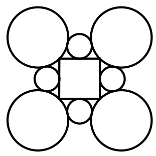
Problems, December 2009

Problem 1. Alphonse and Beth play the following game. A neutral third party alternately tosses a biased penny, which has probability 2/3 of landing heads, and a biased dime, which has probability 1/3 of landing heads. The penny is tossed first. The game ends when two consecutive heads first occur, in which case Alphonse wins, or when two consecutive tails first occur, in which case Beth wins. What is the probability that Alphonse wins?

Problem 2. In the figure below, we have a square, and four congruent circles (large in the picture) that pass through the vertices of the square, and are symmetric with respect to the axes of symmetry of the square. We also have four congruent circles (small in the picture), which are tangent to various lines and "large" circles as shown. Given that "large" circles have radius a, and "small" circles have radius b, what is the side of the square?



Problem 3. Find all solutions in real numbers of the following system of equations:

$$x(y^2+z^2) = 11yz,$$
$$y(z^2+x^2) = 12xz,$$
$$z(x^2+y^2) = 13xy.$$

Problem 4. Find all positive integer solutions of $x^2 - xy - y^2 = \pm 1$. (A proof that you have found them all is needed.)