Solutions to January 2012 Problems

Problem 1. Alan and Beti alternately toss a fair die, with Alan going first. A neutral third party keeps a running tab of the combined sum of all their throws. Whoever first reaches a combined sum divisible by 6 wins. What is the probability that Alan wins?

Solution. Suppose that the current sum is not divisible by 6. What is the probability that the next toss makes it divisible by 6? It is clear that there is exactly one toss that will do the job. For instance, if the current sum leaves a remainder of 4 on division by 6, then only a 2 will make the next sum divisible by 6. So if the current sum is not divisible by 6, the probability that the next sum is divisible by 6 is 1/6.
Now we compute the probability that Alan wins. There are many ways this could happen.
(i) The game could last exactly one toss. The probability of this is 1/6.
(ii) The game could last exactly 3 tosses. The probability of this is (5/6)(5/6)(1/6).
(iii) The game could last exactly 5 tosses. The probability of this is $(5/6)^4(1/6)$.
And so on, forever! The probability that Alan wins is therefore

$$\left(\frac{1}{6}\right)+\left(\frac{5}{6}\right)\left(\frac{1}{6}\right)+\left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right)+\left(\frac{5}{6}\right)^3\left(\frac{1}{6}\right)+\cdots.$$

That is an infinite geometric series with first term 1/6 and common ratio $(5/6)^2$. There is a standard formula for the sum.

Another Way. Let p be the probability that Alan wins. Alan can win in two ways: (i) immediately and (ii) later. The probability that Alan wins immediately is 1/6. Thus the probability he does not win immediately (and possibly ultimately loses) is 5/6.
If Alan does not win immediately, Beti now is in essence "starting," so the probability she wins ultimately is p, and therefore the probability Alan ultimately wins is $1-p$. It follows that

$$p=\frac{1}{6}+\frac{5}{6}(1-p).$$

We now have a linear equation for p. Solve. We get $p = 6/11$.

Problem 2. Triangle ABC is isosceles, and right-angled at C. Point P is 3 units from A and 4 units from B. What is the largest possible distance from P to C?

Solution. Please draw the diagram for me, with CA horizontal, A to the right of A, and CB vertical, with B above C. If we travel from A to B to C, we are travel counterclockwise, the direction most mathematicians prefer.
A point P at distance 3 from A and 4 from B lies at the intersection of two circles. These circles intersect in 0, 1, or 2 points. If there are two points, they are symmetrical about the line AB. Undoubtedly there is a good geometric way of solving the problem. But let's do grind it out it crudely, using coordinates. Let C be the origin, let $A = (e, 0$ and let $B = (0, e)$. We consider all points (x, y) that are the specified distances from P and Q. From the given information,

$$(x-e)^2 + y^2 = 9 \qquad \text{and} \qquad x^2 + (y-e)^2 = 16.$$

We want to maximize $x^2 + y^2$. From our two circle equations, we obtain by adding and subtracting that $2x^2 + 2y^2 - 2e(x+y) + 2e^2 = 25$ and $2e(x-y) = 7$. So we want to maximize $25 - 2e^2 + 2e(x+y)$, or equivalently minimize $e^2 - e(x+y)$. By completing the square we see that for given $x+y$, the minimum is reached when $e = \frac{x+y}{2}$. Solve the system $x - y = \frac{7}{2e}$, $x + y = 2e$ for x and y. We get $x = e + \frac{7}{4e}$ and $y = e - \frac{7}{4e}$. Substitute in almost any of our equations, say $(x-e)^2 + y^2 = 9$. We get after some simplification that $8e^4 - 100e^2 + 49 = 0$. Now we know e.

Problem 3. A regular 49-gon $\mathcal{G}$ is inscribed in a circle. How many of the triangles whose vertices are vertices of $\mathcal{G}$ have the centre of the circle in their interior?

Solution. The triangle ABC has the centre of the circle in its interior if and only if it is acute. Since 49 is odd, no pair of vertices of $\mathcal{C}$ make up a diameter of our circle, so there are no right-angled triangles. The total number of triangles is $\binom{49}{3}$. To count the acute triangles, we count the number of obtuse triangles, and subtract this count from $\binom{49}{3}$.
The vertices of any obtuse triangle can be listed uniquely as A, B, C, where A, B, and C are the vertices of the triangle, in counterclockwise order, and the obtuse angle is at B. There are 49 ways that A can be chosen. For each choice of A, the only restriction on B and C is that each must be among the first 24 vertices of the polygon that lie counterclockwise from A (and that B is the first one met). There are $\binom{24}{2}$ ways to choose B and C, for a total of $49\binom{24}{2}$. Thus the number of acute triangles, and hence of triangles that have the centre of the circle in their interior, is

$$\binom{49}{3} - 49\binom{24}{2}.$$

There are many other ways of counting, so we give an explicit numerical answer. It is 4900.

Problem 4. The positive integer n is called a *perfect power* if there exist integers a and b, with $b > 1$, such that $n = a^b$. For what primes p is $3^p + 4^p$ a perfect power?

Solution. We have the familiar $3^2 + 4^2 = 5^2$, so $3^p + 4^p$ is a perfect power if $p = 2$. We show that there are no other primes such that $3^p + 4^p$ is a perfect power. Any prime > 2 is odd. If m is an odd number, then

$$x^m + y^m = (x+y)(x^{m-1} - x^{m-2}y + x^{m-3}y^2 - x^{m-4}y^3 + \cdots + y^{m-1}).$$

So if p is an odd prime, then $3 + 4$ divides $3^p + 4^p$. Since 7 is prime, any perfect power of the form $3^p + 4^p$ must be of shape $(7a)^e$, where $e > 2$. In particular, 7^2 must divide $3^p + 4^p$. Note that $4 = 7 - 3$, so $3^p + (7-3)^p$. Expand $(7-3)^2$ using the Binomial Theorem. All but the last two terms of the expansion are divisible by 7^2. The only ones that needn't be are the last two terms, namely $(p)(7)(-3)^{p-1}$ and $(-3)^p$. The sum of these, because p is odd, is equal to $(p)(7)(3^{p-1} - 3^p$.

So $3^p + (7-3)^p$ has the same remainder on division by 7^2 as $3^p + (p)(7)3^{p-1} - 3^p$, that is, as $(p)(7)3^{p-1}$. So the only way that $3^p + 4^p$ can be divisible by 7^2 is if $p = 7$.

There are various ways to deal with the remaining possibility $p = 7$. One is to use a calculator. Calculate $3^7 + 4^7$. We get 18571. We know the result must be divisible by 7^2, so divide. We get $18571 = (49)(379)$. This is obviously not a perfect power.

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