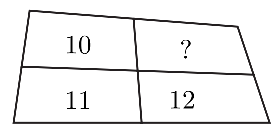
Solutions to January 2006 Problems

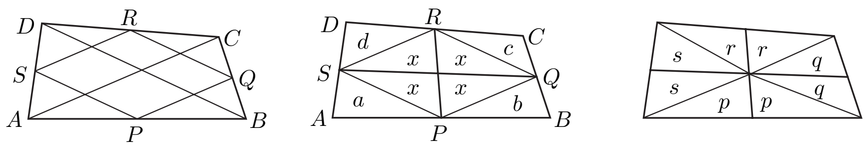
Problem 1. A convex quadrilateral is split into four parts by joining the midpoints of opposite sides as in the picture below. If three of the parts, going counterclockwise, have area 10, 11, and 12, what is the area of the fourth part? (The picture is not drawn to scale.)



Solution. We prove a preliminary result that is interesting in itself.

Lemma (The Varignon Parallelogram)**.** Let $ABCD$ be a quadrilateral, and let P, Q, R, and S be the midpoints of sides AB, BC, CD, and DA. Then $PQRS$ is a parallelogram.

Proof. Look at the left-hand picture in the diagram below. (The quadrilateral is drawn convex, but the lemma holds even if $ABCD$ is not convex.) Draw the diagonals of $ABCD$.



Look first at $\triangle ABC$. The line PQ joins the midpoints of two sides of the triangle, and therefore PQ is parallel to AC. The same argument, applied to $\triangle ACD$, shows that SR is parallel to AC.

Thus PQ and SR are both parallel to AC, and therefore to each other. In the same way, we can show that QR is parallel to PS. It follows that $PQRS$ is a parallelogram. It is called the *Varignon parallelogram* of the quadrilateral $ABCD$. □

Now return to the original problem, and look at the middle picture above. Let the areas of the triangles in the picture be a, b, c, d, and x as shown. Since $PQRS$ is a parallelogram, its diagonals divide $PQRS$ into four triangles of equal area, so the triangles whose areas are labeled "x" really do have the same area.

Because (left-hand picture) P and Q bisect two sides of $\triangle ABC$, the area of $\triangle PBQ$ is one-quarter of the area of $\triangle ABC$. Similarly, the area of $\triangle SRD$ is one-quarter of the area of $\triangle ACD$. Add. We find that $b + d$ is one-quarter of the area of the whole quadrilateral. Similarly, so is $a + c$. It follows that $b + d = a + c$, and therefore

$$(b + x) + (d + x) = (a + x) + (c + x).$$

In our case, $b + x = 12$, $d + x = 10$, and $a + x = 11$. The region we were curious about has area $c + x$, which is $12 + 10 - 11$, that is, 11.

Another Way. Look at the right-hand picture. Let p, q, r, and s be the areas of the triangles as shown. Note that for example the triangles whose areas are labeled "p" really do have the same area, since they have equal bases and heights.

The areas of "opposite" quadrilaterals add up to $p + q + r + s$. It follows that $12 + 10$ is equal to 11 plus the area of our target quadrilateral. That quadrilateral therefore has area 11. Exactly the same argument works if *any* point in the interior of $ABCD$ is joined to the midpoints of the sides!

Comment. The first solution is much more complicated than the second. But as a bonus it shows that $PQRS$ is a parallelogram. Besides being shorter and more natural, the second solution easily generalizes. Let n be an integer, and let $\mathcal{A}$ be a $2n$-sided (planar) polygon. Let O be any point in the interior of $\mathcal{A}$. Divide $\mathcal{A}$ into $2n$ quadrilaterals by joining O to the midpoints of the sides of $\mathcal{A}$. The second argument given above shows that if we know the areas of $2n - 1$ of the quadrilaterals, we can easily compute the area of the remaining one. There is no result close to this for polygons with an odd number of sides.

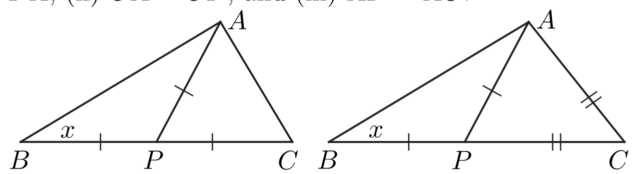
Problem 2. A triangle is isosceles. Suppose there is a line that divides the triangle into two isosceles triangles. What can we conclude about the angles of the original triangle?

Solution. For a while, we will not worry about the fact that the original triangle is isosceles, and will not even draw it as isosceles. It can be dangerous to rely on exact drawings, because it is all too easy to miss a possibility—almost everyone did.

If a line divides a triangle into two *triangles*, the line must pass through a vertex, say A. Let the other two vertices be B and C. Let P be a point on BC, and suppose that AP divides $\triangle ABC$ into two isosceles triangles.

Maybe both $\angle APB$ and $\angle APC$ are $90°$. If not, at least one of them is greater than $90°$. We can if necessary interchange the labels B and C to make sure that $\angle APB$ is greater than or equal to $90°$. Since $\triangle APB$ is isosceles, and $\angle APB$ is "big," it follows that $PA = PB$.

Now look at $\triangle APC$. This triangle is isosceles, so there are three possibilities to examine: (i) $PC = PA$, (ii) $CA = CP$, and (iii) $AP = AC$.



Case (i): Look at the left-hand diagram above. We do some "angle-chasing." Let $\angle ABP = x$ (this is short for the awkward but officially correct "Let the degree measure of $\angle ABP$ be x").

Since $PA = PB$, we have $\angle PAB = x$. But since the angles of a triangle add up to $180°$, we have $\angle APB = 180 - 2x$, and therefore $\angle APC = 2x$.

The angles of $\triangle APC$ add up to $180°$. One of them is $2x$, and the other two are equal. It follows that each is $90 - x$. Thus $\angle BAC = x + (90 - x) = 90$.

There is a more elegant way of seeing this. Note that $PC = PA = PB$. Imagine drawing the circle with center P and BC as a diameter. This circle passes through A, and therefore by a standard result $\angle BAC$ is a right angle.

Finally, we use the fact that $\triangle ABC$ is isosceles. Since $\angle BAC$ is "big" we must have $AB = AC$. Thus each of $\angle ABC$ and $\angle ACB$ has measure $45°$.

Case (ii): Now look at the right-hand picture above. Again, do a routine "angle chase." Exactly as before, we get that $\angle APC = 2x$. Since $CA = CP$, we get $\angle PAC = 2x$, and therefore $\angle BAC = 3x$ and $\angle ACB = 180 - 4x$.

But $\triangle ABC$ is isosceles. There are two possibilities. Maybe $AB = AC$. Then $x = 180 - 4x$, so $x = 36$, and the other two angles of $\triangle ABC$ are $36°$ and $108°$.

Or maybe $BA = BC$. Then $3x = 180 - 4x$, so $x = 180/7$, and the other two angles of $\triangle ABC$ are each $540/7$ degrees. (We can not have $CA = CB$, since angles CAB and CBA are respectively $3x$ and x, so they can not be equal.)

Case (iii): (The picture is not drawn.) We find by an angle chase that $\angle PCA = 2x$ and $\angle CAP = 180 - 4x$.

Now use the fact that $\triangle ABC$ is isosceles. As usual, there are several possibilities to examine. We can not have $AB = AC$, since $\angle ABC$ and $\angle ACB$ are respectively x and $2x$, so can not be equal.

Maybe $BA = BC$. Then $180 - 3x = 2x$, so $x = 36$, and the other two angles are each $72°$. Finally, maybe $CA = CB$. No, that is not possible, for any one of several reasons. For one thing, $PB = CA$, so $CB > CA$.

To sum up, the possibilities are 45–90–45, 36–108–36, 540/7–180/7–540/7, and 72–36–72.

We are not quite finished! We need to check that each of the four types of triangle *can* indeed be split into two isosceles triangles. For that, we need to reverse the arguments used above. That is quite simple. For example, let $\triangle ABC$ have $\angle ABC = \angle BCA = 36°$ (and therefore $\angle CAB = 108°$). Draw line AP so that P is on the segment BC and $\angle PAB = 36°$. It is easy to verify that the line AP splits $\triangle ABC$ into two isosceles triangles.

Another Way. Instead of splitting a "large" isosceles triangle, we can start with a "small" one and analyze how we can glue an isosceles triangle to it to make a "large" one. The detailed case by case analysis is more or less the same as the analysis given above, but a little more efficient.

Problem 3. Find the sum of all the four-digit numbers all of whose digits are odd.

Solution. There are five odd digits, and therefore 5^4 four-digit numbers all of whose digits are odd. In principle, we could list the 625 numbers and add up. With a programmable calculator, we could write a program to list and add. "Technology" is wonderful.

Thinking is better. *Imagine* listing and adding, with the addition done in the usual paper and pencil way, except that we will not do any "carrying," since that introduces an unnecessary complication.

Look first at the problem of adding the units digits. There are 5^3 of our numbers that have a 1 as their units digit, 5^3 that have a 3 as their units digit, and so on. So the units digits add up to

$$5^3(1+3+5+7+9), \qquad \text{that is, to} \qquad 25 \times 5^3.$$

Similarly, the tens digits add up to 25×5^3. But each tens digit counts the number of 10's in the answer, so the tens digits contribute $10 \times 25 \times 5^3$ to the sum. Similarly, the hundreds digits contribute $100 \times 25 \times 5^3$, and the thousands digits contribute $1000 \times 25 \times 5^3$.

Thus the full sum is

$$(1111)(25)(5^3). \tag{1}$$

If we wish, we can now calculate. The numerical answer is 3471875.

Comment. In many ways the numerical answer is less satisfactory than Expression 1. For the expression has *structure*, and the idea and expression easily generalize to variant problems, such as the sum of all six-digit numbers whose digits are all odd, or all six-digit numbers whose digits are chosen from $\{2, 3, 5, 7\}$.

The calculator often turns a structured object into a meaningless jumble of digits. Mathematics is not calculation, it is the discovery and exploitation of structure. In applied problems, we do indeed (ultimately) want a numerical answer. As much as possible, however, we should leave numerical computation to the end. We should not attack problems with calculator in one hand, unless they are essentially trivial and we are racing to finish a test.

Another Way. We break up our four-digit numbers into pairs that have a "nice" sum. Note that $1+9 = 3+7 = 5+5 = 10$. In general, if g is a digit, let $\overline{g} = 10-g$. So $\overline{1} = 9$, $\overline{3} = 7$, $\overline{7} = 3$, and $\overline{5} = 5$.

If the number x has decimal representation $abcd$, where a, b, c, and d are the digits of x from left to right, let $\overline{x}$ be the number that has decimal representation $\overline{a}\overline{b}\overline{c}\overline{d}$. Note that for any of our four-digit numbers x, $\overline{\overline{x}} = x$. Informally, call $\overline{x}$ the *partner* of x. So the partner of the partner of x is x, a good thing for domestic harmony.

We calculate $x + \overline{x}$, using ordinary paper and pencil addition. The result is 11110. And almost everybody's partner is different from herself/himself. Only 5555 is his own partner. Thus there are $(5^4-1)/2$ couples, and each couple has sum 11110, with only the lonely 5555 left out. The required sum is therefore

$$(11110)\left(\frac{5^4-1}{2}\right) + 5555.$$

Now we can calculate, but maybe we should simplify first. Note that $11110 = 2 \times 5555$, so the above expression simplifies to 5555×5^4.

There is a somewhat neater way of doing the same thing. Colour our 5^4 numbers red, and let their sum be S. Make a new collection of our numbers, and colour them blue. Then the sum of all the numbers, red and blue, is $2S$. Pair off all the numbers, by pairing any coloured x with the $\overline{x}$ of the *opposite* colour. Now everyone has a partner, there are 5^4 pairs, each with sum 11110. It follows that $2S = 11110 \times 5^4$, so S is easy to find.

Comment. Or else we could pair off any of our four-digit numbers x with the number $11110 - x$. Since we saw earlier that $x + \overline{x} = 11110$, this is the same pairing as the one described above. The only trouble with this way of defining the pairing is that we then need to show that if all the digits of x are odd, then so are all the digits of $11110 - x$, in other words that $11110 - x$ is one of our numbers. This is not hard, just look at the usual subtraction process.

Some people sent in solutions that assumed that when our numbers are arranged in increasing order, they form an arithmetic progression. They then applied a version of the usual formula for the sum of an arithmetic progression. If we use the fact that there are 625 elements in our sequence, we actually end up with the right answer—for the wrong reason.

Our numbers *do not* form an arithmetic progression. True, at the start come 1111, 1113, 1115, 1117, 1119. But the next term in the arithmetic progression that starts with 1111 and jumps by 2's is 1121, which is *not* one of our numbers. So the *theorem* that gives the sum of an arithmetic progression can not be applied directly. However, the pairing *idea* in the usual proof of the formula for the sum of an arithmetic progression *does* work, and was used in the second argument above.

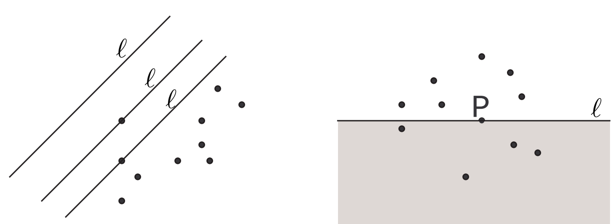
Problem 4. Let $\mathcal{S}$ be a collection of 2006 points in the same plane.

(a) Show that there is a point P in $\mathcal{S}$, and a line passing through P, such that 500 of the points in $\mathcal{S}$ are on one side of the line and 1505 are on the other side.

(b) Suppose that no 3 points of $\mathcal{S}$ lie on the same line. Show that for *any* P in $\mathcal{S}$ there is a line through P such that 1002 of the points in $\mathcal{S}$ are on one side of the line and 1003 are on the other side.

Solution. (a) Draw a line ℓ so that all points of $\mathcal{S}$ are on one side of ℓ. There are many such lines, since $\mathcal{S}$ is a finite set.

Move ℓ very slowly, *parallel* to itself, toward $\mathcal{S}$. After a while, the line ℓ passes through a point P of $\mathcal{S}$. Stop temporarily, and count. Obviously we are not finished, there are 0 points on one side of ℓ and 2005 on the other. The procedure is illustrated in the left-hand diagram of the picture below, with a lot fewer than 2006 points.



Keep going, until ℓ goes through the "next" point P of $\mathcal{S}$. Stop briefly, and count. Now there is 1 point of $\mathcal{S}$ on one side of ℓ, and 2004 on the other. Keep going, until ℓ goes through the "next" point of $\mathcal{S}$. Now there are 2 points of $\mathcal{S}$ on one side of ℓ and 2003 on the other. Keep going. After a while, we reach a point P of $\mathcal{S}$ such that the line ℓ passes through P and has 500 points of $\mathcal{S}$ on one side, and 1505 on the other.

Unfortunately, the above argument is wrong. It is possible that as we move the line ℓ, it passes simultaneously through 2 or more points of $\mathcal{S}$, indeed maybe through all 2006 of them. If the moving line ℓ can pass simultaneously through more than one point of $\mathcal{S}$, then the one by one parade of points that was key to the argument breaks down.

We can rescue the argument. The set $\mathcal{S}$ only has 2006 points. So only finitely many lines pass through two (or more) points of $\mathcal{S}$. In particular, there are only finitely many *slopes* of lines that pass through two or more points of $\mathcal{S}$. In fact, the number of such slopes is between 1 and (2006)(2005)/2, since there are (2006)(2005)/2 ways to *choose* two points from the 2006.

Let m be any number *other* than these finitely many slopes. Instead of starting with an arbitrary line ℓ, start with a line ℓ of slope m such that all points of $\mathcal{S}$ are on one side of ℓ, then use the "moving line" argument given above. As ℓ moves, its slope does not change. The moving line can not pass through more than 1 point of $\mathcal{S}$ at a time, since if it did it would have one of the forbidden slopes. So the argument now works.

Comment. The argument can be extended to show that there are at least *two* points P, and a line ℓ through P, such that there are 500 points of $\mathcal{S}$ on one side of ℓ and 1505 on the other. We can not do better. If all points of $\mathcal{S}$ lie on one line, then there are exactly two points P of $\mathcal{S}$ for which a line through P splits our set in the required way (there are always infinitely many lines that work).

There is nothing special about 2006 and 500. Let $\mathcal{S}$ be a finite set of points, with say n elements, and let a be an integer such that $0 \le a \le n-1$. Then there is a point P of $\mathcal{S}$, and a line ℓ through P, such that a of the points of $\mathcal{S}$ lie on one side of ℓ and $n-a-1$ lie on the other side.

(b) Look at the right-hand picture above. Get a large thin pane of glass, and draw a line ℓ on it. Colour the glass on one side of ℓ transparent red, and on the other side transparent blue. Place the glass on top of the plane that contains our set $\mathcal{S}$, with the line ℓ "passing" through P. If we are very lucky, there are 1002 points of $\mathcal{S}$ on the red side of ℓ, and 1003 on the blue side (or the other way around), and we are finished.

But probably we are not this lucky. For definiteness suppose that there are 1001 or fewer points on the red side. Start rotating the line ℓ (and thus the pane of glass) slowly counterclockwise about P. As ℓ rotates, points of $\mathcal{S}$ other than P will for an instant appear on ℓ, and then move from "red" to "blue," or vice-versa.

Since no three of the points of $\mathcal{S}$ lie on a line, the points move from one colour to the other colour *one at a time.* Note that even though we started with more blue points than red points, the number of red points could *decrease* for a while, or change in complicated ways.

But by the time we have rotated through 180°, all previously red points have become blue, and all previously blue points have become red. So the number of red points has changed from 1001 or fewer to 1004 or more. Since points changed colour one at a time, there must have been a time when there were *exactly* 1002 red points. This completes the proof.

Problem 5. You have 11 thin straight sticks. The length of each stick is a whole number of cm, and each stick has length 88 cm or less. Show that 3 of these sticks can be arranged to form a triangle. (Three positive numbers are the sides of a triangle if the sum of any two is greater than the third. The fact that 11 divides 88 has no bearing on the solution, I think.)

Solution. The argument uses repeatedly the following fact. Let x, y, and z be positive. Then there is a triangle with sides x, y, and z if and only if $x+y>z$, $y+z>x$, and $z+x>y$. More informally, three positive numbers form the sides of a triangle if and only if the sum of any two is greater than the third. The proof is not hard.

Suppose that *no three of the sticks* can be arranged to form a triangle. Call the lengths of the sticks a_1, a_2, a_3, $\dots$, a_{10}, a_{11}, where

$$a_1 \le a_2 \le a_3 \le \cdots \le a_{10} \le a_{11}.$$

It is obvious that $a_1 \ge 1$ and $a_2 \ge 1$. Now look at a_3. Since $a_1 \le a_2 \le a_3$, we have $a_3 + a_2 > a_1$ and $a_3 + a_1 > a_2$. So if $a_2 + a_1 > a_3$, then the sticks of length a_1, a_2, and a_3 form a triangle. But we are assuming that *no three sticks* form a triangle. It follows that $a_3 \ge a_2 + a_1$.

Now look at a_4. Since $a_2 \le a_3 \le a_4$, we have $a_4 + a_3 > a_2$ and $a_4 + a_2 > a_3$. So if $a_3 + a_2 > a_4$, then the sticks of length a_2, a_3, and a_4 form a triangle. But we are assuming that *no three sticks* form a triangle. It follows that $a_4 \ge a_3 + a_2$.

The same argument works in general. Suppose that $2 \le n \le 11$. Since $a_{n-2} \le a_{n-1} \le a_n$, we have $a_{n-1} + a_n > a_{n-2}$ and $a_n + a_{n-2} > a_{n-1}$. So if $a_{n-1} + a_{n-2} > a_n$, then the sticks of length a_{n-2}, a_{n-1}, and a_n form a triangle. But we are assuming that *no three sticks* form a triangle. It follows that $a_n \ge a_{n-1} + a_{n-2}$.

We conclude that if *no three sticks* form a triangle, then $a_n \ge a_{n-1} + a_{n-2}$ for any n such that $2 \le n \le 11$. Now we can start to calculate. We have $a_1 \ge 1$ and $a_2 \ge 1$. If *no three sticks* form a triangle, then $a_3 \ge a_2 + a_1$, so $a_3 \ge 2$.

But if no three sticks form a triangle, then $a_4 \ge a_3 + a_2$, so $a_4 \ge 3$. And if no three sticks form a triangle, then $a_5 \ge a_4 + a_3$, so $a_5 \ge 5$. Similarly, $a_6 \ge 8$, $a_7 \ge 13$, $a_8 \ge 21$, $a_9 \ge 34$, $a_{10} \ge 55$, and $a_{11} \ge 89$.

We have shown that if *no three sticks* form a triangle, then the longest stick has length greater than or equal to 89. But we were told that every stick has length 88 or less. So there are three sticks that form a triangle.

Comment. Fibonacci strikes again! (See the solution to Problem 2 of December 2005.) In general, define the Fibonacci sequence by $F_0 = 0$, $F_1 = 1$, and for every $n > 1$, $F_n = F_{n-1} + F_{n-2}$. The argument above shows that if we have n sticks of integer length, where $n \ge 3$, and every stick has length less than F_n, then some three of the sticks form a triangle.

We perhaps mistakenly specified that every stick has integer length. The proper condition is to say that the length x of any stick satisfies $1 \le x < 89$, without insisting that it be an integer. The argument given above shows that if we have 11 of these sticks, then some three of them form a triangle. The problem specified integer lengths in fear that the number 89 would scream *Fibonacci* to someone familiar with the Fibonacci sequence, and give the argument away.

Here is an interesting related problem. You have 11 thin straight sticks. Each stick has length greater than or equal to 1 cm, and the sum of the lengths of the sticks is less than 1024 cm. Show that you can use *some* of these sticks to make a polygon.

Problem 6. The school cafeteria offers three equally awful lunch choices A, B, and C. Every day, Zoë remembers how bad the immediately previous lunch was, and flips a fair coin to decide between the other two options.

On day 1 of school she had lunch A. Find the probability that she has lunch A on day 100.

Solution. Let p_n be the probability that she has lunch A on day n. We want to find p_{100}. It is good to get some experimental evidence: we calculate p_2, p_3, p_4, and so on for a while, in the hope of seeing what is going on.

The problem has some symmetry: it is clear that, on any day, lunches B and C are equally likely. We don't have perfect three-fold symmetry because Zoë had A on the first day.

Start calculating. We have $p_1 = 1$. On day 2, she must have B or C, so $p_2 = 0$. On day 3, whether she had B or C the day before she chooses A with probability $1/2$, so $p_3 = 1/2$.

Now look at day 4. If she has A on day 3 (probability $1/2$) then she does not have A on day 4. If she has B or C (probability $1/2$) then the probability she has A on day 4 is $1/2$. So the probability she has A on day 4 is $(1/2)(1/2)$. It follows that $p_4 = 1/4$.

A similar calculation gives $p_5 = 3/8$, $p_6 = 5/16$, $p_7 = 11/32$, $p_8 = 21/64$, and $p_9 = 43/128$. With some patience, we could continue all the way to p_{100}. But the prospect is unattractive, and the task becomes very unpleasant if for example we want p_{1000}.

We will first find an expression for p_{n+1} in terms of p_n. This just records in a general way the work we did when we calculated the first few p_k.

On day n, the probability that Zoë eats A is p_n, so the probability she *doesn't* eat A is $1 - p_n$. Thus the probability that she eats A the next day is $(1/2)(1 - p_n)$. We have obtained the Fundamental Recurrence

$$p_{n+1} = (1/2)(1 - p_n). \tag{2}$$

If we stare at the numbers p_n for a while, a possible pattern may leap out. The denominators (so far) are increasing powers of 2, indeed the denominator of p_n seems to be 2^{n-2}, at least if we don't pay attention to p_1 and p_2. And if the denominator is 2^{n-2}, then the numerator (so far) alternates between $(2^{n-2}+1)/3$ and $(2^{n-2}-1)/3$. We could prove by *Mathematical Induction* that this is in fact correct for all n. But that is probably overly fancy. We will avoid Induction.

The p_n seem to approach 1/3. That is reasonable, maybe even obvious. The choice that Zoë made on day 1 should exert less and less influence as time goes on.

Maybe we should find out how close to 1/3 our numbers p_1, p_2, p_3, $\dots$, p_9 are. Let $q_n = 1/3 - p_n$. We get $q_1 = -2/3$, $q_2 = 1/3$, $q_3 = -1/6$, $q_4 = 1/12$, $q_5 = -1/24$, $q_6 = 1/48$, $q_7 = -1/96$, $q_8 = 1/192$, and $q_9 = -1/384$.

There is (so far) an obvious pattern. The q_n look like a geometric progression with common ratio $-1/2$. If the q_n indeed form such a geometric progression, then $q_n = (4/3)(-1/2)^n$ for all n, or, if you prefer, $(4/3)/2^n$ if n is even and $(-4/3)/2^n$ if n is odd.

The pattern is so striking that it is maybe pig headed to doubt it. But mathematics is about *certainty.* Anyway, if things are really this simple, it should be simple to prove it.

Comment. The distinction between *guessing* that a certain pattern continues and *seeing* that it does can be subtle. For the q_n of this problem, and even the p_n, the structure jumps out if we compute half a dozen terms. Someone who says that the result is obvious may be right—but only if she is a professional mathematician.

We work first with q_n, mainly because geometric progressions are familiar. After that will come arguments that work with p_n directly. Let's express the Fundamental Recurrence in terms of the q_n. We have

$$q_n = \frac{1}{3} - p_n \qquad \text{and therefore} \qquad p_n = \frac{1}{3} - q_n.$$

Similarly, $p_{n+1} = 1/3 - q_{n+1}$. Substituting in the Fundamental Recurrence 2, we obtain

$$\frac{1}{3} - q_{n+1} = (1/2)(1 - (\frac{1}{3} - q_n)) = \frac{1}{3} + (1/2)q_n.$$

Simplify: we obtain

$$q_{n+1} = (-1/2)q_n.$$

So indeed the sequence q_1, q_2, q_3, $\dots$ is a geometric sequence with common ratio $-1/2$. Since $q_1 = -2/3$, we conclude that $q_n = (4/3)(-1/2)^n$ for all n. It follows that for all n

$$p_n = \frac{1}{3}\left(1 - \frac{(4)(-1)^n}{2^n}\right).$$

Thus $p_{100} = (1/3)(1 - 1/2^{98})$: p_{100} is indeed very close to 1/3.

Another Way. The first argument relied on experimentation to “guess” what the pattern might be. (I prefer the fancier word “conjecture,” since “guess” sounds too random.) Then came a *proof* that the conjectured formula is indeed correct for all n. We will now produce a formula directly.

In the Fundamental Recurrence 2, get rid of fractions. We get $2p_{n+1} = 1 - p_n$. The “1” in this formula is troublesome. It would be nice to get rid of it, maybe by rewriting the Fundamental Recurrence as $2(p_{n+1} + a) = -(p_n + a)$ for some constant a. The formula would then become $2p_{n+1} = -3a - p_n$, so we want $-3a = 1$, that is, $a = -1/3$.

We have reached $2(p_{n+1} - 1/3) = -(p_n - 1/3)$. It is now "natural" to let $t_k = p_k - 1/3$. We get the recurrence $t_{n+1} = (-1/2)t_n$, so the t_k form a geometric progression with common ratio $-1/2$. The rest is like the first proof.

Note that instead of calculating *values* of p_n for various n, hoping something interesting would turn up, we fooled around with the Fundamental Recurrence, hoping for something interesting.

Another Way. Quite often, *differences* of terms of a sequence behave more nicely than the sequence itself. So it may be a good idea to look at the sequence $p_2 - p_1$, $p_3 - p_2$, $p_4 - p_3$, and so on. For brevity let $d_n = p_{n+1} - p_n$. Calculate, using the known values for the first few p_k. We get $d_1 = -1$, $d_2 = 1/2$, $d_3 = -1/4$, $d_4 = 1/8$, $d_5 = -1/16$, $d_6 = 1/32$, $d_7 = -1/64$, $d_8 = 1/128$. Interesting! Let's assume temporarily that the pattern continues. We would then have $d_k = -2(-1/2)^k$. Note that

$$(p_2 - p_1) + (p_3 - p_2) + (p_4 - p_3) + \cdots + (p_n - p_{n-1}) = p_n - p_1$$

or equivalently,

$$p_n - p_1 = d_1 + d_2 + d_3 + \cdots + d_{n-1}.$$

There is nothing magical about the process: we got the d_k by taking differences, so once we know the d_k we can get the p_k by adding. The geometric series $d_1 + d_2 + d_3 + \cdots + d_{n-1}$ is easy to sum up, and we quickly get a formula for p_n.

It remains to check that the pattern *does* indeed continue. Note from the Fundamental Recurrence that

$$p_{n+2} = (1/2)(1 - p_{n+1})$$

(we have just used $n + 1$ instead of n.) Also,

$$p_{n+1} = (1/2)(1 - p_n).$$

Subtract and simplify. We get

$$p_{n+2} - p_{n+1} = (-1/2)(p_{n+1} - p_n)$$

or equivalently

$$d_{n+1} = (-1/2)d_n.$$

The d_k do indeed form a geometric progression with common ratio $(-1/2)$, and we are finished.

Another Way. The next approach is close in spirit to the calculations that got us to p_9, and to the approach taken by Hank Duan of Pinetree Secondary. We have $p_3 = 1/2$. From Formula 2, $p_4 = (1/2)(1 - 1/2) = 1/2 - 1/4$. Now (crucial point) *do not simplify,* because here simplifying tends to hide the structure. From Formula 2, we obtain

$$p_5 = (1/2)(1 - p_4) = (1/2)(1 - (1/2 - 1/4)) = \frac{1}{2} - \frac{1}{4} + \frac{1}{8}.$$

Again, do not simplify. From Formula 2, we obtain

$$p_6 = (1/2)(1 - p_5) = (1/2)(1 - (1/2 - 1/4 + 1/8)) = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16}.$$

Again, do not simplify. From Formula 2, we obtain

$$p_7 = (1/2)(1-p_6) = (1/2)(1-(1/2-1/4+1/8-1/16)) = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32}.$$

To find p_8, we find $(1/2)(1 - p_7)$. A glance at the expression for p_7 shows that

$$p_8 = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32} - \frac{1}{64},$$

and it is clear that the pattern continues. We are not saying that there *seems* to be a pattern and *hope* it continues. The way we generate the expressions for the p_n shows that the pattern *must* continue. In general (at least when $n \geq 3$)

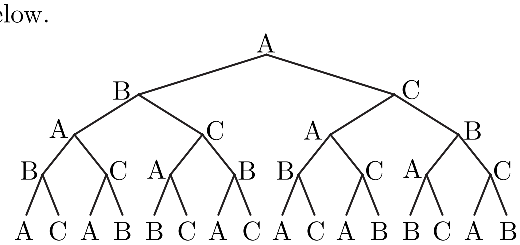
$$p_n = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots + (-1)^{n-3}\frac{1}{2^{n-2}}.$$

So p_n is the sum of a geometric progression with first term 1/2 and common ratio $-1/2$. By a standard formula, the sum is

$$(1/2)\frac{1 - (-1/2)^{n-2}}{1 - (-1/2)}.$$

The above expression simplifies to $(1/3)(1 - (-1)^{n-2}/2^{n-2})$.

Another Way. The previous arguments work directly with probabilities. We can instead work in a more basic way, with a *tree* that represents Zoë's lunch history. Similar arguments were given by Russell Vanderhout of Fraser Heights Secondary and Paul Trakulhoon of Magee Secondary. The tree that traces the history up to day 5 is drawn below.



It is clear that at *any* level n there will be 2^{n-1} "nodes," all equally likely. Let $A(n)$ be the number of "A" nodes at level n. The probability Zoë has lunch A on day n is $A(n)/2^{n-1}$. We will find an expression for $A(n)$.

There are $2^{n-1} - A(n)$ "non-A" nodes at level n. Each gives birth to one A node at level $n + 1$, and therefore

$$A(n+1) = 2^{n-1} - A(n). \tag{3}$$

Calculate, starting from $A(3) = 2$. By the Node Recurrence Formula 3, $A(4) = 4 - 2$. Again by the Node Recurrence Formula, $A(5) = 8 - (4 - 2) = 8 - 4 + 2$. And $A(6) = 16 - 8 + 4 - 2$. There is an obvious pattern, which clearly *must* continue.

So $A(n)$ is the sum of an $n - 2$ term geometric series with first term 2^{n-2} and common ratio $-1/2$. This sum is given by a standard formula.

I think it would be better to treat the cases n odd and n even separately. Using the Node Recurrence Formula, or by a direct count, we can show that $A(n+2) = 2^{n-1} + A(n)$. Then we proceed more or less like before, but jumping by 2's. This avoids the unpleasant minus signs.

Comment. Every day after lunch Zoë is in one of three *states,* (digesting) A, B, or C. Given that she is in a certain state after lunch today, the probability she is in various states tomorrow is given by certain *transition probabilities.* This is a simple example of a *Markov Chain.* Markov chains have many applications, from business to physics.

The problem can be generalized in various ways. We can stick to 3 meal choices, but change the transition probabilities associated with each of A, B, and C. This general problem can be solved, but the solution uses a fair amount of machinery.

Or else we can keep the symmetry of the original problem, but with m meals available. If Zoë has one of them today, tomorrow she chooses one of the others, with all choices equally likely. Here the analysis is more or less the same as the ones we gave for the case $m = 3$.