci-dessus, avec $|\dot{F}/\dot{F}^2| = 2^n$, nous construisons une action de $E_n = (\mathbb{Z}/2)^n$ sur $X_F \cong (\mathbb{S}^1)^r$, $r = n + \binom{n}{2} - \dim \mathrm{H}^2(G_F, \mathbb{F}_2)$, telle que $\mathrm{H}^*(\mathcal{G}_F, \mathbb{F}_2)$ puisse s'exprimer comme une extension

$$0 \longrightarrow (\zeta_1, \ldots, \zeta_r) \longrightarrow \mathrm{H}^*(\mathcal{G}_F, \mathbb{F}_2) \longrightarrow \mathrm{H}^*_{E_n}(X_F, \mathbb{F}_2) \longrightarrow 0,$$

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où $\zeta_1, \ldots, \zeta_r \in \mathrm{H}^2(\mathcal{G}_F, \mathbb{F}_2)$ constituent une suite régulière. Nous montrons que cette action est libre si et seulement si F est non-formellement réel, auquel cas $\mathrm{H}^*(\mathcal{G}_F, \mathbb{F}_2)$ est Cohen–Macaulay et X_F/E_n est une forme d'espace euclidien telle que la sous-algèbre engendrée par les éléments unidimensionels dans $\mathrm{H}^*(X_F/E_n, \mathbb{F}_2)$ soit précisément la cohomologie galoisienne modulo 2 de F.

Au §4 nous décrivons la construction pour un certain nombre d'exemples : en particulier, pour le corps $\mathbb{Q}_2$ nous obtenons une variété compacte de dimension 5 avec série de Poincaré donnée par $1 + 3t + 6t^2 + 6t^3 + 3t^4 + t^5$ (4.2). Nous présentons un calcul général pour les corps superpythagoriciens (4.3) et une méthode pour construire la cohomologie des W-groupes universels $W(n)$ (voir 4.5). Finalement, nous discutons brièvement la possibilité d'un calcul général de $\mathrm{H}^*(\mathcal{G}_F, \mathbb{F}_2)$ en utilisant les suites spectrales.

Nous avons essayé d'indiquer les principaux ingrédients utilisés dans les démonstrations de nos résultats. Les détails seront publiés ailleurs. Sauf indication du contraire, les coefficients sont pris dans $\mathbb{F}_2$ et seront supprimés à partir de maintenant.

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1. Preliminaries on Galois groups

Let F denote a field of characteristic different from 2. We denote its quadratic closure by F_q and the associated Galois group by G_F. The W-group of F is defined (*see* [11]) as the quotient group

$$\mathcal{G}_F = G_F/G_F^4[G_F^2, G_F]. \tag{1.1}$$

This Galois group is known to contain important information about F; indeed, the main result in [11] is that if $WF \cong WL$ then $\mathcal{G}_F \cong \mathcal{G}_L$, and the converse is true given a mild assumption (*). Let $\Phi(\mathcal{G}_F) \subset \mathcal{G}_F$ denote its Frattini subgroup, then it is also an elementary Abelian 2-group and $\mathcal{G}_F$ can be expressed as a central extension

$$1 \longrightarrow \Phi(\mathcal{G}_F) \longrightarrow \mathcal{G}_F \longrightarrow E \longrightarrow 1, \tag{1.2}$$

where $E \cong (\dot{F}/\dot{F}^2)^*$, the Pontrjagin dual of $\dot{F}/\dot{F}^2$.

The field F is said to be *formally real* if -1 cannot be expressed as a sum of squares. Using some Galois theory, we prove:

THEOREM 1.3. – *If $|\mathcal{G}_F| > 2$, then F is not formally real if and only if every element of order 2 in $\mathcal{G}_F$ is central.*

In terms of the extension (1.2), this implies that $\Phi(\mathcal{G}_F)$ is the unique maximal elementary Abelian subgroup. In the formally real case it can be shown that $\Phi(\mathcal{G}_F)$ is an index 2 subgroup of any maximal elementary Abelian subgroup of $\mathcal{G}_F$.

Given a set Ω there exists a unique central extension of type (1.2), denoted $\mathrm{W}(\Omega)$, that maps onto any W-group $\mathcal{G}_F$ with $\mathcal{G}_F/\Phi(\mathcal{G}_F) \cong \mathrm{W}(\Omega)/\Phi(\mathrm{W}(\Omega)) \cong E_\Omega$, an elementary Abelian 2-group with a minimal set of generators in 1–1 correspondence with the elements in Ω. This group is called the *universal* W*-group* on Ω (note that it is the W-group of a certain field, *see* [4]) and for F as above fits into a central extension $1 \longrightarrow V \longrightarrow \mathrm{W}(\Omega) \longrightarrow \mathcal{G}_F \longrightarrow 1$, where $V \subset \Phi(\mathrm{W}(\Omega))$.

In the special case when $|\dot{F}/\dot{F}^2| = 2^n$, the group $\mathcal{G}_F$ is a quotient of $\mathrm{W}(n)$, the universal W-group on n generators. In this case $|\mathcal{G}_F| < \infty$, $\mathcal{G}_F/\Phi(\mathcal{G}_F) \cong (\mathbb{Z}/2)^n$ and $\Phi(\mathcal{G}_F) \cong (\mathbb{Z}/2)^r$, where $r = n + \binom{n}{2} - \dim \mathrm{H}^2(G_F)$. From now on, whenever $|\dot{F}/\dot{F}^2| = 2^n$, E_n will denote the group $\mathcal{G}_F/\Phi(\mathcal{G}_F)$, n its rank, and r the rank of $\Phi(\mathcal{G}_F)$.

2. Cohomology of W-groups

To begin we identify the group extension (1.2) using cohomological and field-theoretic data. Recall that if $\ell : \dot{F}/\dot{F}^2 \rightarrow K_1F/2K_1F$ is the canonical isomorphism between $\dot{F}/\dot{F}^2$ written multiplicatively and additively respectively, then Milnor's K-theory mod 2 (*see* [10]) can be expressed as $\mathbb{F}_2[x_i \mid i \in \Omega]/I_F$, where $\{x_i \mid i \in \Omega\}$ are one-dimensional polynomial generators which constitute a basis for $K_1F/2K_1F$, and I_F is the ideal generated by the quadratic polynomials corresponding to $\ell(a)\ell(1-a)$, for $a \in \dot{F}/\dot{F}^2$, $a \neq 1$. Let $\mathcal{B}_F$ denote the subspace of $\mathrm{H}^2(E)$ spanned by these polynomials. From the five term exact sequence associated to (1.2) we obtain an injective map $\delta : \mathrm{H}^1(\Phi(\mathcal{G}_F)) \longrightarrow \mathrm{H}^2(E)$. By analyzing the defining extension for $\mathcal{G}_F$ we prove:

THEOREM 2.1. – *The image of δ is the subspace $\mathcal{B}_F \subset \mathrm{H}^2(E)$ and if $|\dot{F}/\dot{F}^2| < \infty$, then $\mathcal{G}_F$ is the uniquely determined central extension associated to this subspace. Moreover, if $\mathcal{R} \subset \mathrm{H}^*(\mathcal{G}_F)$ denotes the subring generated by one-dimensional classes, then $\mathcal{R} \cong \mathrm{H}^*(F)$, the* mod 2 *Galois cohomology of the field F.*

The proof of the first part of Theorem 2.1 uses results due to Merkurjev [9]; for the second part, we apply the Milnor conjecture to identify the Galois cohomology explicitly (this unpublished result is due to Voevodsky [13]; the second statement in Theorem 2.1 is the only result in our paper which relies on it).

An important point to make is that $\mathrm{H}^*(\mathcal{G}_F)$ can contain substantially more cohomology than $\mathrm{H}^*(F)$ alone, and in fact its structure reflects properties of F. This is well illustrated when $|\dot{F}/\dot{F}^2| < \infty$, an assumption which we make in what follows. One basic result is:

THEOREM 2.2. – *There exist polynomial classes $\zeta_1, \ldots, \zeta_r \in \mathrm{H}^2(\mathcal{G}_F)$ which form a regular sequence. If F is non-formally real, then $\mathrm{H}^*(\mathcal{G}_F)$ is free and finitely generated over $\mathbb{F}_2[\zeta_1, \ldots, \zeta_r]$ (in particular Cohen–Macaulay). In the formally real case, $\mathrm{H}^*(\mathcal{G}_F)$ has depth equal to r or $r+1$ and Krull dimension equal to $r+1$.*

Using the results in [2], we can also prove:

THEOREM 2.3. – *Let F be a non-formally real field. Then there exist non-zero classes $x \in \mathrm{H}^*(\mathcal{G}_F)$ which restrict trivially on all proper subgroups of $\mathcal{G}_F$. Any such class must be exterior, i.e., $x^2 = 0$.*

Theorem 2.3 indicates that the usual detection methods for computing cohomology will not work for these W-groups. In contrast, for formally real fields detection can occur and in fact the element $[-1] \in \mathrm{H}^1(\mathcal{G}_F)$ plays a key role. Using [3] we obtain:

THEOREM 2.4. – *Let F be a formally real field, then the following three statements are equivalent:*

(1) *$[-1] \in \mathrm{H}^1(\mathcal{G}_F)$ is not a zero divisor.*

(2) *F is Pythagorean (**) and $H^*(\mathcal{G}_F)$ is Cohen–Macaulay.*

(3) *F is Pythagorean and $\mathrm{H}^*(\mathcal{G}_F)$ is detected on its elementary Abelian subgroups.*

3. Topological models

We now address the problem of computing the quotient algebra $\mathrm{H}^*(\mathcal{G}_F)/(\zeta_1, \ldots, \zeta_r)$. To do this, we construct a so called *topological model*, its main properties are summarized in:

THEOREM 3.1. – *Given a field F with $|\dot{F}/\dot{F}^2| = 2^n$, there exists an action of E_n on $X_F \cong (\mathbb{S}^1)^r$ with the following properties:*

(1) *E_n only has cyclic isotropy subgroups;*

(2) *the action is free if and only if F is non-formally real; and*

(3) $\mathrm{H}^*(\mathcal{G}_F)/(\zeta_1, \ldots, \zeta_r) \cong \mathrm{H}^*_{E_n}(X_F)$.

The term appearing in (3) is the *equivariant cohomology*, or equivalently the mod 2 cohomology of the Borel construction $X_F \times_{E_n} EE_n$. In case (2) we obtain the cohomology of a compact r-dimensional manifold, X_F/E_n (a Euclidean space form), such that the subring generated by one-dimensional classes is isomorphic to $\mathrm{H}^*(F)$.

The proof of Theorem 3.1 can be sketched as follows. First we find a $\mathbb{Z}E_n$ lattice M and a class $\alpha \in \mathrm{H}^2(E_n, M)$ such that $M/2M$ has a trivial E_n-action, and under the mod 2 reduction map, α is mapped to the defining class of the extension in $\mathrm{H}^2(E_n, \Phi(\mathcal{G}_F))$. Then we realize α by constructing an *explicit* E_n action on a product of r circles (X_F) such that α corresponds to the discrete group $\pi_1(X_F \times_{E_n} EE_n)$. Note that if $p_F(t)$ is the Poincaré series for the cohomology of $\mathcal{G}_F$, and $q_F(t)$ the one for the equivariant cohomology $\mathrm{H}^*_{E_n}(X_F)$, then $p_F(t) = q_F(t)/(1-t^2)^r$. We should also point out that the geometry of the action reflects the field theory in other ways besides condition (2). For example, if F is Pythagorean, one can find a hyperplane $H \subset E_n$ which acts freely on X_F; this corresponds to the index two subgroup $\mathcal{G}_{F(\sqrt{-1})} \subset \mathcal{G}_F$. In the non-formally real case $\mathrm{H}^*(X_F/E_n)$ can be identified with a basis for $\mathrm{H}^*(\mathcal{G}_F)$ as a free $\mathbb{F}_2[\zeta_1, \ldots, \zeta_r]$-module. The model is not unique, but any two of them have the same mod 2 cohomology.

4. Examples and final remarks

4.1. The 4-dimensional Euclidean space forms

There are 74 possible 4-dimensional Euclidean space forms (*see* [14]), deciding which of these can arise from our construction is not easy. However, as we are only interested in cohomology, for us it suffices to identify what possible cohomology can arise from space forms of the form X_F/E_n. Using the very special nature of $\mathcal{G}_F$ we can show that in fact only two possible Poincaré series can occur:

(1) Take $F = \mathbb{F}_p((t_1))((t_2))((t_3))$ with p congruent to 3 mod 4. In this case, $n = 4$, $\dim \mathrm{H}^2(G_F) = 6$ and from our formula we see that $r = n + \binom{n}{2} - \dim \mathrm{H}^2(G_F) = 4$, and hence $q_F(t) = 1 + 4t + 6t^2 + 4t^3 + t^4$.

(2) Take $F = K((t))$, where $\mathcal{G}_K = \mathrm{W}(2)$. Then $n = 3$, $\dim \mathrm{H}^2(G_F) = 2$, hence $r = 4$ and $q_F(t) = 1 + 3t + 4t^2 + 3t^3 + t^4$.

4.2. Example: $F = \mathbb{Q}_2$

In this case, the vectors $[-1]$, $[2]$, $[5]$ form a basis for $\dot{F}/\dot{F}^2$. This field is non-formally real, hence the E_3 action will be free. Let $E_3 = \langle e_1, e_2, e_3 \rangle$, we now define the action on $X_{\mathbb{Q}_2}$, using complex coordinates:

$$e_1(z_1, z_2, z_3, z_4, z_5) = (z_1, z_2, \overline{z_3}, \overline{z_4}, -z_5), \quad e_2(z_1, z_2, z_3, z_4, z_5) = (-z_1, z_2, -\overline{z_3}, z_4, -\overline{z_5}),$$

$$e_3(z_1, z_2, z_3, z_4, z_5) = (z_1, -z_2, z_3, -\overline{z_4}, \overline{z_5})$$

and the orbit space has Poincaré series $q_F(t) = 1 + 3t + 6t^2 + 6t^3 + 3t^4 + t^5$.

4.3. Superpythagorean fields

We first recall from [7] that a formally real field F is said to be *superpythagorean* if it satisfies the following condition. For any subset S of F containing $\dot{F}^2$ but such that $-1 \notin S$, if S is a subgroup of index 2 in $\dot{F}$, then S is an ordering on F. It is easy to see that superpythagorean fields are Pythagorean. Nice examples of such fields are given by $F_n = \mathbb{R}((t_1))((t_2)) \cdots ((t_{n-1}))$, the field of iterated power series over $\mathbb{R}$. In this case $|\dot{F}_n/\dot{F}_n^2| = 2^n$, with a basis given by the classes $[-1]$, $[t_1]$, $\ldots$, $[t_{n-1}]$. Denote the associated W-group by $S(n)$. Making an appropriate choice of generators arising from Galois theory, one can prove that if F is *any* superpythagorean field with $|\dot{F}/\dot{F}^2| = 2^n$, then in fact $\mathcal{G}_F \cong S(n)$. Taking the space X_{F_n} constructed as before for F_n, we have $X_{F_n} \simeq (\mathbb{S}^1)^{n-1}$ with an action of $E_n = (\mathbb{Z}/2)^n$. If we write $\mathrm{H}^*(E_n) \cong \mathbb{F}_2[x_1, \ldots, x_n]$, then we obtain:

$$\mathrm{H}^*_{E_n}(X_{F_n}) \cong \mathrm{H}^*(F_n) \cong \mathbb{F}_2[x_1, \ldots, x_n]/x_2(x_1 + x_2), \ldots, x_n(x_1 + x_n)$$

and $q_{F_n}(t) = \frac{(1+t)^{n-1}}{(1-t)}$. The class $x_1 = [-1]$ is non-nilpotent in $\mathrm{H}^*(\mathcal{G}_F)$, and corresponds to a cyclic isotropy subgroup.

If F is superpythagorean with infinite square class group, $\mathcal{G}_F$ can be expressed as an inverse limit of finite groups, each of which is isomorphic to some $S(n)$. Choosing a basis $\mathcal{B}$ for $\dot{F}/\dot{F}^2$ which contains $x_1 = [-1]$, one can use this to express the cohomology of $\mathcal{G}_F$ as a limit of the cohomology of these finite groups and from there show (*see* [12]) that $\mathcal{R} \cong \mathrm{H}^*(F) \cong \mathbb{F}_2[x_i \mid x_i \in \mathcal{B}]/M$, where M is the ideal generated by the set $\{x_i(x_1 + x_i) \mid x_i \in \mathcal{B}\}$.

4.4. The universal W-groups $\mathrm{W}(n)$

Given their defining properties, these are perhaps the most interesting of all W-groups. Using the well-known theorem of Tsen–Lang, for any natural number n we can construct an infinite algebraic extension K_n of $\mathbb{C}(t)$ such that $\mathcal{G}_{K_n} = \mathrm{W}(n)$. In what follows we fix n and denote $F = K_n$. The model in this instance is deceptively simple. If $E_n = \langle e_1, \ldots, e_n \rangle$, let $X_F = (\mathbb{S}^1)^{n+\binom{n}{2}}$ with coordinates z_{ij}, then we define:

$$e_\ell(z_{ij}) = \begin{cases} -z_{ij} & \text{if } i = j = \ell; \\ \overline{z}_{ij} & \text{if } i = \ell, \ j \neq \ell; \\ -\overline{z}_{ij} & \text{if } j = \ell, \ i \neq \ell; \\ z_{ij} & \text{otherwise.} \end{cases}$$

This is, of course, a free action, and the problem is now to compute the cohomology of the orbit space X_F/E_n. This seems to be an interesting problem in its own right.

We construct cohomology classes by a somewhat unusual method. We replace X_F/E_n by a different model M_n and compute its rational cohomology; we sketch how this goes. Let M_n be the $n + \binom{n}{2}$ dimensional manifold described as a fiber bundle over $(\mathbb{S}^1)^n$ with fiber $(\mathbb{S}^1)^{\binom{n}{2}}$ and k-invariants $y_i y_j$, $i \neq j$, where $\mathrm{H}^*((\mathbb{S}^1)^n, \mathbb{Z}) \cong \Lambda(y_1, \ldots, y_n)$. Then there is a homomorphism $\pi_1(M_n) \longrightarrow \mathrm{W}(n)$ which induces an isomorphism $\mathrm{H}^*(\mathrm{W}(n))/(\zeta_{ij}) \cong \mathrm{H}^*(M_n) \cong \mathrm{H}^*(X_F/E_n)$ and so we have an inclusion $\mathrm{H}^*(M_n, \mathbb{Z})/\mathrm{Torsion} \otimes \mathbb{F}_2 \hookrightarrow \mathrm{H}^*(X_F/E_n)$. The term on the left can be computed using a rational Eilenberg–Moore spectral sequence and Lie algebra homology (*see* [8], [6]) from which we obtain the following explicit combinatorial formula in:

THEOREM 4.5. – *If $\mathcal{G}_F = \mathrm{W}(n)$ and $q_F(t) = 1 + a_1 t + \cdots + a_r t^r$, then*

$$a_i \geq \sum_{p+q=i} \sum_{Y_\lambda} \prod_{(s,t) \in Y_\lambda} \frac{n + t - s}{h(s,t)},$$

where Y_λ ranges over all symmetric, $p + 2q$-box, p-hook Young diagrams, and $h(s,t)$ denotes the hooklength of the box (s,t).

In particular we can verify that Theorem 4.5 gives an equality for $i \leq 3$, whence we obtain that $a_1 = n$, $a_2 = n(n+1)(n-1)/3$, and $a_3 = n(n^2 - 1)(3n - 4)(n + 3)/60$.

Determining whether Theorem 4.5 always gives rise to an equality is an interesting problem, equivalent to showing that the homology of the integral Koszul complex $K(\mathbb{Z}) = \Lambda(u_{ij}) \otimes$

$\Lambda(y_1, \ldots, y_n)$ with $d(u_{ij}) = y_i y_j$ is 2-torsion free. More generally, for finite W-groups one might apply the mod 2 Eilenberg–Moore spectral sequence directly to the extension (1.2), using the explicit differentials described in [5]. For all examples known to us it will indeed collapse at E_2; in particular, the second statement in Theorem 2.1 can be expressed in terms of such a collapse along one of the edges of the spectral sequence (which will always occur). A global collapse theorem would, of course, be highly desirable, as it would allow us to express $H^*(\mathcal{G}_F)$ as a certain Tor algebra (up to filtration).

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(*) To be precise, we have: if F and L are fields such that $WF \cong WL$, then $\mathcal{G}_F \cong \mathcal{G}_L$; moreover, the converse is true under the further assumption that F and L have the same level if $\langle 1, 1\rangle_F$ is universal.

(**) Recall that F is said to be *Pythagorean* if $F^2 + F^2 = F^2$.

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