Solutions to January 2008 Problems

Problem 1. For any real number x, let $\lfloor x \rfloor$ be the greatest integer which is less than or equal to x. It is easy to verify that $\lfloor \sqrt{40} \rfloor = 6$, $\lfloor \sqrt{4400} \rfloor = 66$, $\lfloor \sqrt{444000} \rfloor = 666$, and $\lfloor \sqrt{44440000} \rfloor = 6666$. State and prove the general result that these computations suggest.

Solution. Let f_n be the number whose decimal representation consists of n consecutive 4, followed by n consecutive 0. Let s_n be the number whose decimal representation consists of n consecutive 6. We will show that for every integer $n \geq 1$,

$$\lfloor \sqrt{f_n} \rfloor = s_n.$$

The equation above is equivalent to the double inequality

$$s_n^2 \leq f_n < (s_n + 1)^2.$$

which turns out to be fairly easy to prove.

We first show that $s_n^2 \leq f_n$. Look at a concrete example, say $n = 5$. Imagine finding the square of s_5 by using the usual paper and pencil multiplication process. First we multiply 66666 by 6. We get 399996, which is less that 400000. Then we multiply 66666 by 60, getting something that is less than 4000000. Continue in the usual way, and add up. We get that $s_5^2 < f_5$. The idea works, with essentially no change, for any n.

We could also proceed in a more formal way, by first finding a "closed form" expression for s_n. The number whose decimal expansion consists of n consecutive 9 is $10^n - 1$. It follows that $s_n = (6/9)(10^n - 1)$.

Thus $s_n^2 = (4/9)(10^n - 1)^2$. But

$$(10^n - 1)^2 = 10^{2n} - 2 \times 10^n + 1 < 10^{2n} - 10^n.$$

We conclude that $s_n^2 < (4/9)(10^{2n} - 10^n) = (10^n)(4/9)(10^n - 1) = f_n$.

Now we prove the second inequality, namely $f_n < (s_n + 1)^2$. It is simplest to prove the stronger result $f_n < s_n(s_n + 1)$. Again we proceed informally, by examining a particular case, say $n = 5$. Imagine multiplying 66667 by 66666, using the ordinary paper and pencil process. First multiply 66667 by 6. We get 400002, which is bigger than 400000. Now multiply 66667 by 60. We get a number bigger than 4000000. Continue, and add up. We get a number which is bigger than f_5. The same idea clearly works for any n.

If we wish, we can write up a more formal sounding version, by using the fact that $s_n = (2/3)(10^n - 1)$. We will show that $(s_n + 1)^2 > f_n$, by proving the stronger result $s_n(s_n + 1) > f_n$. Note that $s_n + 1 = (2/3)(10^n) + 1/3 > (2/3)(10^n)$. It follows that

$$s_n(s_n + 1) > (2/3)(10^n - 1)(2/3)(10^n) = f_n.$$

Problem 2. Simplify $\left(\cos\frac{\theta}{2}\right)\left(\cos\frac{\theta}{4}\right)\left(\cos\frac{\theta}{8}\right)\cdots\left(\cos\frac{\theta}{512}\right)\left(\cos\frac{\theta}{1024}\right)$.

Solution. We will use repeatedly the trigonometric identity

$$\sin 2\phi = 2\cos\phi\sin\phi.$$

Let x be the value of our expression. Multiply x (on the right) by $\sin(\theta/1024)$. The term $\cos(\theta/1024)$ combines with $\sin(\theta/1024)$ to yield $(1/2)\sin(\theta/512)$. But the term $\cos(\theta/512)$ combines with $\sin(\theta/512)$ to yield $(1/2)\sin(\theta/256)$. And the term $\cos(\theta/256)$ combines with $\sin(\theta/256)$ to yield $(1/2)\sin(\theta/128)$. Continue in this way. After a while we find that

$$x\sin\frac{\theta}{1024} = \frac{\sin\theta}{2^{10}}.$$

We conclude that

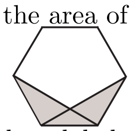
$$x = \frac{\sin\theta}{2^{10}\sin\frac{\theta}{1024}}.$$

This gives us a reasonably simple expression for x.

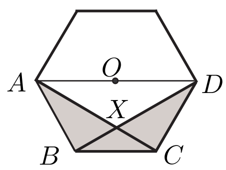
The expression is not quite correct: if $\sin(\theta/1024) = 0$, our expression for x involves division by 0. For completeness, we deal with this exceptional case. Note that $\sin(\theta/1024) = 0$ precisely if $\theta = 1024n\pi$ for some integer n. Then $\cos(\theta/512)$, $\cos(\theta/256)$, …, $\cos(\theta/2)$ are all equal to 1, so our expression for x simplifies to $\cos(\theta/1024)$, or more simply to -1 if $\theta/(1024\pi)$ is an odd integer, and to 1 if $\theta/(1024\pi)$ is an even integer.

Comment 1. The multiplication by $\sin(\theta/1024)$ plays a role analogous to the use of a catalyst in initiating a chemical reaction. Note that after the catalyst has done its work, it is removed (we divide by $\sin(\theta/1024)$).

Problem 3. The figure below is a regular hexagon of area 1. Find the area of the shaded region.



Solution. In order to talk about the hexagon, it is useful to introduce labels as in the figure below.



We first calculate the area of $\triangle ABC$. This triangle can be viewed as having base BC, and height equal to the (perpendicular) distance between lines BC and AD.

Let O be the center of the hexagon—that is, the midpoint of AD. Then $\triangle BCO$ has the same base BC as $\triangle ABC$, and the same height, hence the same area.

But $\triangle BCO$ is one of the 6 equilateral triangles that together "make up" our hexagon. So $\triangle BCO$ has area $1/6$. It follows that $\triangle ABC$ also has area $1/6$. We can conclude immediately, by symmetry, that $\triangle BCD$ also has area $1/6$.

It is easy to see that the area of the shaded region is equal to the area of $\triangle ABC$, *plus* the area of $\triangle BCS$, *minus* the area of triangle BCX. Thus it only remains to find the area of $\triangle BCX$.

Look now at $\triangle ABC$ and $\triangle ACX$. View the first as having base CA and the second as having base CX (we have to twist our necks a bit). Then they have the same height. We show that CA is 3 times as long as CX. This is easy. For $\triangle BCX$ is similar to $\triangle DAX$. It is easy to see that DA has twice the length of BC, and therefore XA is twice the length of CX, which implies that CA is 3 times the length of CX.

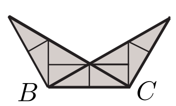
We conclude that the area of $\triangle BCX$ is 1/3 of the area of $\triangle ABC$, that is, 1/18. It follows that the shaded region has area $1/6 + 1/6 - 1/18$, or more simply 5/18.

Another Way. There is an easier way to find the area of $\triangle BCX$. Note as before that $\triangle BCX$ is similar to $\triangle DAX$, and that DA is twice BC. So the height of $\triangle BCX$, viewed as having base BC, is 1/2 the height of $\triangle DAX$, viewed as having base DA. It follows that the height of $\triangle BCX$ is 1/3 of the perpendicular distance between BC and DA. So $\triangle BCX$ has area 1/3 the area of $\triangle BCO$, that is, 1/18.

Another Way. The shaded region is the quadrilateral $ABCD$ with $\triangle DAX$ removed. It is clear that $ABCD$ has area 1/2. To find the area of $\triangle DAX$, we use as before the two similar triangles BCX and DAX. The scaling factor from BCX to DAX is 2. It follows that the to conclude that the height of $\triangle DAX$, viewed as having base DA, is 2/3 of the perpendicular distance between BC and DA.

Thus the height of $\triangle DAX$ is 2/3 the height of $\triangle BCO$, while its base is 2 times the base of $\triangle BCO$. It follows that the area of $\triangle DAX$ is (2/3)(2)(1/6), that is, 4/18. We conclude that the shaded region has area $1/2 - 4/18$, which is 5/18.

Another Way. Please refer to the first argument, and look at the picture below. The shaded region has been dissected into 10 small congruent triangles (they are all "30-60-90").



We had seen that $\triangle ABC$ has area 1/6. It consists of 6 of our small triangles, while the whole shaded region has 10, so the shaded area is (1/6)(10)/6, which is 10/36.

Problem 4. Thirty students each took pass/fail tests #1 and #2. At least one doubly unfortunate student failed both. Find a simple *expression* for the number of different ways this could have happened. (Here, *which* individuals pass which test(s) matters.)

Solution. I forgot to mention that the students are at the Incentives School, where a student gets 1 dollar for passing test #1, and 2 dollars for passing test #2. Line up the students in a row, and give to each of them her/his reward.

Note that once we know how much money a person gets, we know what tests she passed.

If we don't worry about the condition that at least one person failed both tests, there are 4^{30} ways of assigning the grades (handing out the money). But some of these ways are "bad," in that no one fails both tests, no one gets \$0. How many bad patterns are there? In a bad pattern, everyone gets 1, 2, or 3 dollars, so there are 3^{30} bad patterns. Thus the number of good patterns is $4^{30} - 3^{30}$.

Another Way. We can use a probabilistic version of the above argument. Imagine that for any student the Pass/Fail outcome of the two tests is like tossing a fair tetrahedral die, with the possible results on the two tests inscribed on the faces. So for example one of the faces has "Pass #1, Fail #2" written on it.

For any student, the probability of not failing both tests is 3/4. So the probability that no one fails both is $(3/4)^{30}$. The probability that at least one person fails both is therefore $1 - (3/4)^{30}$.

We calculate this probability another way. The total number of possible outcomes is 4^{30}. Let N be the number of outcomes in which at least one person fails both. Then N is the number we are trying to find. We have

$$\frac{N}{4^{30}} = 1 - (3/4)^{30},$$

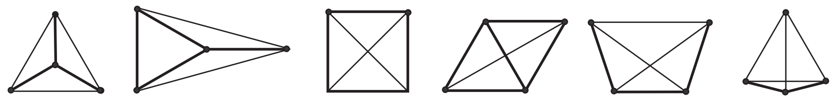
and therefore $N = 4^{30}(1 - (3/4)^{30})$.

Comment 2. Note that we are *not* saying that it is actually the case that the *distribution* of the results on the tests is the same as the distribution obtained by tossing our tetrahedral die 30 times. Of course it isn't. The point is that we have set up a probability situation in which the number N we are looking for comes up naturally. Usually, we use counting to find probabilities. But on occasion, we can use probability to do counting. The above example is not really a good one, since the probabilistic solution is very close to the first solution. But there are examples where a sophisticated probabilistic argument is the only known way of counting.

Problem 5. Find all configurations of 4 distinct points in the plane such that exactly two different numbers occur as distances between pairs of these points. (One such configuration consists of the 4 vertices of a square. The two different distances are the length of an edge and the length of a diagonal.)

Solution. We give the list with only an informal justification that the list is complete. Look at the 4 points. There are a couple of "degenerate" possibilities to consider. Maybe all 4 points lie on a line. It is easy to see that we then have more than two distances between pairs of points. A bit of playing around also shows that if 3 of the points lie on a line and the third is off the line, there are more than two distances between pairs. So we can confine attention to configurations with no 3 points on a line.

There are now two geometrically distinct possibilities to consider. Connect all 6 pairs of points. Viewed from "outside," we either have a triangle or a quadrilateral. (In formal terms, the convex hull of the 4 points is either a triangle or a (convex) quadrilateral.) The results are summarized in the figure below. Heavy lines are "short" and lighter lines are "long."



1 2 3 4 5 6

Suppose that the convex hull is a triangle. This triangle cannot have 3 unequal sides. Maybe it is equilateral. The "fourth" point must be at the center of the triangle. That's our first configuration that works, call it Configuration #1. Or maybe the outer triangle is isosceles but not equilateral. A little sketching then shows that there is only one configuration that works. Call it Configuration #2.

Now we deal with configurations in which the convex hull is a quadrilateral. There are several possibilities to consider: (i) all "outer" lengths are equal; (ii) exactly 3 outer lengths are equal (iii) there are two pairs of equal outer lengths, with the equal outer lengths adjacent to each other; (iv) there are two pairs of equal outer lengths, with the equal outer lengths oppposite to each other.

Look at case (i). We have a rhombus. Maybe the two diagonals are equal, in which case we have a square, Configuration #3. Or maybe one of the diagonals is equal to an outer length. In that case, our configuration consists of 2 equal equilateral triangles glued to each other along an edge. Call this Configuration #4.

Look at case (ii). Some sketching shows that the only configuration that has a chance of working looks like what we have called Configuration #5. Is there such a configuration? Yes, and it was drawn in Problem 3, November 2007. Just take 4 of the vertices of a regular pentagon. (If you thrown in the fifth vertex, you have a configuration of 5 points that determine only 2 distances. This number (5) is the maximum possible size of a set of points in the plane that determines 2 distances. Quite a bit of work has been done on analogous problems in high dimensional Euclidean space.)

Look at case (iii). Then unequal outer sides meet at two of our points. The diagonal that joins these points will be equal to one or the other of the outer pair, so it will form an equilateral triangle with this pair. We end up with Configuration #6.

In case (iv), opposite sides are equal in pairs, so we have a non-rhombus parallelogram. But then one of the diagonals is forced to be larger than either of the sides, which gives more than two distances.