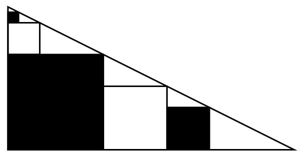
Problems, January 2010

Problem 1. For what values of m are the four roots of the equation

$$x^4 - (2m+4)x^2 + m^2 = 0$$

all real and in arithmetic progression?

Problem 2. The picture shows five squares, three of them shaded. Let z be the side of the largest square, and let x and y be the sides of the other two shaded squares. Show that $\sqrt{x} + \sqrt{y} = \sqrt{z}$.



Problem 3. (a) Show that if A is a positive integer, there exist a positive integer n, and non-negative integers $a_1, a_2, \ldots, a_n$ such that $a_k \le k$ for all k and

$$A = a_1 \cdot 1! + a_2 \cdot 2! + a_3 \cdot 3! + \cdots + a_n \cdot n!$$

(example: $77 = 1 \cdot 1! + 2 \cdot 2! + 0 \cdot 3! + 3 \cdot 4!$).

(b) Let r be a rational number such that $0 < r < 1$. Show that there exist a positive integer n, and non-negative integers $a_1, a_2, \ldots, a_n$ such that $a_k \le k$ for all k and

$$r = \frac{a_1}{2!} + \frac{a_2}{3!} + \cdots + \frac{a_n}{(n+1)!}.$$

Problem 4. Let $a_1, a_2, \ldots, a_7$ be integers. Use the Pigeonhole Principle to show that there are numbers $\epsilon_1, \epsilon_2, \ldots, \epsilon_7$ such that:

1. any ϵ_i is equal to 1, 0, or -1,
2. not all the ϵ_i are 0, and
3. $\epsilon_1 a_1 + \epsilon_2 a_2 + \cdots + \epsilon_7 a_7$ is divisible by 100.

http://www.pims.math.ca/education/math_problems/
http://www.math.ubc.ca/~adler/problems/