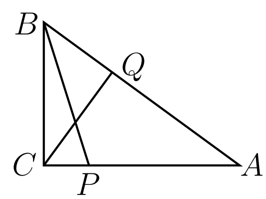
Problems, January 2011

Problem 1. (a) Let n be even, and let $\mathcal{P}$ be a regular n-sided polygon which is inscribed in a circle of radius 1. Show that for any point A on that circle, the sum of the squares of the distances from A to the vertices of $\mathcal{P}$ is $2n$. (b) What about if n is odd? (This is probably quite a bit harder.)

Problem 2. Suppose that a and b are positive integers, and that n is an integer greater than 1. Show that if $a^n + b^n$ is a power of 2, then $a = b = 2^e$ for some non-negative integer e.

Problem 3. Triangle ABC is right-angled at C, CQ is an altitude of $\triangle ABC$, and $AQ = 1$. Point P on CA has the property that $AP = BP = 1$. Find an exact expression for the length of AB.



Problem 4. A fair die has the numbers 0, 0, 0, 0, 0, and 1 written on its six faces. This die is tossed n times. Find a simple expression for the probability that the sum of the n numbers obtained is odd. The expression should not involve the summation operator $\sum$.