

Problems, November 2007

Problem 1. It is not hard to verify with a calculator that

$$\frac{5^3 + 1^3}{5^3 + 4^3} = \frac{5 + 1}{5 + 4}, \quad \frac{67^3 + 41^3}{67^3 + 26^3} = \frac{67 + 41}{67 + 26}, \quad \frac{124^3 + 43^3}{124^3 + 81^3} = \frac{124 + 43}{124 + 81}.$$

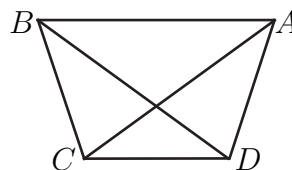
But it is easier to “cancel” the 3’s from numerator and denominator! Explain why cancelling the 3’s gives the right answer in these and similar cases.

Problem 2. Suppose that $0 < x < y$. Which is larger,

$$\frac{1 + x + x^2 + \cdots + x^{100}}{1 + x + x^2 + \cdots + x^{101}} \quad \text{or} \quad \frac{1 + y + y^2 + \cdots + y^{100}}{1 + y + y^2 + \cdots + y^{101}}?$$

(This is straightforward only if it is approached the right way.)

Problem 3. In quadrilateral $ABCD$, $AB = AC = BD$ and $BC = CD = DA$. Find AB/BC .



Problem 4. At the beginning, bowl A contains 2 cups of sugar, and nothing else, and bowl B contains 1 cup of flour, and nothing else.

Xavier transfers 1 cup of stuff from A to B, mixes the contents of B thoroughly, then transfers 1 cup of stuff from B to A, and mixes the contents of A thoroughly. That’s the end of the first *cycle*. Xavier then transfers 1 cup of stuff from A to B, mixes the contents of B thoroughly, transfers one cup of stuff from B to A, and mixes thoroughly. That’s the end of the second cycle. Xavier continues. (a) How much flour is in bowl A at the end of the 4-th cycle? (b) Find a simple expression for the amount of flour in bowl A at the end of the n -th cycle, and prove that your expression is correct.

Problem 5. Alicia types all the numbers from 1 to 10^5 inclusive, and Beti types all the numbers from 1 to 10^6 , inclusive. (a) Show that the total number of digits that Alicia types is equal to the total number of 0’s that Beti types. (b) Generalize, with 10^k and 10^{k+1} in lieu of 10^5 and 10^6 .