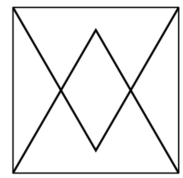
Solutions to October 2011 Problems



Problem 1. Two equilateral triangles are erected on opposite sides of a 1×1 square as shown. Find an exact expression for the area of the region that is common to these two triangles.

Solution. Let a be the area we are trying to find. It is a standard and easily proved fact that an equilateral triangle of side 1 has area $\sqrt{3}/4$.

Consider the two triangles on the left and right side of the square. Each is isosceles, with two angles equal to $30°$, and "long" side equal to 1. Drop a perpendicular from the vertex that has angle $120°$ to the long side. This perpendicular has length $1/(2\sqrt{3})$, so each of our side triangles has area $1/(4\sqrt{3}$.

Add up the areas of the two equilateral triangles, and the two side triangles. We get the area of the square, plus a, since our region of interest is counted twice. It follows that

$$1 + a = \frac{\sqrt{3}}{2} + \frac{1}{2\sqrt{3}} = \frac{2}{\sqrt{3}}$$

and therefore

$$a = \frac{2}{\sqrt{3}} - 1$$

Another Way. Draw a short horizontal line that divides the rhombus whose area we want into two equilateral triangle. The big equilateral triangles have height $\sqrt{3}/2$. A look at the diagram shows that our small equilateral triangles have height $\sqrt{3}/2 - 1/2$. Scaling of linear dimensions by a scaling factor r scales areas by the factor r^2. To get from a big equilateral triangle to one of the small ones, we scale linear dimensions by the factor

$$\frac{\frac{\sqrt{3}-1}{2}}{\frac{\sqrt{3}}{2}} = \frac{\sqrt{3}-1}{\sqrt{3}}.$$

The area of one of one of our large equilateral triangles is $\sqrt{3}/4$. So the combined area of our two small equilateral triangles is

$$2\frac{\sqrt{3}}{4}\left(\frac{\sqrt{3}-1}{\sqrt{3}}\right)^2,$$

which can be simplified to $(2\sqrt{3} - 3)/3$

Problem 2. Find (with proof) the product of all the real solutions of the equation

$$x^{101} - 4x^{99} + x^{98} - 4x^{96} + x^{95} - 4x^{93} + \cdots + x^5 - 4x^3 + x^2 - 4 = 0.$$

Solution. Let $P(x)$ be the polynomial on the left. It is almost obvious that the polynomial $x^2 - 4$ divides $P(x)$. Do the division. We find that

$$P(x) = (x^2 - 4)(x^{99} + x^{96} + x^{93} + \cdots + x^3 + 1).$$

The polynomial $P(x)$ has the obvious roots $x = \pm 2$. For additional real roots, we look for real roots of the equation

$$x^{99} + x^{96} + x^{93} + \cdots + x^3 + 1 = 0.$$

Note that

$$(x^{99} + x^{96} + x^{93} + \cdots + x^3 + 1)(x^3 - 1) = x^{102} - 1.$$

(this comes from the more familiar $(1+y+\cdots+y^{n-1})(1-y) = 1-y^n$). Since the only real roots of the right-hand side are $x = \pm 1$, these are the only possible real roots of $(x^{99}+x^{96}+x^{93}+\cdots+x^3+1)(x^3-1)$. But -1 is not a root of $x^3 - 1$, and 1 is not a root of $x^{99} + x^{96} + x^{93} + \cdots + x^3 + 1$, we conclude that -1 is the only real root of $x^{99} + x^{96} + x^{93} + \cdots + x^3 + 1$.

So the real roots of $P(x)$ are ± 2 and -1. Their product is 4.

Problem 3 (Modified)**.** Find, with proof, the number of ways that 36 can be represented as a sum of one or more positive even integers. Here the order of summation matters. So for example $36 = 6 + 6 + 20 + 4$ is to be counted as different from $36 = 20 + 6 + 6 + 4$. (The original question asked for the sum of one or more *odd* integers, and is harder.)

Solution. From any representation of 36 as a sum of positive even numbers, such as $36 = 20+6+6+4$, we can get a representation of 18 as a sum of positive integers by dividing everything by 2, as in $18 = 3 + 3 + 10 + 2$. Conversely, from a representation of 18 as a sum of positive integers, we can get a representation of 36 as a sum of positive integers by multiplying through by 2.

Let $f(k)$ be the number of ways to represent k as a sum of 1 or more positive integers. We want to find $f(18)$. In principle, we can write a program that lists all representations of 18 as an ordered sum of positive integers. But let's see what we can do by hand.

It is often useful to experiment. It is clear that $f(1) = 1$. Note that $f(2) = 2$ because $2 = 2 = 1+1$. We have $f(3) = 4$, because $3 = 3 = 2+1 = 1+2 = 1+1+1$. Similarly, $f(4) = 8$ since $4 = 4 = 3+1 = 1+3 = 2+2 = 2+1+1 = 1+2+1 = 1+1+2 = 1+1+1+1$. (We should really format these representations better, by using separate lines, but that would take too much space.)

For 5, note that $5 = 5 = 4+1 = 1+4 = 3+2 = 2+3 = 3+1+1 = 1+3+1 = 1+1+3 = 2+2+1 = 2+1+2 = 1+2+2 = 2+1+1+1 = 1+2+1+1 = 1+1+2+1 = 1+1+1+2 = 1+1+1+1+1$. Count. We find that $f(5) = 16$.

We could go on to do an explicit listing of the representations of 6, and maybe 7. The number of representations of n grows rapidly with n. But there is an obvious pattern to the $f(n)$. For the first few n, $f(n) = 2^{n-1}$. If the pattern continues, we have $f(18) = 2^{17}$. This is the answer to our problem, except for one important missing detail. We need to *prove* that $f(n) = 2^{n-1}$ for all n. There are many possible arguments. We give the simplest, and may on later editing give others.

The argument is of a type sometimes called *Stars and Bars.* Imagine you have n "stars" ($*$) laid out in a row, with some space between consecutive stars, like this.

$$* \quad * \quad * \quad * \quad * \quad * \quad * \quad * \quad * \quad * \quad * \quad * \quad * \quad * \quad * \quad * \quad * \quad *$$

Then there are $n - 1$ *spaces* between consecutive stars. Starting at the leftmost such space, decide whether or not to put a *bar* ($|$) in that space. As a very simple example, we could do as follows:

$$* \quad |* \quad * \quad * \quad |* \quad * \quad |* \quad |* \quad * \quad * \quad * \quad * \quad * \quad * \quad * \quad * \quad |* \quad *$$

In the example above, there are 18 stars, and we have put a total of five bars. Now count from the left how many stars there are up to the first bar, from the first bar to the second bar, and so on. In this case we get 1, 3, 2, 1, 9, 2. This corresponds to the decomposition

$$18 = 1 + 3 + 2 + 1 + 9 + 2.$$

It is clear that every decomposition of n as an ordered sum of positive integers corresponds to a unique placement of bars (possibly none). So now we count the number of ways of placing bars (possibly none) in the $n-1$ spaces. At the leftmost space, we have 2 choices, Y or N. For *each* such choice, we have 2 choices at the next space, for a total so far of 2×2. For *each* choice for the first two spaces, there are 2 choices at the third space, for a total so far of $2 \times 2 \times 2$. Continue. For our $n-1$ spaces, we have a total of 2^{n-1} choices.

Problem 4. Prove that the only integer solution of the equation

$$x^2 + y^2 + z^2 + w^2 = 8(xy + yz + xz)$$

is given by $x = y = z = w = 0$.

Solution. It is useful to get information on the remainder when a perfect square is divided by 8. It is clear that if n is even, say $n = 2k$, then n^3 is divisible by 8. Thhe more interesting case is n odd, say $n = 2k + 1$. Then

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 4(k)(k+1) + 1.$$

But one of k and $k+1$ is even, and the other is odd. It follows that $k(k+1)$ is even, and therefore $4k(k+1)$ is divisible by 8. It follows that the remainder when $(2k+1)^2$ is divided by 8 is 1.

Now we turn to our equation. If one or more of x, y, z, or w is odd, then the remainder when $x^2 + y^2 + z^2 + w^2$ on division by 8 canot be 0. (As an aside, *whatever* x, y, z and w are, the remainder of $x^2 + y^2 + z^2 + w^2$ on division by 8 cannot be 7. This observation is useful elsewhere.)

We have seen that if one or more of our variables is odd, $x^2 + y^2 + z^2 + w^2$ cannot have remainder 0 on division by 8. But $8(xy + yz + zx$ is obviously divisible by 8. So unless all our variables are even, the equation $x^2 + y^2 + z^2 + w^2 = 8(xy + yz + zx$ cannot hold.

Let $x = 2x_1$, $y = 2y_1$, $z = 2z_1$, and $w = 2w_1$. Substitute in our original equation, and simplify. After not much work, we arrive at

$$x_1^2 + y_1^2 + z_1^2 + w_1^2 = 8(x_1y_1 + y_1z_1 + x_1z_1).$$

Of course, this looks exactly like our initial equation. Exactly as before, we conclude that each of the variables is even. Let $x_1 = 2x_2$, $y_1 = 2y_2$, and so on. After the same manipulations as before, we get

$$x_2^2 + y_2^2 + z_2^2 + w_2^2 = 8(x_2y_2 + y_2z_2 + x_2z_2).$$

Continue. We have found that each of our variables by 2, then that it is divisible by 4, then that it is divisible by 8, and so on forever. But the only integer that is divisible by every power of 2 is 0. So the only solution of our equation is $x = y = z = w = 0$.

http://www.pims.math.ca/education/math_problems/
http://www.math.ubc.ca/~adler/problems/