Solutions to January 2010 Problems

Problem 1. For what values of m are the four roots of the equation

$$x^4 - (2m + 4)x^2 + m^2 = 0$$

all real and in arithmetic progression?

Solution. Which arithmetic progression? We could let the terms of the arithmetic progression be a, $a+d$, $a+2d$, and $a+3d$. This is the conventional choice, but not the best choice. It is better to let the "common difference" be $2d$, and to let the terms be $a-3d$, $a-d$, $a+d$, and $a+3d$. Note that the sum of the roots of the equation $x^4-(2m+4)x^2+m^2=0$ is 0 (the negative of the coefficient of x^3). The only way that the sum of the terms of our arithmetic progression can be 0 is if $a = 0$.

If the roots of our quartic are r_1, r_2, r_3, and r_4, then the symmetric function

$$r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4$$

is equal to the coefficient of x^2. Since our roots are $-3d$, $-d$, d, and $3d$, the symmetric function is equal to $-10d^2$, so $10d^2 = 2m + 4$.

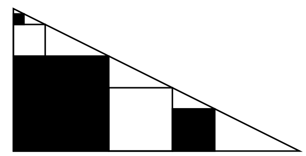
The product of the roots is m^2, but it is also $9d^4$, so $9d^4 = m^2$. We must therefore have

$$\frac{(2m + 4)^2}{10^2} = \frac{m^2}{9}.$$

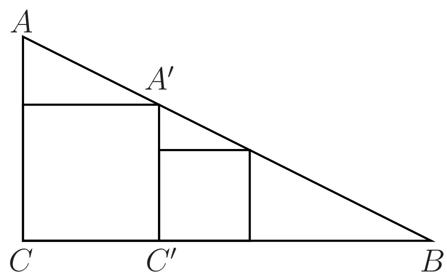
A little simplification yields the equation $4m^2 - 9m - 9 = 0$. This has the roots $m = -3/4$ and $m = 3$.

We are not quite finished. So far, all we know is that *if* the roots are in arithmetic progression, then m can take on no value other than $-3/4$ and 3. This does not ensure that for these particular values of m, the roots are real and in arithmetic progression. But it is straightforward to solve the quartic equation for these two values of m, and verify that indeed the roots are real and in arithmetic progression.

Problem 2. The picture shows five squares, three of them shaded. Let z be the side of the largest square, and let x and y be the sides of the other two shaded squares. Show that $\sqrt{x} + \sqrt{y} = \sqrt{z}$.



Solution. The diagram below is the original picture, with some of the details stripped out, and some labelling to make the argument clear.



Let a be the base of $\triangle ABC$, and let b be the height. If z is the side of the "big" square, then $\triangle A'BC'$ has base $a - z$ and height z. Because $\triangle A'BC'$ is similar to $\triangle ABC$, we have $(a - z)/z = a/b$. From this we conclude that

$$z = \frac{ab}{a+b}.$$

More importantly, the height of $\triangle A'BC'$ is therefore $ab/(a + b)$, while the height of $\triangle ABC$ is b. It follows that $\triangle A'BC'$ is $\triangle ABC$ scaled by a linear scale factor $a/(a+b)$. Thus if z' is the side of the smaller square in the picture above, then

$$z' = z\frac{a}{a+b}.$$

Now go back to the original diagram of the problem. The shaded square to the right of the big shaded square has been obtained from the big shaded square by scaling *twice* by the factor $a/(a + b)$. Call its side x. We conclude that

$$x = z\frac{a^2}{(a+b)^2}.$$

Similarly, let y be the side of the other small shaded square. By symmetry we have

$$y = z\frac{b^2}{(a+b)^2}.$$

From these two equations, we have

$$\sqrt{x}+\sqrt{y}=\sqrt{z}\frac{a}{a+b}+\sqrt{z}\frac{b}{a+b}=\sqrt{z}.$$

Problem 3. (a) Show that if A is a positive integer, there exist a positive integer n, and non-negative integers $a_1, a_2, \ldots, a_n$ such that $a_k \le k$ for all k and

$$A = a_1\cdot 1! + a_2\cdot 2! + a_3\cdot 3! + \cdots + a_n\cdot n!$$

(example: $77 = 1\cdot 1! + 2\cdot 2! + 0\cdot 3! + 3\cdot 4!$).

(b) Let r be a rational number such that $0 < r < 1$. Show that there exist a positive integer n, and non-negative integers $a_1, a_2, \ldots, a_n$ such that $a_k \le k$ for all k and

$$r = \frac{a_1}{2!} + \frac{a_2}{3!} + \cdots + \frac{a_n}{(n+1)!}.$$

Solution. (a) We show how to obtain numbers $a_1, a_2, \ldots, a_n$ that "work." We obtain them one at a time, "backwards," starting with a_n. (If the argument looks mysterious, because of all the symbols and subscripts, try to follow the computations by using an explicit value of A.)

Let n be any integer such that $(n+1)! > A$, and let $A_n = A$. Consider the number $A_n/(n!) = A/(n!)$. Since $(n+1)! > A_n$, we have $A_n/(n!) < n+1$. Let a_n be the greatest integer which is less than or equal to $A_n/(n!)$. Then $0 \le a_n \le n$. Also,

$$a_n \le \frac{A_n}{n!} < a_n + 1,$$

from which we conclude that

$$a_n\cdot n! \le A_n < a_n\cdot n! + n!.$$

Now consider the number $A_{n-1} = A_n - a_n\cdot n! = A - a_n\cdot n!$. From the above inequality, we have $0 \le A_{n-1} < n!$.

Continue, obtaining a_{n-1} by doing the same thing with A_{n-1} that we did with A_n. Let

$$A_{n-2} = A_{n-1} - a_{n-1}\cdot(n-1)! = A - a_n\cdot n! - a_{n-1}\cdot(n-1)!.$$

Keep going, after a while hitting A_0. Since a_1 is the greatest integer which is less than or equal to $A_1/1!$, we have $a_1 = A_1$ and therefore $A_0 = 0$. It follows that

$$A - a_n\cdot n! - a_{n-1}(n-1)! - \cdots - a_1\cdot 1! = 0,$$

which gives our desired representation.

Comment 1. It is not hard to show that the representation obtained above is essentially unique. To be precise, suppose that $n! \leq A < (n+1)!$. Then there exist unique integers a_1, a_2, a_n such that $a_k \leq k$ for all $k \leq n$ and

$$A = a_1 \cdot 1! + a_2 \cdot 2! + a_3 \cdot 3! + \cdots + a_n \cdot n!$$

The above representation is usually called the *Cantor Factorial Representation* of A.

The Cantor Factorial Representation can be generalized. Let b_1, b_2, b_3, and so on be an infinite sequence of integers, all greater than 1. Let $B_0 = 1$, $B_1 = B_0 b_1$, $B_2 = B_1 b_2$, $B_3 = B_2 b_3$, and so on. Then if A is a positive integer, there exist a positive integer n, and non-negative integers a_0, a_1, $\dots$, a_n such that $a_k < b_k$ for all k and

$$A = a_0 \cdot B_0 + a_1 \cdot B_1 + a_2 \cdot B_2 + \cdots + a_n \cdot B_n!$$

Moreover, this representation is essentially unique. The familiar decimal representation of integers is obtained by taking all the b_k equal to 10, and more generally the base b representation is the case $b_k = b$ for all k. The Cantor Factorial Representation is obtained by taking $b_k = k + 1$.

(b) We hunt for clues as to what a suitable n might be. Suppose the sought for result is true, and r can be represented in the desired way. Then $r \cdot (n+1)!$ is an integer. Moreover, suppose n is the *least* positive integer such that $r \cdot (n+1)!$ is an integer. Multiply r by $(n+1)!$. We get

$$r \cdot (n+1)! = a_1 \frac{(n+1)!}{2!} + a_2 \frac{(n+1)!}{3!} + \cdots + a_{n-1} \frac{(n+1)!}{n!} + a_n.$$

Note that every term on the right but the last one is divisible by $n+1$. Moreover, a_n is not divisible by $n+1$, for if it were, then every term would be divisible by $n+1$, which would imply that $r \cdot n!$ is an integer. Since $a_n \leq n$, this forces us to conclude that (if r can be represented in the desired way) a_n is the *remainder* when $r \cdot (n+1)!$ is divided by $n+1$.

Now we have all of the ingredients for a formal proof by induction. Say that a rational r is in the class R_n if $0 < r < 1$ and $r \cdot (n+1)!$ is an integer. We start by observing that a rational r in the class R_1 is representable. For r to be in R_1 means that $r \cdot 2!$ is an integer. This forces $r = 1/2$, and $1/2$ can obviously be represented in the desired way.

Suppose now that we know that whenever r is in the class R_{n-1}, then r is representable, using quotients at most up to $n!$. We show that if r is in the class R_n, then r is representable, using quotients at most up to $(n+1)!$. The only potential problem occurs if r is in the class R_n but not in R_{n-1}, meaning that $r \cdot (n+1)!$ is an integer but $r \cdot n!$ is not.

Let a_n be the remainder when $r \cdot (n+1)!$ is divided by $n+1$. Then a_n is not divisible by $n+1$ (else $r \cdot n!$ would be an integer, meaning r is in the class R_{n-1}). So $0 < a_n \leq n$. Note also that $r \cdot (n+1)! \geq a_n$.

Define the rational number s by

$$s = r - \frac{a_n}{(n+1)!}.$$

If $s = 0$, we have an easy representation of r in the desired form. If $s \neq 0$, we have $0 < s < 1$.

We show next that $s \cdot n!$ is an integer. Recall that $r \cdot (n+1)!$ is an integer b, and a_n is the remainder when b is divided by $n+1$. So $b - a_n$ is divisible by $n+1$. But

$$b - a_n = r \cdot (n+1)! - a_n = \left(r - \frac{a_n}{(n+1)!}\right) \cdot (n+1)! = s \cdot (n+1)!.$$

So $s \cdot (n+1)!$ is divisible by $n+1$, meaning that $s \cdot n!$ is an integer.

We have therefore shown that s is in the class R_{n-1}. Thus, by the induction hypothesis, s has a representation using denominators at most up to $n!$, from which it follows immediately that r has a representation using denominators at most up to $(n+1)!$. This completes the induction argument.

Comment 2. We have given a very formal induction argument. It could, and likely should, have been made much more informal. Take a concrete rational number r, say $4/5$. The smallest n such that $r \cdot (n+1)!$ is an integer is $n = 4$. Look at the remainder when $r \cdot 5!$ is divided by 5. Since $r \cdot 5! = 96$, the remainder is 1. So in the representation of $4/5$, we should have a term of the shape $1/5!$. Now look at $4/5 - 1/5!$. This is $19/4!$. Call it s. The smallest n for which $s \cdot (n+1)!$ is an integer is $n = 3$. The remainder when $s \cdot 4!$ is divided by 4 is 3. Thus we should have a term of $3/4!$ in the representation. Now look at $19/4! - 3/4!$. This is $2/3$. Call it t. The remainder when $t \cdot 3!$ is divided by 3 is 2, so we should have a $2/3!$ term in the representation. Let $u = t - 2/3!$. Then $u = 1/2$. We have obtained the representation

$$\frac{4}{5} = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \frac{1}{5!}$$

It turns out that representations of the type described above are essentially unique. In analogy with "infinite" decimals, they can be extended to infinite "sums." With some effort the idea can be used to give factorial-type representations for all real numbers.

Problem 4. Let a_1, a_2, ..., a_7 be integers. Use the Pigeonhole Principle to show that there are numbers ϵ_1, ϵ_2, ..., ϵ_7 such that:

1. any ϵ_i is equal to 1, 0, or -1,
2. not all the ϵ_i are 0, and
3. $\epsilon_1 a_1 + \epsilon_2 a_2 + \cdots + \epsilon_7 a_7$ is divisible by 100.

Solution. Let $(\delta_1, \delta_2, \dots, \delta_7)$ be a 7-tuple of numbers, where each δ_i is equal to 0 or 1. Define $S(\delta_1, \dots, \delta_7)$ by

$$S(\delta_1, \dots, \delta_7) = \delta_1 a_1 + \cdots + \delta_7 a_7.$$

Let $R(\delta_1, \dots, \delta_7)$ be the remainder when $S(\delta_1, \dots, \delta_7)$ is divided by 100.

Note that there are 128 7-tuples $(\delta_1, \dots, \delta_7)$, and only 100 possible values of $R(\delta_1, \dots, \delta_7)$. It follows by the Pigeonhole Principle that there are two distinct 7-tuples $(\delta_1, \dots, \delta_7)$ and $(\delta'_1, \dots, \delta'_7)$ such that

$$R(\delta_1, \dots, \delta_7) = R(\delta'_1, \dots, \delta'_7)$$

(indeed there is quite a bit more repetition than this, since 128 is substantially larger than 101).

It follows that $S(\delta_1, \dots, \delta_7) - S(\delta'_1, \dots, \delta'_7)$ is divisible by 100. Thus

$$(\delta_1 - \delta'_1)a_1 + \cdots + (\delta_7 - \delta'_7)a_1$$

is divisible by 100.

Now let $\epsilon_i = \delta_i - \delta'_i$. Each ϵ_i is equal to 1, 0, or -1. Not all the ϵ_i are 0, since the 7-tuples $(\delta_1, \dots, \delta_7)$ and $(\delta'_1, \dots, \delta'_7)$ are different. So we have shown that there are ϵ_1, ..., ϵ_7 with the desired properties.

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http://www.math.ubc.ca/~adler/problems/