Solutions to November 2006 Problems

Problem 1. A six-sided convex polygon is inscribed in a circle. All its angles are equal. Show that its sides need not be equal. What can be said about seven-sided equal-angled inscribed convex polygons? Generalize.

Solution. Look at the left-hand circle in Figure 1. Draw a radius to this circle. Now, moving counterclockwise, draw from the center of the circle a sequence of radii which make an angle to the preceding radius that alternates between x and y, where $x + y = 120°$. In the picture, $x = 75°$ and $y = 45°$.

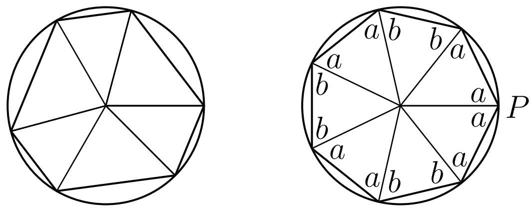

Figure 1: Equal-angled Hexagons and Heptagons

A bit of angle-chasing shows that for every choice of x and y we get an equal-angled inscribed hexagon. It is not hard to show that this is in fact the only way to get convex equal-angled inscribed hexagons.

If $x \neq y$, then the sides are not all equal. By letting $x + y = 360°/n$ with $x \neq y$, we obtain in the same way a convex equal-angled inscribed $2n$-gon whose sides are not all equal.

Now suppose that a convex equal-angled heptagon has been inscribed in the right circle of Figure 1. Draw line segments from the center of the circle to the vertices of the heptagon, and let one of these vertices be P.

Start at P, and let the first angle we meet be a degrees, as in the picture. Then as we travel counterclockwise, the next angle must be a, and the one after that b, where $a + b = 900/7$. The one after that must also be b. Go on all the way around. When we get back to P, we can see that if $a \neq b$ there is a conflict. It follows that the heptagon must be regular. Exactly the same argument works for convex equal-angled convex n-gons where n is odd: every such n-gon has equal sides.

Problem 2. Find an *exact* expression for the smallest positive real number x such that $\cos 3x + \sin 2x = 0$.

Solution. In mathematical work, $\sin u$ is the sine of u, where u is measured in *radians*, not in degrees. Recall, perhaps, how degree measure and radian measure are related: π radians are equal to 180 degrees. It will turn out that the answer to our problem (in degrees) is 54. But to be technically correct, we must convert to radians, and a 54 degree “angle” has radian measure $54\pi/180$, or more simply $3\pi/10$.

Even though $3\pi/10$ is the (only) right answer, we take into account the fact that some of you are not yet familiar with radian measure. So we will work in "high-school" style, in degrees.

We can let microprocessors do much of the work, either by reading the answer off a graph or by using the *Solve* button on a calculator. Or else we can use even a simple scientific calculator, and a crude informal numerical procedure, or a more sophisticated numerical procedure such as *Newton's Method*, to approximate the root.

If we work in degree notation, we fairly quickly find that the answer is either $54°$ or very close to that. Unfortunately, the graphing calculator, the *Solve* button, or numerical procedures can never tell us that we have found the solution *exactly*. So a calculation of the type described above, followed by the statement that the answer is $54°$, is not adequate.

But in this case, once we have *conjectured* that the answer is exactly $54°$, *verification* of the conjecture is quick. For if $x = 54°$, then $3x = 162°$, and $2x = 108°$.

By basic symmetry properties of the sine and cosine function, the cosine of 162 degrees is the negative of the cosine of 18 degrees. Also, the sine of 108 degrees is equal to the sine of 72 degrees, which in turn is equal to the cosine of 18 degrees. So we are finished: the answer is $54°$, or more properly it is $3\pi/10$.

Another Way. Or else we can bring out the machinery of trigonometric identities. The identity $\cos 3x = \cos 2x \cos x - \sin 2x \sin x$, together with the usual double angle identities, yields after a while $\cos 3x = 4\cos^3 x - 3\cos x$.

Thus we can rewrite the original equation as $4\cos^3 x - 3\cos x + 2\cos x \sin x = 0$. The common factor $\cos x$ produces the obvious solution $x = \pi/2$ (90 degrees). It will turn out that this is not the *smallest* positive solution.

Now look for solutions of $4\cos^2 x - 3 + 2\sin x = 0$. This can be rewritten as $4\sin^2 x - 2\sin x - 1 = 0$. By the Quadratic Formula, the solutions are $\sin x = (1 \pm \sqrt{5})/4$. So we want the least positive solution of $\sin x = (1 + \sqrt{5})/4$.

Perhaps it is time to go to the calculator. We find that x is approximately 0.942477796 radians. Out of curiosity, we might go to degrees. To the limit of calculator accuracy, the result seems to be $54°$, or close to that.

Interesting but not conclusive. We have found a numerical solution, and if we believe that answers to all problems must be 'nice,' we may even believe that the answer really is $54°$. But we need to *prove* this. The verification that $54°$ is correct can be done exactly as in the first solution.

This solution was much more complicated than the first one. But as a bonus we found an exact expression for the sine of the $54°$ angle.

Another Way. We need to find where the two curves $y = \cos 3x$ and $y = -\sin 2x$ meet. As was remarked earlier, technically the meaning of $\sin u$ is the sine of u, where u is given in radians. But since radian measure is less familiar, we will work with degrees. A casual sketch shows that the two curves first meet at a point a little short of $60°$.

Recall that $\cos(90° + u) = -\sin u$. So we want the first positive solution of $\cos 3x = \cos(90° + 2x)$. If we bear in mind that the answer is a little short of

$60°$, we can see that the angles $3x$ and $90° + 2x$ should be symmetrical about $180°$, and therefore

$$90° + 2x - 180° = 180° - 3x, \qquad \text{so} \qquad x = 54°.$$

The geometrical approach we used works for $\cos ax + \sin bx = 0$, where a and b are any real numbers, and for other closely related equations.

Comment. From the second solution, we found as a bonus that the sine of the $54°$ angle is $(1+\sqrt{5})/4$. The argument used trigonometric identitities. There is a more elementary (and I think more attractive) argument that uses the geometry of the regular pentagon.

The left-hand picture in Figure 2 is a regular pentagon with the diagonals drawn in. If you remove the sides of the pentagon, you are left with just the diagonals, which form the (regular) *pentagram*, a figure that some think has occult significance.

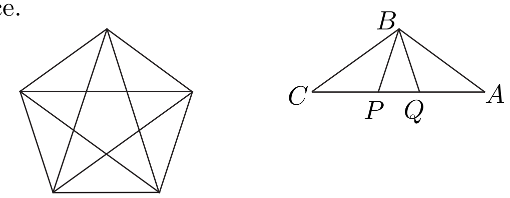

Figure 2: Regular Pentagons

In order not to desecrate the pentagram with labels, we separate out the top part, to form the right-hand picture in Figure 2. Now we do some angle chasing. It is a standard fact that (the measure of) $\angle ABC$ is $108°$, so $\angle CAB$ and $\angle BCA$ are each $36°$. By symmetry (look back at the pentagon on the left) $\angle PBC$ and $\angle QBA$ are each $36°$, and therefore $\angle PBQ$ is also $36°$ (interesting: the diagonals trisect $\angle ABC$).

From the above calculations, it follows that $\angle APB = \angle CQB$ (they are each $72°$), so $AP = CQ = AB$. Note also that $\triangle PBC$ is similar to $\triangle ABC$.

Assume that $AB = 1$, and let $CA = x$. By similarity, we have

$$\frac{1}{x} = \frac{QA}{1} = \frac{CP}{1} = \frac{x - AP}{1} = \frac{x-1}{1}.$$

So $1/x = x - 1$, or equivalently $x^2 - x - 1 = 0$. It follows that $x = (1 + \sqrt{5})/2$.

Now it is easy to calculate the trigonometric functions for $54°$, and a number of related angles. For instance, drop a perpendicular BX from B to the line CA. Then $\angle ABX$ is $54°$, and therefore the sine of the $54°$ angle is AX/AB. But $AX = (1 + \sqrt{5})/4$ and $AB = 1$, and we are finished.

We saw that $(1 + \sqrt{5})/2$ is the ratio of the diagonal of a regular pentagon to the side of that pentagon. This number crops up in an amazing number of places in mathematics. It is sometimes called the *divine proportion*, or the *golden ratio*. A quick web search will show that a tremendous amount of nonsense has been written about it.

Problem 3. Suppose that $f(7) = 10^5$, and that $f(n+2) = f(n+1) + f(n)$ for every non-negative integer n. How many ordered pairs (a, b) of positive integers are there such that $f(0) = a$ and $f(1) = b$?

Solution. Let $f(0) = a$ and $f(1) = b$. Then $f(2) = a + b$, $f(3) = a + 2b$, $f(4) = 2a + 3b$, $f(5) = 3a + 5b$, $f(6) = 5a + 8b$, and $f(7) = 8a + 13b$.

Thus our problem is equivalent to finding how many ordered pairs (x, y) of positive integers satisfy the equation

$$8x + 13y = 100000.$$

(We changed the names of the variables from a and b to x and y, for the sake of familiarity.)

If, in the above equation, instead of 100000 we had a smallish number, such as 300, we could do a simple search for solutions. But with 100000 we need to be more systematic. Note first that the equation has the obvious (though non-positive) solution $x = 12500$, $y = 0$.

Any integer solution has the shape $x = 12500 - r$, $y = s$, where r and s are integers. Substituting in the original equation, we get

$$8(12500 - r) + 13s = 100000, \qquad \text{or equivalently} \qquad 8r = 13s.$$

But $8r = 13s$ if and only if $r = 13t$ and $s = 8t$ for some integer t. Thus all integer solutions of the original equation are given by

$$x = 12500 - 13t, \qquad y = 8t,$$

where t ranges over the integers. Now we determine the t for which both x and y are positive. Clearly we want $t < 12500/13$ and $t > 0$. But $12500/13$ lies between 961 and 962, so the required t range from 1 to 961. There are therefore 961 possible values of t, and hence 961 ordered pairs (a, b).

Another Way. Instead of working 'forwards' from $f(0)$ and $f(1)$, we work backwards from $f(7)$. For convenience let $N = 100000$. Let $f(6) = z$. This single variable z then determines the entire sequence.

For example, $f(5) + z = N$, so $f(5) = N - z$. Similarly, $f(4) + f(5) = z$, and therefore $f(4) = z - (N - z) = 2z - N$. A similar argument gives $f(3) = (N - z) - (2z - N) = 2N - 3z$, $f(2) = (2z - N) - (2N - 3z) = 5z - 3N$, $f(1) = 5N - 8z$, and $f(0) = 13z - 8N$.

We want to have $f(1) > 0$ and $f(0) > 0$, or equivalently

$$\frac{8N}{13} < z < \frac{5}{8}.$$

Finally, note that $8N/13 \approx 61538.46$ and $5N/8 = 62500$. Thus z can be any integer from 61539 to 62499. There are 961 such integers, and they give rise to 961 ordered pairs (a, b).

Comment. Both approaches generalize without much trouble, the second approach more pleasantly than the first. We can replace the 10^5 and 7 of the problem by any positive integers M and k.

Problem 4. We have 99 pennies on a table, all of them with heads facing up. (a) An *allowed move* consists of taking *exactly* 7 pennies, and turning them upside down. Can we get all the pennies with heads facing down using only allowed moves? (b) What about if an allowed move consists of taking exactly 8 pennies and turning them upside down?

Solution. (a) Note that $98 = 14 \times 7$. So if we can get coin #1 heads down, with the rest heads up, then it is easy to get the rest of the coins heads down in 14 additional allowed moves. For convenience, instead of using the conventional H and T for heads and tails, we will use 1 and 0. And call the operation of turning k coins upside down a k-turn. We are working with 7-turns.

One way of getting coin #1 heads down, with the rest heads up, uses coin #1 and 10 more coins, say the first 11 coins. At the start, they are in configuration 11111111111. A 7-turn (coins #2 to #8) gives 10000000111. Another (#5 to #11) gives 10001111000. And a final one (#1 to #4, #9 to #11) gives 01111111111.

Comment. There are many other ways of doing the task. The general strategy we used can be carried out for k-turns where k is odd.

(b) The job cannot be done if an allowed move consists of turning 8 pennies upside down. In fact, it cannot even be done if an allowed move consists of turning upside down an even number of pennies, where the number can change from move to move.

In part (a), we showed that something can be done by more or less explicitly describing how to do it. That approach is unlikely to work to prove a 'negative' proposition, since somehow we must examine all possible procedures and show that none of them work. We will work indirectly, by showing that after any number of allowed moves, the number of coins that are heads up is odd, and therefore cannot ever become 0.

It is easy to see that any 8-turn can be accomplished by using four 2-turns. Though we do not need this, it is also easy to see that any 2-turn can be accomplished by using two 8-turns. For suppose we want to turn a and b upside down. Let S be a collection of 7 coins that does not include a or b. Turn a and the coins in S upside down. Then turn b and the coins in S upside down. These two 8-turns have the net effect of turning a and b upside down, and nothing else.

Now we show that any 2-turn does not change the parity of the number of coins that are heads up. If the 2-turn involves a coin that is heads up and a coin that is heads down, the number of coins that are heads up does not change. If the 2-turn involves two coins that are heads down, then the number of coins that are heads up increases by 2. And if the 2-turn involves two coins that are heads up, then the number of coins that are heads up decreases by 2. Thus in any case, if we start with an odd number of coins heads up, the number of coins that are heads up after any number of 2-turns must be odd.

Problem 5. Call a word w over the alphabet $\mathcal{A}$ *odd* if each letter of $\mathcal{A}$ occurs an odd number of times in w. Find a simple expression for the number of 101-letter odd words over the alphabet $\{a, b, c\}$.

Solution. Let n be an odd integer. Let $f(n)$ be the number of "odd" words of length n. We want to calculate $f(101)$. The number 101 is so large that it is more or less necessary to attack the problem of finding $f(n)$ for general (odd) n.

This problem is hard! There are some high-powered ways of getting at $f(n)$, such as generating functions. There are less high-powered ways that involve tedious computations with binomial coefficients. We will use much less fancy techniques.

We could start by looking at values of n much smaller than 101, such as 3, 5, 7, 9, and maybe a few more. For these n, we can with some effort calculate $f(n)$, and maybe form a plausible conjecture about the general case. But in these solutions we skip writing out these preliminary explorations, and go directly to the general case.

Our first approach involves obtaining a simple *recurrence* formula for $f(n+2)$ in terms of $f(n)$. This kind of approach is often useful in combinatorial work. In many combinatorial problems, an explicit formula can be difficult to find, or may not even exist, but fairly pleasant recurrences turn up quite often.

How do we obtain an odd word of length $n+2$? We can append aa, bb, or cc to an odd word of length n. That gives us $3f(n)$ odd words.

Or else we can (i) append ab or ba to a not-odd word of length n that has an even number of a's and of b's; or (ii) append bc or cb to a not-odd word of length n that has an even number of b's and of c's; or (iii) append ca or ac to a not-odd word of length n that has an even number of c's and of a's. This takes care of all possibilities, for n is odd, so if a word is not-odd, it must have an even number of each of two of the letters a, b, or c, and an odd number of the other letter.

By symmetry, one-third of the not-odd words of length n have an even number of a's and of b's. It is easy to see that there is a total of 3^n words of length n, and therefore there are $(1/3)(2)(3^n - f(n))$ odd words of type (i). And there are just as many odd words of types (ii) and (iii), for a total of $(2)(3^n - f(n))$.

So the total number of odd words of length $n+2$ is $3f(n) + 2(3^n - f(n))$, and therefore

$$f(n+2) = (2)(3^n) + f(n). \qquad (1)$$

Clearly $f(1) = 0$. It follows from Equation (1) that $f(3) = (2)(3^1)$, and then that $f(5) = (2)(3^1)+(2)(3^3)$, and $f(7) = (2)(3^1)+(2)(3^3)+(2)(3^5)$. Equation (1) makes it clear that the process goes on in a simple way, and that

$$f(101) = 2(3^1 + 3^3 + 3^5 + \cdots + 3^{99}).$$

We may notice that $f(101)$ is the sum of a geometric series, with first term $(2)(3)$ and common ratio 3^2. And if someone supplies us with the magic closed form expression for the sum, we end up concluding that

$$f(101) = \frac{(2)(3)(3^{100} - 1)}{3^2 - 1}.$$

Another Way. This is a variant of the first solution, which differs from it in two important ways. We study odd n and even n at the same time. And we use some elementary probability theory to do the counting. (Usually, we use counting to find probabilities.)

Let n be a positive integer. Call a word of length n over the alphabet $\{\mathrm{a},\mathrm{b},\mathrm{c}\}$ *good* if the number of a's, of b's, and of c's in the word are all even or all odd. (Note the change in the definition of *good*. But for n odd it coincides with the previous definition, since then the number of a's, b's, and c's cannot be all even.)

Let $p(n)$ be the probability that a "randomly" chosen n-letter word is good. We obtain an expression for $p(n+1)$ in terms of p_n.

What is the probability that a randomly chosen word of length $n+1$ is good? The word obtained by deleting the last letter must be bad (two evens and an odd, or two odds and an even). And given a bad word of length n, there is exactly one letter which we can append to it to make a good word of length $n+1$. So the probability that a word of length $n+1$ is good is $(1-p(n))(1/3)$. We conclude that

$$p(n+1) = \frac{1}{3} - \frac{p(n)}{3}. \tag{2}$$

Note that $p(1) = 0$. Using Equation (2), we conclude that $p(2) = 1/3$, $p(3) = 1/3 - 1/9$, $p(4) = 1/3 - (1/3)(1/3 - 1/9) = 1/3 - 1/9 + 1/27$, and so on.

We can now find a closed form formula for $p(n)$ by summing the appropriate geometric series. The usual formula gives

$$p(n) = (1/3)\frac{1-(-1/3)^{n-1}}{4/3}. \tag{3}$$

Since $p(n) = f(n)/3^n$, we conclude that $f(n) = 3^n p(n)$. After simplifying a bit, we obtain

$$f(n) = \frac{(3)(3^{n-1} + (-1)^n)}{4}. \tag{4}$$

Finally, if n is odd then $(-1)^n = -1$, and $f(n) = (3/4)(3^{n-1} - 1)$.

Another Way. This last solution will be by far the most "elementary" (but not the easiest). Let n be odd, and let W be the set of *all* words of length n over the alphabet $\{\mathrm{a},\mathrm{b},\mathrm{c}\}$.

Let A consist of the words in W in which the letter a occurs an odd number of times, and b and c each occur an even number of times. Define B and C analogously. Finally, let T consist of the words in W in which all three of a, b, and c occur an odd number of times. It is clear that every word in W is in A or B or C or T.

By symmetry, A, B, and C each have the same size. We now show that there is one more word in A than there is in T.

If w is a word in T, define $h(w)$ as follows. Locate the first letter (from the left) in w which is *not* a. If the letter is b, we obtain $h(w)$ by replacing *this occurrence* of b with c. If the letter is c, we obtain $h(w)$ by replacing this occurrence of c with b. Since b and c occurred an odd number of times in w, they each occur an even number of times in $h(w)$, so $h(w)$ is in A.

It is easy to see that given $h(w)$, we can uniquely identify w. It is also obvious that if v is any word in A which has at least one occurrence of b or c, then there is a w in T such that $h(w) = v$.

In formal language, h is a one-to-one function from T to A, and the range of h consists of all but one point in A. It follows that there is *one more* word in A than there is in T.

Finally, let x be the number of words in T. Then each of A, B, and C contains $x + 1$ words. But there are 3^n words in W, so $4x + 3 = 3^n$, and therefore $x = (3^n - 3)/4$.

Comment. More or less the same approach works for even n. Let E be the set of words that have an even number of occurrences of a, b, and c. Let A' be the set of words that have an even number of occurrences of a, and an odd number of occurrences of each of b and c. Define B' and C' analogously. It turns out that there is one *more* word in E than there is in A' (or B', or C'). So if x is the number of words in E, then $x = (3^n - 1)/4$.

People in combinatorics attach value to what they call a *combinatorial proof* of a counting formula. The idea is that in finding the size of something, we should as much as possible rely on explicit mappings to sets of known size. Many combinatorial results do not have any known combinatorial proof.

The last argument was purely combinatorial, clean, 'easy'. Maybe not surprisingly, I only found it after I knew the 'answer' from much uglier arguments! The last argument uses a minimal amount of machinery, and we can actually *see* why the count is what it is. With other techniques, we may push our way to the answer, so we know it is correct, but in a sense we do not know *why* it is correct.