

Solutions to January 2011 Problems

Problem 1. (a) Let n be even, and let \mathcal{P} be a regular n -sided polygon which is inscribed in a circle of radius 1. Show that for any point A on that circle, the sum of the squares of the distances from A to the vertices of \mathcal{P} is $2n$. (b) What about if n is odd? (This is probably quite a bit harder.)

Solution. (a) Divide the n vertices of \mathcal{P} into $n/2$ diametrically opposite pairs. If $\{U, V\}$ is such a pair, then since UV is a diameter, $\triangle UAV$ is right-angled at A . Thus, by the Pythagorean Theorem, $(AU)^2 + (AV)^2 = (UV)^2 = 4$. (In the special case where A is one of U or V , we don't quite have a triangle, but even more easily we have $(AU)^2 + (AV)^2 = 4$.) Now sum over all $n/2$ pairs. Our sum is $(n/2)(4)$, that is, $2n$.

(b) We can take the n vertices of the circle to be $(\cos \theta_k, \sin \theta_k)$ where $\theta_k = 2\pi k/n$ ($k = 0, 1, 2, \dots, n-1$.) Let A have coordinates $(\cos \alpha, \sin \alpha)$. Then the distance from $(\cos \alpha, \sin \alpha)$ to $(\cos \theta_k, \sin \theta_k)$ is equal to

$$\sqrt{(\cos \alpha - \cos \theta_k)^2 + (\sin \alpha - \sin \theta_k)^2}.$$

Thus the sum we are looking for is

$$\sum_{k=0}^{n-1} ((\cos \alpha - \cos \theta_k)^2 + (\sin \alpha - \sin \theta_k)^2).$$

Expand the squares and simplify, using the fact that $\cos^2 x + \sin^2 x = 1$. After a while we obtain

$$2n - 2 \sum_{k=0}^{n-1} (\cos \alpha \cos \theta_k + \sin \alpha \sin \theta_k).$$

We complete the calculation by showing that $\sum \cos \theta_k$ and $\sum \sin \theta_k$ are both 0. The easiest way to do this is to use the fact that $\cos \phi + i \sin \phi = e^{i\phi}$. But we can also prove the results by using trigonometric identities. The sum involving sines is very easy. All we need is the fact that $\sin(2\pi - \phi) = -\sin \phi$. The sum involving cosines is more difficult. Let $\beta = \pi/n$. We want to calculate

$$\cos 0 + \cos 2\beta + \cos 4\beta + \dots + \cos 2(n-1)\beta.$$

Multiply each term by $\sin \beta$. Now note that

$$2 \sin \beta \cos 2k\beta = \sin(2k+1)\beta - \sin(2k-1)\beta.$$

(This fact is readily checked by using the ordinary formulas for the sine of a sum and difference of two numbers). Now we can use the above fact to simplify

$$2 \sin \beta \cos 0 + 2 \sin \beta \cos 2\beta + 2 \sin \beta \cos 4\beta + \dots + 2 \sin \beta \cos 2(n-1)\beta.$$

We get

$$[\sin \beta - \sin(-\beta)] + [\sin 3\beta - \sin \beta] + [\sin 5\beta - \sin 3\beta] + \cdots + [\sin(2n-1)\beta - \sin(2n-3)\beta].$$

Add up, noting the cancellations. There is almost complete collapse. We end up with $-\sin(-\beta) + \sin(2n-1)\beta$, that is, $-\sin(-\pi/n) + \sin(2\pi - \pi/n)$, which is easily seen to be 0.

Comment. A much tidier-looking version of the same argument uses some basic vector manipulations. Put the origin O at the centre of the circle. Let the vertices of \mathcal{P} be V_1, V_2, \dots, V_n . We can think of A and the V_i as vectors (“arrows”) from O to A and the V_i .

The distance from A to V_i is the length of the vector $V_i - A$. In symbols, it is $\|V_i - A\|$. So we want

$$\|V_1 - A\|^2 + \|V_2 - A\|^2 + \cdots + \|V_n - A\|^2.$$

But in general $\|X - Y\|^2 = \|X\|^2 + \|Y\|^2 - 2X \cdot Y$. So our sum of squares of distances is

$$\|V_1\|^2 + \|V_2\|^2 + \cdots + \|V_n\|^2 + n\|A\|^2 - 2A \cdot (V_1 + V_2 + \cdots + V_n).$$

Each of the $\|V_i\|^2$ is equal to 1, the square of the radius of the circle, as is $\|A\|^2$. By symmetry, the sum of the V_i is the 0-vector. Thus the above sum is equal to $2n$.

Problem 2. Suppose that a and b are positive integers, and that n is an integer greater than 1. Show that if $a^n + b^n$ is a power of 2, then $a = b = 2^e$ for some non-negative integer e .

Solution. Let 2^e be the largest power of 2 that divides both a and b . Then $a = 2^e c$, $b = 2^e d$, and at least one of c and d is odd. We will show that in fact $c = d = 1$, which completes the proof.

Since $a^n + b^n$ is a power of 2, $2^{en} c^n + 2^{en} d^n$ is a power of 2. It follows that $c^n + d^n$ is a power of 2. Since c and d are positive, $c^n + d^n$ is even. But then since at least one of c and d is odd, both are odd.

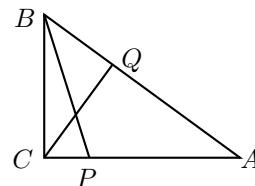
We show that n cannot be even. If n is even, then c^n is a perfect square, and thus has remainder 1 on division by 4. The same remark applies to d^n . So $c^n + d^n$ has remainder 2 on division by 4. But $c^n + d^n$ is a power of 2 that is greater than 2, so it has remainder 0 on division by 4.

So n is odd. Thus

$$c^n + d^n = (c + d)(c^{n-1} - c^{n-2}d + \cdots + d^{n-1}).$$

Note that $c^{n-1} - c^{n-2}d + \cdots + d^{n-1}$ is a sum of an odd number of odd terms. If at least one of c and d is greater than 1, then $c^n + d^n > c + d$. It follows that $c^{n-1} - c^{n-2}d + \cdots + d^{n-1}$ is an odd number greater than 1, so $c^n + d^n$ cannot be a power of 2. We conclude that $c = d = 1$.

Problem 3. Triangle ABC is right-angled at C , CQ is an altitude of $\triangle ABC$, and $AQ = 1$. Point P on CA has the property that $AP = BP = 1$. Find an exact expression for the length of AB .



Solution. There are many approaches. Probably the one that involves the least thinking is to let $t = CP$ and use the Pythagorean Theorem to compute everything in sight. Then some pair of similar triangles and some careful algebra give us t , and then AB .

For fun we will avoid the Pythagorean Theorem, and most of the algebra. Let $z = AB$. Triangles ABC and ACQ can be viewed as having the same height CQ , and bases z and 1. Thus the ratio of

their areas is $z/1$. But the triangles are similar, with corresponding sides AC and AQ , so the ratio of their areas is $(AC)^2/1^2$. It follows that $(AC)^2 = z$, so $AC = \sqrt{z}$.

Now drop a perpendicular from P to AB , meeting AB at R (not shown). Then $\triangle APR$ is similar to $\triangle ABC$. It follows that $AR/AP = AC/AB$. Note that $AR = z/2$. So $(z/2)/1 = \sqrt{z}/z$. This can be rewritten as $z^{3/2} = 2$, so $z = 2^{2/3}$.

Problem 4. A fair die has the numbers 0, 0, 0, 0, 0, and 1 written on its six faces. This die is tossed n times. Find a simple expression for the probability that the sum of the n numbers obtained is odd. The expression should not involve the summation operator \sum .

Solution. Let p_n be the required probability. We find an expression for p_{n+1} in terms of p_n . If the die is tossed $n+1$ times, the sum of the $n+1$ numbers obtained can be odd in two different ways: (1) The sum of the first n numbers is odd, and the $(n+1)$ -th toss is a 0 or (2) The sum of the first n numbers is even, and the $(n+1)$ -th toss is a 1.

The probability of (1) is $5p_n/6$, and the probability of (2) is $(1-p_n)/6$. It follows that

$$p_{n+1} = \frac{5p_n}{6} + \frac{1-p_n}{6} = \frac{1}{6} + \frac{2p_n}{3}.$$

Note that $p_1 = 1/6$. Thus $p_2 = 1/6 + (1/6)(2/3)$, and $p_3 = 1/6 + (1/6)(2/3) + (1/6)(2/3)^2$, and in general

$$p_n = \frac{1}{6} + \frac{1}{6} \left(\frac{2}{3}\right) + \cdots + \frac{1}{6} \left(\frac{2}{3}\right)^{n-1}.$$

The above geometric progression can be evaluated in the usual way. We obtain

$$p_n = \frac{1}{2} - \frac{1}{2} \left(\frac{2}{3}\right)^n.$$

Another Way. We solved the recurrence using a *general* strategy, one that will work with $x_{n+1} = a + bx_n$, where $b \neq 0$. In our case, we can take a shortcut. It is intuitively clear that p_n is close to $1/2$ when n is large. So let $p_n = 1/2 - x_n$. Then

$$\frac{1}{2} - x_{n+1} = \frac{1}{6} + \frac{2}{3} \left(\frac{1}{2} - x_n\right).$$

This simplifies to $x_{n+1} = (2/3)x_n$. But it is easy to see that $x_1 = 1/3$, so $x_n = (1/3)(2/3)^{n-1}$.

Another Way. The sum is odd if and only if there is an odd number of 1's. The probability that there are k 1's is $\binom{n}{k}(1/6)^k(5/6)^{n-k}$. We want to find the sum of these probabilities over all odd k . So we want to find a simple expression for

$$\binom{n}{1}(5/6)^{n-1}(1/6)^1 + \binom{n}{3}(5/6)^{n-3}(1/6)^3 + \binom{n}{5}(5/6)^{n-5}(1/6)^5 \cdots$$

By the Binomial Theorem,

$$1 = (5/6 + 1/6)^n = \binom{n}{0}(5/6)^n(1/6)^0 + \binom{n}{1}(5/6)^{n-1}(1/6)^1 + \binom{n}{2}(5/6)^{n-2}(1/6)^2 + \binom{n}{3}(5/6)^{n-3}(1/6)^3 + \cdots$$

and, again by the Binomial Theorem,

$$(4/6)^n = (5/6 - 1/6)^n = \binom{n}{0}(5/6)^n(1/6)^0 - \binom{n}{1}(5/6)^{n-1}(1/6)^1 + \binom{n}{2}(5/6)^{n-2}(1/6)^2 - \binom{n}{3}(5/6)^{n-3}(1/6)^3 + \dots$$

Subtract, and divide by 2. We obtain

$$(1/2)(1 - (4/6)^n) = \binom{n}{1}(5/6)^{n-1}(1/6)^1 + \binom{n}{3}(5/6)^{n-3}(1/6)^3 + \binom{n}{5}(5/6)^{n-5}(1/6)^5 + \dots$$

The expression on the right, and therefore the one on the left, is our desired probability.