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Torsion in equivariant cohomology

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§0. Introduction

Let X be a finite dimensional G–CW complex, where G is a finite group. Swan [S] introduced the notion of equivariant Tate Cohomology motivated by the fact that it vanishes for free actions and that it is torsion over $\mathbf{Z}$. This simplifies and strengthens certain cohomological arguments involving spectral sequences.

In this framework, a natural question arises: what is the minimum integer m which annihilates $\hat{H}^*_G(X)$? In this paper we will show that, roughly speaking, the torsion in $\hat{H}^*_G(X)$ quantifies the nature of the isotropy subgroups of G cohomologically. More precisely,

THEOREM 3.1. *Let X be a finite dimensional G–CW complex. Then*

$$\exp \hat{H}^*_G(X) \;\Big|\; \prod_{i=1}^{r(X)} \exp y_i$$

where $y_1, \ldots, y_{r(X)} \in H^*(G, \mathbf{Z})$, $r(X) = \max\{p\text{-rank } G_\sigma \mid G_\sigma$ *is an isotropy subgroup*$\}$ *and p ranges over all prime divisors of* $|G|$.

The proof is based on a recent result due to Carlson [C2] concerning the exponent of $\mathbf{Z}G$-modules. His techniques apply readily to our geometric situation by considering the cellular chain complex of X as a graded permutation module over $\mathbf{Z}G$. The main tools are from complexity theory: we summarize what we need in §2.

For elementary abelian groups, the result can in fact be sharpened to

THEOREM 4.1. *Let X be a finite dimensional G–CW complex, where* $G = (\mathbf{Z}/p)^r$. *Then*

$$\exp \hat{H}^*_G(X) = \max\{|G_\sigma| \mid G_\sigma \text{ is an isotropy subgroup}\}.$$

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This can be thought of as an exponent version of a theorem due to Quillen [Q], which states that the asymptotic growth rate of equivariant cohomology with $\mathbf{F}_p$ coefficients is determined by its p-elementary abelian isotropy subgroups. The main difference is that the torsion information lies within a finite range of dimensions.

As a corollary of the proof we obtain that for p-elementary abelian groups the size of the largest isotropy subgroup is determined by the exponent of $\hat{H}^0_G(*)\to\hat{H}^0_G(X)$.

In terms of ordinary equivariant cohomology we obtain the following result:

COROLLARY 4.5. *Let X be a finite dimensional G–CW complex, $G=(\mathbf{Z}/p)^r$. If $i>\dim X$, then there exists an isotropy subgroup $G_\sigma\subset G$ such that*

$$\exp H^i(X\times_G EG,\mathbf{Z})\mid |G_\sigma|.$$

We recover a result due to Browder [B] for homology manifolds with an orientation-preserving $(\mathbf{Z}/p)^r$ action and in particular a generalization of his estimate on the rank of symmetry, namely:

COROLLARY 4.2. *Let X be a connected finite dimensional G–CW complex, $G=(\mathbf{Z}/p)^r$. Then*

$$|G|/\max\{|G_\sigma|\}\ \Big|\ \prod_{i=1}^{\infty}\exp\hat{H}^{-i-1}(G,H^i(X,\mathbf{Z}))$$

as G_σ ranges over all the isotropy subgroups of G.

Finally we include an application of our techniques to exhibit the cohomology classes of order p^{n+1} in $H^*(E_n,\mathbf{Z})$, where E_n is the extra-special p-group of order p^{2n+1}, p odd, all of those elements have exponent p.

The paper is organized as follows: in §1 we describe the main properties of equivariant Tate Cohomology; in §2 we give the basic definitions and concepts needed from complexity theory; in §3 we prove our main theorem and in §4 the applications are given.

The author is indebted to J. Carlson for inspiring and motivating this work.

§1. Equivariant Tate (co)homology

In this section we will describe the main properties of equivariant Tate (co)homology for a finite dimensional G–CW complex. G will be finite throughout.

DEFINITION 1.1. A complete resolution over $\mathbf{Z}G$ is an acyclic complex $F_* = (F_i)_{i\in\mathbf{Z}}$ of projective $\mathbf{Z}G$-modules, together with a map $\epsilon : F_0 \to \mathbf{Z}$ such that $\epsilon : F_*^+ \to \mathbf{Z}$ is a resolution in the usual sense, $F_*^+ = (F_i)_{i\geq 0}$.

Let X be a G–CW complex, with cellular integral chain complex $C_*(X)$.

DEFINITION 1.2. (1) The equivariant Tate homology of X is defined as

$$\hat{H}_i^G(X) = H_i(F_* \otimes_G C_*(X))$$

where F_* is a complete resolution.

(2) The equivariant Tate cohomology of X is defined as

$$\hat{H}_G^i(X) = H^i(\mathrm{Hom}_G(F_*, C^*(X)))$$

where F_* is a complete resolution.

The usual properties of Tate (co)homology apply to these groups, and in particular they are torsion over $\mathbf{Z}$. The following proposition relates them to ordinary equivariant (co)homology.

PROPOSITION 1.3. *If* $i > \dim X$, *then*

$$\hat{H}_G^i(X) = H^i(X \times_G EG, \mathbf{Z}), \qquad \hat{H}_i^G(X) = H_i(X \times_G EG, \mathbf{Z}).$$

Proof. We have a short exact sequence of complexes

$$0 \to \bar{F}_*^- \to F_* \xrightarrow{\varphi} F_*^+ \to 0.$$

In the long exact homology sequence associated to the above after tensoring with $C_*(X)$ over $\mathbf{Z}G$, it is clear that for $i \geq \dim X$

$$H_i(\bar{F}_*^- \otimes_G C_*(X)) = 0.$$

Hence φ induces the desired isomorphism; the argument for cohomology is analogous.

The main advantage of Tate (co)homology (first introduced by Swan [S]) is that it vanishes for free actions. This can be deduced from the second of two spectral sequences available to compute $\hat{H}_G^i(X)$ (analogous for homology)

$$E_2^{p,q} = \hat{H}^p(G, H^q(X, \mathbf{Z})) \Rightarrow \hat{H}_G^{p+q}(X)$$
$$E_1^{p,q} = \hat{H}^q(G, C^p(X)) \Rightarrow \hat{H}_G^{p+q}(X).$$

These arise from the two filtrations on the double complex $\mathrm{Hom}_G(F_*, C^*(X))$. We quote a result due to Adem [A] which we will use later on

THEOREM 1.4. *If X is a connected finite dimensional G–CW complex, then*

$$|G|/\exp \operatorname{im} \epsilon^* \ \Big| \ \prod_{i=1}^{\infty} \exp \hat{H}^{-i-1}(G, H^i(X, \mathbf{Z}))$$

where $\epsilon^* : \mathbf{Z}/|G| \to \hat{H}^0_G(X)$ is induced by the augmentation.

§2. Complexity and cohomological varieties

We recall the notions of complexity theory necessary in the proof of the main theorem.

Let K be a field of characteristic $p > 0$. For a finite group G, let $H(G, K) = H^*(G, K)$ if $p = 2$ and $H(G, K) = \sum_{n \geq 0} H^{2n}(G, K)$ if p is odd; denote by $V_G(K)$ its maximal ideal spectrum.

If M is a finitely generated KG-module, $\mathrm{Ext}^*_{KG}(M, M)$ is a finitely generated module over $H(G, K)$.

DEFINITION 2.1. Let M be a KG-module, then $V_G(M)$ is the collection of all maximal ideals of $H(G, K)$ that contain $J(M)$, the annihilator in $H(G, K)$ of $\mathrm{Ext}^*_{KG}(M, M)$.

$V_G(M)$ is called the cohomological variety of M.

Now let $P_* \to M$ be a minimal projective resolution of M over KG. The complexity of M is the well defined integer

$$cx_G(M) = \min \left\{ s \geq 0 \mid \lim_{n \to \infty} \frac{\dim P_n}{n^s} = 0 \right\}.$$

The following is a list of properties of $V_G(M)$ which we will need later on (we refer to [Be], [C1] for more details).

PROPOSITION 2.2
1. $V_G(M) = \{0\} \Leftrightarrow M$ is projective.
2. $\dim V_G(M) = cx_G(M)$.
3. $V_G(M_1 \oplus M_2) = V_G(M_1) \cup (V_G(M_2)$.

4. $V_G(M_1 \otimes M_2) = V_G(M_1) \cap V_G(M_2)$.
5. $V_G(K) = p$-rank of G, where char$(K) = p$.

Similarly if $\gamma \in H(G, K)$, we define $V_G(\gamma) =$ Subvariety of $V_G(K)$ consisting of ideals which contain γ.

Now let X be a $G-CW$ complex with isotropy subgroups $\{G_\sigma\}_{\sigma \in S}$.

DEFINITION 2.3. The cohomological isotropy variety of X at p is $V_G(X)_p = \bigcup_{\sigma \in S} V_G(\mathbf{F}_p[G/G_\sigma])$.

Clearly, by 2.2 dim $V_G(X)_p = \max\{p - \text{rank } G_\sigma\}$. These cohomological varieties carry the necessary information to extract our main result about the torsion in $\hat{H}^*_G(X)$.

§3. The main theorem

THEOREM 3.1. *Let X be a finite dimensional $G-CW$ complex. Then there exist classes $\xi_i \in H^{s_i}(G, \mathbf{Z})$ $i = 1, \ldots, r(X)$, such that*

$$\exp \hat{H}^*_G(X) \,\Big|\, \prod_{i=1}^{r(X)} \exp \xi_i$$

where

$$r(X) = \max_{\sigma, p} \{p - \text{rank } G_\sigma\}.$$

Proof. Let $\delta_p : H^*(G, \mathbf{Z}) \to H^*(G, \mathbf{F}_p)$ and denote $M = \oplus_\sigma \mathbf{Z}[G/G_\sigma]$; clearly

$$V_G(X)_p = V_G(M/pM) \quad \text{and} \quad r(X) = \max_{p \mid |G|} \{cx_G(M/pM)\}.$$

By a result due to Carlson [C2] we may choose $\xi_1, \ldots, \xi_{r(X)} \in H^*(G, \mathbf{Z})$ such that

$$\left(\bigcap_{i=1}^{r(X)} V_G(\delta_p(\xi_i))\right) \cap V_G(M/pM) = \{0\}$$

for all $p \mid |G|$.

It is not hard to see that the ξ_i can be represented by maps $\hat{\xi}_i$

$$0 \to L_i \to \Omega^{s_i}(\mathbf{Z}) \xrightarrow{\hat{\xi}_i} \mathbf{Z} \to 0.$$

Here $\Omega^{s_i}(\mathbf{Z})$ is a dimension-shift (torsion free) of $\mathbf{Z}$, i.e. $\hat{H}^k(G, \Omega^{s_i}(\mathbf{Z})) \cong \hat{H}^{k-s_i}(G, \mathbf{Z})$.

One can also verify (see [C1]) that $V_G(\delta_p(\xi_i)) = V_G(L_i/pL_i)$. Now from 2.2(4) it follows that

$$V_G(L_1 \otimes \cdots \otimes L_{r(X)} \otimes M/pM) = \{0\}.$$

Hence $L_1 \otimes \cdots \otimes L_{r(X)} \otimes M$ is projective (2.2.1) and so each summand $L_1 \otimes \cdots \otimes L_{r(X)} \otimes \mathbf{Z}[G/G_\sigma]$ is too. We conclude that the $\mathbf{Z}G$-(co)chain complex $L_1 \otimes \cdots \otimes L_{r(X)} \otimes C^*(X)$ is made up of projective $\mathbf{Z}G$-modules (twisting by orientation characters does not matter).

Now for each $i = 1, \ldots, r(X)$ we have a short exact sequence of $\mathbf{Z}G$-(co)chain complexes:

$$\begin{aligned} 0 \to C^*(X) \otimes L_1 \otimes \cdots \otimes L_i \to C^*(X) \otimes L_1 \otimes \cdots \otimes L_{i-1} \otimes \Omega^{s_i}(\mathbf{Z}) \\ \xrightarrow{1\otimes\cdots\otimes 1\otimes\hat{\xi}_i} C^*(X) \otimes L_1 \otimes \cdots \otimes L_{i-1} \to 0. \end{aligned}$$

(3.2)

We examine $1 \otimes \cdots \otimes \hat{\xi}_i$ in Tate cohomology:

$$\hat{H}^k(G, C^*(X) \otimes L_1 \otimes \cdots \otimes L_{i-1} \otimes \Omega^{s_i}(\mathbf{Z})) \to \hat{H}^k(G, C^*(X) \otimes L_1 \otimes \cdots \otimes L_{i-1}).$$

By the obvious dimension-shifting, we have that

$$\hat{H}^k(G, C^*(X) \otimes L_1 \otimes \cdots \otimes L_{i-1} \otimes \Omega^{s_i}(\mathbf{Z})) \cong \hat{H}^{k-s_i}(G, C^*(X) \otimes L_1 \otimes \cdots \otimes L_{i-1})$$

and the map $(1 \otimes \cdots \otimes 1 \otimes \hat{\xi}_i)^*$ represents cup product by $\xi_i \in H^{s_i}(G, \mathbf{Z})$. Clearly then we have that $\exp im(1 \otimes \cdots \otimes 1 \otimes \hat{\xi}_i)_*$ divides $\exp \xi_i$.

Now from the sequence 3.2 we derive that

$$\exp \hat{H}^*(G, C^*(X) \otimes L_1 \otimes \cdots \otimes L_{i-1}) / \exp \hat{H}^*(G, C^*(X) \otimes L_1 \otimes \cdots \otimes L_i)$$

divides $\exp \xi_i$.

Multiplying out these relations for $i = 1, \ldots, r(X)$ we obtain

$$\exp \hat{H}^*_G(X)/\exp \hat{H}^*(G, C^*(X) \otimes L_1 \otimes \cdots \otimes L_{r(X)}) \;\Big|\; \prod_{i=1}^{r(X)} \exp \xi_i$$

Using the fact that $C^*(X) \otimes L_1 \otimes \cdots \otimes L_{r(X)}$ is projective and the second spectral sequence in §1 it is clear that $\hat{H}^*(G, C^*(X) \otimes L_1 \otimes \cdots \otimes L_{r(X)}) \equiv 0$, thus completing the proof.

From the proof it is apparent that the classes $\xi_i \in H^*(G, \mathbf{Z})$ depend on how the isotropy subgroups are related to G cohomologically. In general this may be very complicated, but when G is p-elementary abelian, it is not. The following corollary illustrates how torsion in the equivariant cohomology quantifies the size of the isotropy subgroups; this will be made more precise in the following section.

COROLLARY 3.2. *Let X be a finite dimensional $G-CW$ complex, where $G = (\mathbf{Z}/p)^r$. Then*

$$\exp \hat{H}^*_G(X) \mid \max_{\sigma} \{|G_\sigma|\}.$$

§4. Applications and Examples

Let X be a finite dimensional $G-CW$ complex. There is an obvious equivariant map $X \to *$, which induces a map of G-chain complexes $C_*(X) \xrightarrow{\varepsilon} \mathbf{Z}$. This map factors through $C_0(X)$, yielding a commutative triangle:

$$\begin{tikzcd} C_*(X) \arrow[rr, "\varepsilon"] & & \mathbf{Z} \\ & C_0(X) \arrow[ul, "i"] \arrow[ur, "\varepsilon^0"'] & \end{tikzcd}$$

Let S denote a set of 0-cells in X representing the G-orbits; then in Tate Cohomology the above diagram induces

$$\begin{tikzcd} \hat{H}^*_G(X) \arrow[dr, "i^*"] & & \hat{H}^*(G, \mathbf{Z}) \arrow[ll, "\varepsilon^*"'] \arrow[dl, "(\varepsilon^0)^*"] \\ & \bigoplus_{\sigma \in S} \hat{H}^*(G_\sigma, \mathbf{Z}) & \end{tikzcd}$$

where $(\varepsilon^0)^*$ is the usual map induced by the augmentation, from which we deduce that for all σ in S

$$|G_\sigma| \mid \exp \hat{H}^*_G(X).$$

Using equivariant subdivision, it follows that the above holds for any isotropy subgroup, and so we have

$$\operatorname{lcm}\{|(G_\sigma)|\} \mid \exp \hat{H}^*_G(X).$$

For elementary abelian groups, 3.1 and the preceding remarks combine to yield.

THEOREM 4.1. *If $G = (\mathbf{Z}/p)^r$ and X is a finite-dimensional G–CW complex, then*

$$\max\{|G_\sigma|\} = \exp \operatorname{im} \varepsilon^* = \exp \hat{H}^0_G(X) = \exp \hat{H}^*_G(X).$$

Given a G–CW complex X, where $G = (\mathbf{Z}/p)^r$, we have shown that

$$\hat{H}^0_G(pt) \to \hat{H}^0_G(X)$$

measures the size of the largest isotropy subgroup. This can be estimated using the first spectral sequence in §2: the only differentials involved are

$$E_r^{-r\,r-1} \to E_r^{0,0} \qquad r \geq 2.$$

The term $E_\infty^{0,0}$ is the image of the map $\hat{H}^0(G, H^0(X)) \to \hat{H}^0_G(X)$ and the map induced by ε^0 factors through it. As in 1.4 we have

COROLLARY 4.2. *If X is a connected, finite dimensional G–CW complex, $G = (\mathbf{Z}/p)^r$, then*

$$|G|/\max_\sigma\{|G_\sigma|\} \,\Big|\, \prod_{i=1}^{\infty} \exp \hat{H}^{-i-1}(G, H^i(X)).$$

This was proved by Browder [B] for orientation preserving $(\mathbf{Z}/p)^r$-actions on homology manifolds, using the following result, which we recover using our methods:

THEOREM 4.3. *If $G = (\mathbf{Z}/p)^r$ acts cellularly on a homology manifold M^n*

preserving orientation, then

$$|G|/\max\{|G_\sigma|\} = |H^n(M, \mathbf{Z})/j^*H^n(M \times_G EG, \mathbf{Z})|$$

where $j: M \to M \times_G EG$.

Proof. Using duality it is not hard to see that

$$|H^n(M, \mathbf{Z})/j^*H^n(M \times_G EG, \mathbf{Z})| = |G|/\exp \hat{H}^*_G(M).$$

An application of 4.3 completes the proof.

For groups that are not elementary abelian, 4.3 fails. Browder [B] has constructed an example of a cellular $\mathbf{Z}/p^2$-action on $X = S^2 \times S^{2n-1}$ such that it preserves orientation, $\operatorname{im} j^* \neq 0 \bmod p$ $(j: X \to X \times_{\mathbf{Z}/p^2} E\mathbf{Z}/p^2)$ but $X^{\mathbf{Z}/p^2} = \emptyset$. This means that $\exp \hat{H}^*_{\mathbf{Z}/p^2}(X) = p^2$ but still $X^{\mathbf{Z}/p^2} = \emptyset$. We also have

COROLLARY 4.4

Krull Dimension of $H^*(X \times_G EG, \mathbf{F}_p) = \max\limits_{E \subset G} \{\log_p (\exp \hat{H}^0_E(X))\}$

as E ranges over all p-elementary abelian subgroups of G.

The significance of 4.4 is that asymptotic information about $H^*(X \times_G EG, \mathbf{F}_p)$ can be obtained from a single Tate Cohomology group. In terms of ordinary equivariant cohomology we have

COROLLARY 4.5. *Let X be a G–CW complex $G = (\mathbf{Z}/p)^r$. Then, if $i > \dim X$, there exists an isotropy subgroup $G_\sigma \subset G$ such that*

$$\exp H^i(X \times_G EG, \mathbf{Z}) \mid |G_\sigma|.$$

EXAMPLE 4.6. We now apply Theorem 4.3 to obtain cohomology classes for the extra-special p-groups with elements of exponent p, for p odd. Denote by E_n the one of order p^{2n+1}, described by:

Generators: $x_1, \ldots, x_n, y_1, \ldots, y_n, c$

Relations:
$$[x_i, y_j] = 1 \quad \text{for} \quad i \neq j$$
$$[x_i, y_i] = c$$
$$[x_{i_1}, x_{i_2}] = [y_{i_1}, y_{i_2}] = 1$$
$$x_i^p = y_j^p = 1 \quad \text{for} \quad 1 \leq i, j \leq n$$
c central.

Let T denote the one-dimensional unitary representation of $K \subset E_n$, the subgroup generated by $x_1, \ldots, x_n, c$, determined by

$$x_1, \ldots, x_n \mapsto 1 \quad \text{and} \quad c \mapsto e^{2\pi i/p}.$$

Then $V = \mathbf{C}E_n \otimes_K T$ is unitary, and E_n acts cellularly on $X = S(V)$. This E_n-space was used by Thomas in [Th] for K-theory calculations.

Notice that $\langle c \rangle$ acts freely on X, hence

$$\hat{H}^*_{E_n}(X) \cong \hat{H}^*_{E_n/\langle c \rangle}(X/\langle C \rangle).$$

The elements $x_1, \ldots, x_n, y_1, \ldots, y_n$ map to a basis of the quotient group $E_n/\langle c \rangle \cong (\mathbf{Z}/p)^{2n}$. The isotropy subgroups are all of rank $\leq n$; hence we conclude that $\exp \hat{H}^*_{E_n}(X) = p^n$.

Using the first spectral sequence described in §1 we obtain an exact sequence

$$\hat{H}^{2p^n-1}_{E_n}(X) \to \mathbf{Z}/p^{2n+1} \xrightarrow{d} \hat{H}^{2p^n}(E_n, \mathbf{Z}).$$

Hence $d(\mu_X) = \xi$ is an element of exponent at least p^{n+1}. However, as this is the upper bound for $\exp \bar{H}^*(E_n, \mathbf{Z})$, it *has* this exponent. It is the p^n-th Chern class of the representation V, and by its construction, ξ^i has highest exponent for all $i \geq 1$. (Carlson [C2] has supplied an algebraic argument to locate classes of this exponent, Tezuka and Yagita [T–Y] have done this using Brown–Peterson Cohomology).

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