Solutions to October 2009 Problems

Problem 1. Alphonse and Beth are mathematicians and partners. In their absence, a third person (Gamay) places the King of Spades, the Queen of Hearts, and the 2 of Clubs face down, *in a row,* in random order.

Alphonse enters the room and turns one of the cards face up. Then he turns another card face up. If one of the cards Alphonse turned face up is the King of Spades, Gamay gives Alphonse $1.

Gamay then turns Alphonse's two cards face down, and Beth enters the room. She turns one of the cards face up. Then she turns another card face up. If one of the cards Beth turned face up was the Queen of Hearts, Gamay gives Beth $1. If Alphonse and Beth together use best strategy, what is the probability they get a combined total of $2?

Solution. Think of the cards (from left to right) as Card 1, Card 2, and Card 3. Consider the following strategy.

Alphonse turns Card 1 face up. If Card 1 is the Queen of Hearts, he turns up Card 2. If Card 1 is the 2 of Clubs, he turns up Card 3. (If Card 1 is the King of Spades, he turns up anything, or doesn't bother.)

Beth turns Card 2 face up. If Card 2 is the King of Spades, Beth turns up Card 1. If Card 2 is the 2 of Clubs, Beth turns up Card 3. (If Card 2 is the Queen of Hearts, she turns up anything.)

The cards may have been dealt out in any one of six orders. In the obvious shortcut notation, these are KQ2, K2Q, QK2, Q2K, 2KQ, and 2QK. Each of these orders is by assumption equally likely, so each has probability 1/6 of occurring.

If the order is KQ2, Alphonse and Beth each clearly win $1. If the order is K2Q, then Alphonse clearly wins $1. But since Beth's first chosen card was the 2, her strategy tells her to turn up Card 3, which is the Queen, so she also wins $1.

If the order is QK2, then Alphonse's first card was the Queen, so his strategy tells him to turn up Card 2 on his second try. This is the King, so he wins $1. And Beth's first chosen card is the King, and her strategy tells her to turn up Card 1 on her second try. This is the Queen, so she wins $1.

If the order is Q2K, then on his first try Alphonse gets the Queen, and on his second he gets the 2, so he wins nothing. A similar analysis shows that Beth also wins nothing.

If the order is 2KQ, again neither Alphonse nor Beth wins. And if the order is 2QK, both Alphonse and Beth win $1.

Note that we have arranged things so that Alphonse wins if and only if Beth wins, and that this happens in 4 of the 6 cases. So the couple wins \$2 with probability 4/6. To put it another way, it is clear that Alphonse wins \$1 with probability 2/3. And since Beth wins \$1 if and only if Alphonse does, the probability that between them they win \$2 is also 2/3.

It is clear that this is a best strategy, in the sense that it maximizes the probability of winning \$2. Whatever the strategy, the probability that Alphonse gets the King of Spades is 2/3. So the probability that Alphonse gets the King of Spades *and* Beth gets the Queen of Hearts cannot be made greater than 2/3, and 2/3 is what our strategy achieved.

Comment. Let's see what happens if Alphonse and Beth are strangers to each other, and act independently. Then Alphonse wins \$1 with probability 2/3, as does Beth, so the probability that between them they win \$2 is (2/3)(2/3), that is, 4/9, which is substantially less than the 2/3 probability we obtained if they use the proper strategy.

Comment. Whether or not our heroes use a strategy, if they play the game many times, on average Alphonse wins 2/3 of a dollar, and so does Beth. So between them, independently of whether any strategy is used, on the average they win 4/3 of a dollar. So though the strategy did improve substantially the probability of winning \$2 between them, the strategy had absolutely no effect on their average winnings as a couple. A similar remark can be made about gambling "systems" in general. Such a system (such as doubling one's bet if one loses) can change one's probability of winning, but it will have no effect on the average gain (or more accurately, loss).

Comment. Should we trust Gamay to arrange the cards in random order? The game was played a few times, and she quickly noticed the strategy that Alphonse and Beth use. Gamay has quick hands, and may decide to give Alphonse and Beth a lot more losing deals than would be provided by "randomness."

Alphonse and Beth can foil Gamay as follows. Before each deal, they choose at random one of the 6 permutations of the numbers 1, 2, and 3. Let this permutation be a, b, c. Then they use the basic strategy described above, but substituting a for 1, b for 2, and c for 3. By doing this they can make sure that on average they get \$2 between them with probability 4/6, whether or not Gamay cheats. (Such *randomized* strategies are very important in Game Theory.)

Problem 2. Let $f(x, y) = x^2 - 2xy + 3y^2 - 4x + 5y$. What is the smallest value of $f(x, y)$ as x and y range independently over the real numbers? (Proof is needed that the value given indeed is the smallest.)

Solution. We use a variant of "completing the square:"

$$\begin{aligned} x^2 - 2xy + 3y^2 - 4x + 5y &= (x - y - 2)^2 + 2y^2 + y - 4 && (1) \\ &= (x - y - 2)^2 + 2(y - 1/4)^2 - 33/8. && (2) \end{aligned}$$

It is obvious that $(x-y-2)^2$ and $(y-1/4)^2$ are always non-negative. So $f(x, y)$ will reach a minimum of $-33/8$ if we can arrange to make each of $x - y - 2$ and

$y - 1/4$ equal to 0. But this happens precisely if $y = 1/4$ and $x = 9/4$. So the smallest possible value of $f(x, y)$ is $-33/8$.

Problem 3. The altitudes from two vertices of a triangle have length 3 and 6 respectively. What are the possible values of the length of the altitude from the third vertex? (Proof is needed that the claimed possible values are indeed possible, and that no others are.)

Solution. Let $\triangle ABC$ have sides a, b, and c. Suppose that the altitude to the side of length a is 3, the altitude to the side of length b is 6, and the altitude to the side of length c is h. Then by computing the area of $\triangle ABC$ in three different ways, we find that

$$3a = 6b = hc.$$

Since $3a = 6b$, we have $a = 6k$, $b = 3k$ for some positive k. But side c must be less than $a + b$ (Triangle Inequality), else the triangle would not "close up." Thus $c < 9k$.

Now from the fact that $3a = hc$, we conclude that

$$h = \frac{18k}{c} > \frac{18k}{9k} = 2.$$

It follows that the altitude from the third vertex must be greater than 2.

Note that we must also have $b + c > a$, that is, $c > 3k$. From $3a = hc$, we then conclude as in the previous paragraph that $h < 6$. Thus h must satisfy the inequalities

$$2 < h < 6.$$

So values of h that are ≤ 2 are impossible, as are values ≥ 6. Next we will show that all values of h that are not explicitly forbidden by the above inequalities are indeed possible. So we will show that if $2 < h < 6$, there is a triangle whose altitudes are 3, 6, and h.

Let $a = 18t/3$, $b = 18t/6$, and $c = 18t/h$, where $2 < h < 6$. It is not hard to check that $a + b > c$, $b + c > a$, and (automatically) $a + c > b$. So there is a triangle with sides a, b, and c. Suppose that the area of this triangle is A. Then if x, y, and z are the altitudes to sides a, b, and c, we have

$$ax = by = cz = 2A,$$

and therefore $x : y : z = 3 : 6 : h$.

But x, y, z must not merely be in the right *proportion*: they must be exactly 3, 6, and h. To show this can be done, we use a scaling argument. Imagine first that t is very close to 0. Then the altitude to a is very close to 0, and in particular is less than 3. Now let t grow, until it becomes very large. When t is very large, then the altitude to a is very large, and in particular is greater than 3. So at some value of t, the altitude to a is exactly 3. For that value of t, the altitude to b is 6 and the altitude to c is h.

Comment. We can find an expression for the exact value of t that works. Recall Heron's Formula, which says that if a triangle has sides a, b, and c, then its area A is given by

$$A = \sqrt{s(s-a)(s-b)(s-c)},$$

where s is the semiperimeter $(a+b+c)/2$.

Recall that we need to have $ch = 2A$. Substituting in Heron's Formula, and simplifying a little, we find that we need

$$18t = 2(9t)^2\sqrt{\left(\frac{1}{3}+\frac{1}{6}+\frac{1}{h}\right)\left(-\frac{1}{3}+\frac{1}{6}+\frac{1}{h}\right)\left(\frac{1}{3}-\frac{1}{6}+\frac{1}{h}\right)\left(\frac{1}{3}+\frac{1}{6}-\frac{1}{h}\right)}.$$

Now it is easy to solve explicitly for t in terms of h.

Problem 4. Define the Fibonacci sequence F_0, F_1, F_2, and so on by $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. Let m be a positive integer. Show that there is a positive integer $N = N(m)$ such that m divides F_N.

Solution. Let $G(n)$ be the remainder when $F(n)$ is divided by m. Then $G(n)$ can take on at most m values, namely 0, 1, …, $m-1$. We look at the sequence $G(0)$, $G(1)$, $G(2)$, $G(3)$, and so on. For example, if $m = 5$, the sequence goes 0, 1, 1, 2, 3, 0, 3, 3, 1, 4, 0, 4, 4, 3, 2, 0, 2, 2, 4, 1, 0, 1, 1, 2, 3, and so on.

An important thing to note is that since $F(n+2) = F(n+1) + F(n)$, the remainder when $F(n+2)$ is divided by m is completely determined when the remainder when $F(n+1)$ is divided by m and the remainder when $F(n)$ is divided by m are known. More specifically, $G(n+2)$ is the remainder when the sum $G(n+1) + G(n)$ is divided by m. (This has a practical computational advantage. To find $G(n+2)$ requires only a simple calculation using the "small" numbers $G(n+1)$ and $G(n)$. We do not need to know $F(n+2)$, which rapidly becomes huge.)

Observe that in our computed example with $m = 5$, the number m divides $G(n)$ when $n = 0$ (of course), but also at $n = 5$, $n = 10$, $n = 15$, and $n = 20$. Unfortunately, what happens at $n = 5$, 10, and 15 is a bit misleading: it is what happens at $n = 20$ which is the most useful.

Note that if $m = 5$, then $G(20) = G(0) = 0$ *and* $G(21) = G(1) = 1$. Since $G(n+2)$ is completely determined by $G(n+1)$ and $G(n)$, it follows, *without any computation* that $G(22)$ *must* be equal to $G(2)$, $G(23)$ must be equal to $G(3)$, and so on, that in general $G(20+k) = G(k)$. Thus there is cycling, and $G(n)$ can be easily determined for any n as long as we know $G(0)$ up to $G(19)$. In particular, we can see that if $m = 5$ then $G(n)$ is divisible by 5 for infinitely many values of n.

We now turn from the case $m = 5$ to the general case. Look at the ordered pair $(G(n), G(n+1))$. Since $G(k)$ can take on at most m values, the ordered pair $(G(n), G(n+1))$ can take on at most m^2 distinct values. In fact it is easy to see that except in the trivial case $m = 1$, the pair $(G(n), G(n+1))$ cannot take on the value $(0, 0)$, so there are at most $m^2 - 1$ different values if $m > 1$.

Since the pair $(G(n), G(n+1))$ can take on only finitely many values, there is an integer a and a positive integer q such that the pair $(G(a), G(a+1))$ is identical to the pair $(G(a+q), G(a+q+1))$. But since $G(n+2)$ is completely determined by $G(n+1)$ and $G(n)$, it follows that if $n \geq a$, then $G(n+q) = G(n)$, so *after a while* the sequence $G(0)$, $G(1)$, $G(2)$, and so on exhibits cycling.

This is not quite enough: for example the sequence 0, 1, 1, 3, 4, 3, 4, 3, 4, and so on exhibits cycling after a while. We would like to show that the sequence $G(0)$, $G(1)$, $G(2)$ and so on exhibits *pure* cycling, that is, cycling from the beginning. Recall that we had shown that there is an a such that the pair $(G(a), G(a+1))$ is identical to the pair $(G(a+q), G(a+q+1))$. We need to show that we can take $a = 0$.

To do this, we show that we can "step backwards," and that for example if $a \geq 1$, and the pair $(G(a), G(a+1))$ is identical to the pair $(G(a+q), G(a+q+1))$, then the pair $(G(a-1), G(a))$ is identical to the pair $(G(a+q-1), G(a+q))$. We already know that $G(a) = G(a+q)$. So it is enough to show that $G(a-1) = G(a+q-1)$.

Note that for any $k \geq 1$, we have $F(k-1) + F(k) = F(k+1)$. So $F(k-1) = F(k+1) - F(k)$, and so $G(k-1)$ is the remainder when $F(k+1) - F(k)$ is divided by m. This remainder is completely determined by the remainders when $F(k+1)$ and $F(k)$ are divided by m. Thus $G(k-1)$ is completely determined once we know $G(k)$ and $G(k+1)$.

Recall that $(G(a), G(a+1))$ is identical to the pair $(G(a+q), G(a+q+1))$. Since in general $G(k-1)$ is completely determined once we know $G(k)$ and $G(k+1)$, it follows that $G(a-1) = G(a+q-1)$. Thus the "beginning" of the cycling can always be pulled back by 1, so it can be pulled back all the way to the beginning of the Fibonacci sequence.

Now we are essentially finished. The sequence $G(0)$, $G(1)$, $G(2)$, and so on is cyclic from the beginning. In particular, there is a cycle length q such that $G(0) = G(q) = G(2q)$, and so on. But $F(0) = 0$, so $F(0)$ is trivially divisible by m, that is, $G(0) = 0$. It follows that $G(q)$, $G(2q)$, and so on are all equal to 0, that is, $F(q)$, $F(2q)$, and so on are all divisible by m.

Comment. For a detailed discussion of Fibonacci numbers, one could start with the Wikipedia article and the references it gives. Some simple questions have not yet been fully answered. For example, we have shown that the length of the period is no greater than m^2, but an exact expression for the period in terms of m is not known.

http://www.pims.math.ca/education/math_problems/
http://www.math.ubc.ca/~adler/problems/