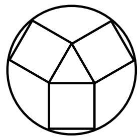
Problems, January 2009

Problem 1. Find all pairs (x, y) of real numbers such that

$$x^2 + xy + y^2 = 8 \qquad \text{and} \qquad x - xy + y = 2.$$

Problem 2. Squares are erected externally on the sides of an equilateral triangle with sides 3. What is the radius of the smallest circle that contains the resulting figure?



Problem 3. Let a and b be positive integers. Show that $\sqrt{3}$ lies between $\dfrac{a}{b}$ and $\dfrac{a+3b}{a+b}$.

Problem 4. (a) Show that it is not possible to find 2008 *odd* integers x_1, x_2, ..., x_{2008} such that

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_{2008}} = \frac{1}{2009}.$$

(b) The above equation obviously has a solution in positive integers (let $x_k = 2008 \cdot 2009$ for all k.) Show that it has a solution where the x_k are *distinct* positive integers.