Solutions to January 2009 Problems

Problem 1. Find all pairs (x, y) of real numbers such that

$$x^2 + xy + y^2 = 8 \qquad \text{and} \qquad x - xy + y = 2.$$

Solution. *Suppose* that x and y are numbers such that the two equations hold. Then $xy - (x + y) = -2$. It follows that

$$(x^2 + xy + y^2) + (xy - (x + y)) = 8 + (-2),$$

and therefore

$$(x + y)^2 - (x + y) - 6 = 0.$$

Let $s = x + y$. Then $s^2 - s - 6 = 0$. We are lucky, the expression $s^2 - s - 6$ factors in a simple way—but there would be no difficulty even if it did not. We get $s = 3$ or $s = -2$.

Deal first with the case $x + y = s = 3$. Then $xy = 3 - 2 = 1$. Thus

$$x^2 - 2xy + y^2 = (x^2 + xy + y^2) - 3xy = 5$$

and therefore $x - y = \pm\sqrt{5}$. Now we use the equation $x + y = 3$ to conclude that $x = \frac{3+\sqrt{5}}{2}$ and $y = \frac{3-\sqrt{5}}{2}$ or $x = \frac{3-\sqrt{5}}{2}$ and $y = \frac{3+\sqrt{5}}{2}$. (The original equations are symmetrical in x and y, so solutions must exhibit the same symmetry.)

Suppose next that $x + y = -2$. Then $xy = -4$, and therefore $(x - y)^2 = 20$. Thus $x - y = \pm 2\sqrt{5}$. This gives $x = -1 + \sqrt{5}$ and $y = -1 - \sqrt{5}$ or $x = -1 - \sqrt{5}$ and $y = -1 + \sqrt{5}$.

These are the correct answers, but we are *not quite* finished. The logic of what we have done so far goes as follows: *If* the pair (x, y) satisfies our equations, *then* the pair (x, y) cannot be anything other than the pairs we have listed. That does *not* necessarily mean that our 4 pairs satisfy the equations. So we *must* check in some way that they do. A straightforward way to do this is to substitute our numbers in the original equations, and check that they work. Or else we can quickly see that each step in our calculation is reversible.

Comment. When one is solving equations or systems of equations, it is very common to omit the verification (or reversibility) step at the end. This is in principle wrong, and though no solutions are missed, it is possible to end up with non-solutions, traditionally called *extraneous* solutions.

Another Way. A reasonable way to solve a system of two equations is to use one equation to express one variable in terms of the other, and then substitute. From $x - xy + y = 2$, we obtain $y = (x - 2)/(x - 1)$. Substituting for y in the other equation, we obtain

$$x^2 + \frac{x(x-2)}{x-1} + \frac{(x-2)^2}{(x-1)^2} = 8.$$

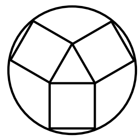
Clear denominators. We obtain $x^4 - x^3 - 9x^2 + 14x - 4 = 0$. This is a *quartic* equation. There is an explicit "formula" for the roots of a quartic, due initially to Ferrari (16th century, nothing to do with automobiles.) That formula is quite unwieldy. However, because $x + y$ turned out to have very simple values, our quartic factors simply as a product of quadratics:

$$x^4 - x^3 - 9x^2 + 14x - 4 = (x^2 - 3x + 1)(x^2 + 2x - 4).$$

(I cheated, by using the already calculated roots to find the quadratics.) Now the roots are easy to write down by using the Quadratic Formula.

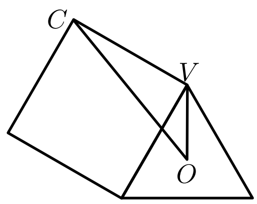
If instead of $x^2 + xy + y^2 = 8$ we had started with (say) $x^2 + xy + y^2 = 7$, the quartic we obtain would be much more painful to tackle. Anyway, substituting violates a generally useful principle. The original equations are symmetric in x and y. Symmetry is our friend, it is usually a poor idea to abandon it.

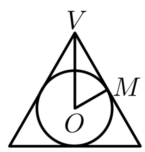
Problem 2. Squares are erected externally on the sides of an equilateral triangle with sides 3. What is the radius of the smallest circle that contains the resulting figure?



Solution. We strip away two of the squares, and take the opportunity to make the picture larger. Let O be the centre of the triangle. (There are many notions of the centre of a triangle, but all these notions agree for equilateral triangles. For definiteness let O be the centre of the inscribed circle of the triangle.)

Let C be a corner of a square which is not one of the vertices of the triangle. By symmetry, the radius we are looking for is the distance OC. Let V be as shown. We obtain the picture on the left. For later use, we make a copy of the triangle, obtaining the picture on the right.





Note that $\angle CVO = 90^\circ + 30^\circ = 120^\circ$. Thus if we can calculate VO, then OC can be found using the Cosine Law.

To find VO, look at the picture above and to the right. We have $VM = 3/2$, and then by familiar properties of 30-60-90 triangles, $VO = \sqrt{3}$. Finally, we apply the Cosine Law:

$$(OC)^2 = (CV)^2 + (VO)^2 - 2(CV)(VO)\cos 120^\circ.$$

This gives

$$(OC)^2 = 9 + 3 + 3\sqrt{3}, \qquad \text{and therefore} \qquad OC = \sqrt{12 + 3\sqrt{3}}.$$

Problem 3. Let a and b be positive integers. Show that $\sqrt{3}$ lies between $\dfrac{a}{b}$ and $\dfrac{a+3b}{a+b}$.

Solution. We can divide the problem into two cases. Either (i) $a/b \le \sqrt{3}$ or (ii) $a/b > \sqrt{3}$. (In fact, we cannot have $a/b = \sqrt{3}$, that is, $\sqrt{3}$ is irrational, but we will not need to know that.)

Case (i) Since $a/b \le \sqrt{3}$, by squaring we find that $a^2 \le 3b^2$. We want to show that $(a+3b)/(a+b)$ is on the "other side" of $\sqrt{3}$, that is, we want to show that $(a+3b)/(a+b) \ge \sqrt{3}$. Equivalently, we want to show that

$$\left(\frac{a+3b}{a+b}\right)^2 \ge 3.$$

Equivalently, we want to show that

$$(a+3b)^2 \ge 3(a+b)^2.$$

Expand both sides. We want to show that

$$a^2 + 6ab + 9b^2 \ge 3a^2 + 6ab + 3b^2,$$

or equivalently that

$$2a^2 \le 6b^2.$$

But this follows easily from the fact that $a^2 \le 3b^2$.

The argument for Case (ii) is very similar, and is omitted.

Comment. With a fair bit of additional work with inequalities, one can show that generally $(a+3b)/(a+b)$ is significantly closer to $\sqrt{3}$ than a/b is.

Start for example with $a = b = 1$. Then $a + 3b = 4$, $a + b = 2$. So we get the new approximation $4/2$ for $\sqrt{3}$. For convenience simplify this to $2/1$. Now let $a = 2$, $b = 1$. Then $(a+3b)/(a+b) = 5/3$, giving us the new approximation $5/3$. Setting $a = 5$, $b = 3$ gives the new approximation $7/4$. Continue. Lifting ourselves up by our own bootstraps, we get successively the approximations $19/11$, $26/15$, and $71/41$. Note that this already gets us quite close to $\sqrt{3}$. (But there are far more efficient ways to approximate $\sqrt{3}$, for example the technique usually, and not really correctly, called *Newton's Method*.)

The idea generalizes. In essentially the same way, one can show that if $k > 1$, and a and b are positive, then $\sqrt{k}$ lies between a/b and $(a+kb)/(a+b)$.

Problem 4. (a) Show that it is not possible to find 2008 *odd* integers x_1, x_2, $\dots$, x_{2008} such that

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_{2008}} = \frac{1}{2009}.$$

(b) The above equation obviously has a solution in positive integers (let $x_k = 2008 \cdot 2009$ for all k.) Show that it has a solution where the x_k are *distinct* positive integers.

Solution. (a) Imagine that the equation has a solution $(a_1, a_2, \cdots, a_{2008})$, with all the a_k odd. Then

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{2008}} = \frac{1}{2009}.$$

Clear denominators by multiplying both sides by 2009 times the product of all the a_k. The right-hand side is then simply the product of the a_k, so it is odd. The left-hand side is a sum of 2008 products. Each term is a product of odd numbers, so it is odd. But the sum of 2008 odd numbers is even. So after denominators have been cleared, the right-hand side is odd, and the left-hand side is even, which is impossible.

(b) We will show that there are distinct positive integers y_1, y_2, $\dots$, y_{2008} such that

$$\frac{1}{y_1} + \frac{1}{y_2} + \cdots + \frac{1}{y_{2008}} = 1.$$

If we do that, then setting $x_k = 2009y_k$ gives a solution of the original problem.

Since 2008 is awfully big, let's solve the problem with a lot fewer y_k. Can we express 1 as the sum of the reciprocals of a few distinct integers? One can't help bumping many times into the fact that $1/2 = 1/3 + 1/6$. It follows that

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1. \tag{1}$$

We have 3 reciprocals of distinct integers, not 2008, but it's a start. We try to push up on the number of reciprocals. One idea is to divide both sides of the Equation 1 by 6. We get

$$\frac{1}{12} + \frac{1}{18} + \frac{1}{36} = \frac{1}{6}.$$

By substituting the above expression for $1/6$ in Equation 1, we obtain

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{12} + \frac{1}{18} + \frac{1}{36} = 1.$$

We now have an expression for 1 as a sum of 5 reciprocals of distinct positive integers. Now use the same idea over again. Divide both sides of Equation 1 by 36. We obtain a representation of $1/36$ as a sum of 3 reciprocals. Substituting for $1/36$ in the above equation, we obtain a representation of 1 as a sum of 7 reciprocals. We can imagine continuing, but it is clear that the idea will not quite work. We can now easily get a representation of 1 as a sum of 9 reciprocals, then 11 reciprocals, and so on. So we can get to 2007 reciprocals in this way, but not 2008.

However, a simpler idea works. Recall that $1/2 = 1/3 + 1/6$. Divide both sides by 3. We obtain $1/6 = 1/9 + 1/18$. Substituting for $1/6$ in Equation 1, we obtain

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{9} + \frac{1}{18} = 1.$$

Now substitute for $1/18$, by dividing both sides of $1/2 = 1/3 + 1/6$ by 9. This gives a representation of 1 as a sum of 5 reciprocals. Continue. The number of reciprocals increases by 1 each time, so (after a while) we can express 1 (and hence $1/2009$) as a sum of 2008 reciprocals.

Comment. There is a *very large* mathematical literature on related problems. A fraction of the form $1/n$ is called a *unit fraction*, or sometimes an *Egyptian fraction*. This is because mathematicians in ancient Egypt solved problems that involved fractions by expressing the fractions as sums of Egyptian fractions with distinct denominators. Their reason for doing this is mysterious to me, and to others more expert in the field. Because of the use of unit fractions, Egyptian arithmetic was quite unwieldy, though it was aided by tables of representations of frequently occurring fractions. The fact that a reasonably sophisticated culture has such an awkward grasp of fractions may indicate that general fractions are not nearly as "natural" a concept as they appear nowadays.

http://www.pims.math.ca/education/math_problems/
http://www.math.ubc.ca/~adler/problems/