Solutions to November 2007 Problems

Problem 1. It is not hard to verify with a calculator that

$$\frac{5^3+1^3}{5^3+4^3}=\frac{5+1}{5+4},\qquad \frac{67^3+41^3}{67^3+26^3}=\frac{67+41}{67+26},\qquad \frac{124^3+43^3}{124^3+81^3}=\frac{124+43}{124+81}.$$

But it is easier to "cancel" the 3's from numerator and denominator! Explain why cancelling the 3's gives the right answer in these and similar cases.

Solution. The left-hand sides of the three examples are of the form

$$\frac{n^3+a^3}{n^3+b^3},\qquad \text{where}\qquad a+b=n.$$

We will show that the (in general) spurious "cancellation" of the 3's in fact works for any such n, a, and b. (Not *quite* true. When $b=-n$, both sides are undefined. It is a matter of taste whether we then call them equal.)

Factor the numerator and the denominator, using the standard identity $x^3+y^3=(x+y)(x^2-xy+y^2)$. We obtain

$$\frac{n^3+a^3}{n^3+b^3}=\frac{(n+a)(n^2-na+a^2)}{(n+b)(n^2-nb+b^2)} \tag{1}$$

Finally, we show that $n^2-na+a^2=n^2-nb+b^2$, so that we can (legitimately) cancel them in the right-hand side of Equation 1. Of course we must make sure that $n^2-nb+b^2\neq 0$. By completing the square or otherwise, it is not hard to show that $n^2-nb+b^2=0$ only when $n=b=0$, which is ruled out by the fact that $b\neq -n$.

Showing that $n^2-na+a^2=n^2-nb+b^2$ is easy. For example, we can substitute $n-a$ for b in the second expression, and simplify.

Problem 2. Suppose that $0<x<y$. Which is larger,

$$\frac{1+x+x^2+\cdots+x^{100}}{1+x+x^2+\cdots+x^{101}}\qquad\text{or}\qquad\frac{1+y+y^2+\cdots+y^{100}}{1+y+y^2+\cdots+y^{101}}\,?$$

(This is straightforward only if it is approached the right way.)

Solution. We are familiar with the "formula" for the sum $1+r+r^2+\cdots+r^n$ (if $r\neq 1$). Since we know the formula, it is tempting to use it. However, using the formula, though it appears to "simplify" things, leads to technical complications.

Instead, let $f(t)=(1+t+\cdots+t^{100})/(1+t+\cdots+t^{101})$. We show that $f(x)>f(y)$ by showing that $f(x)-f(y)>0$. We have

$$f(x)-f(y)=\frac{1+x+\cdots+x^{100}}{1+x+\cdots+x^{101}}-\frac{1+y+\cdots+y^{100}}{1+y+\cdots+y^{101}}.$$

"Cross-multiply." We obtain the expression

$$\frac{(1+\cdots+x^{100})(1+\cdots+y^{101})-(1+\cdots+x^{101})(1+\cdots+y^{100})}{(1+\cdots+x^{101})(1+\cdots+y^{101})}.$$

The denominator is positive, so we concentrate on the numerator. There is a lot of cancellation, and we obtain

$$(1+x+\cdots+x^{100})y^{101}-(1+y+\cdots+y^{100})x^{101},$$

which can be rewritten as

$$(y^{101}-x^{101})+xy(y^{100}-x^{100})+x^2y^2(y^{99}-x^{99})+\cdots+x^{100}y^{100}(y-x).$$

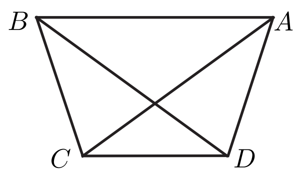
But since $y>x>0$, all the summands in the expression above are positive. This completes the proof that $f(x)-f(y)>0$.

Another Way. We sketch a somewhat more complicated approach. In the notation of the first solution, $1/f(x)=1+x^{n+1}/(1+\cdots+x^n)$, with a similar expression for $1/f(y)$. We will show that $1/f(x)<1/f(y)$, or equivalently that $g(x)<g(y)$ where $g(t)=t^{n+1}/(1+\cdots+t^n)$. To do this, we show that $g(t)$ is an increasing function. In the expression for $g(t)$, divide "top" and "bottom" by t^{n+1}. We get

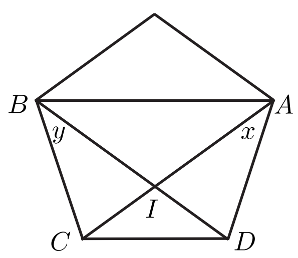
$$g(t)=\frac{t^{n+1}}{1+t+\cdots+t^n}=\frac{1}{\frac{1}{t^{n+1}}+\frac{1}{t^n}+\cdots+\frac{1}{t}}.$$

As t grows, each term in the denominator decreases, and the numerator does not change, so $g(t)$ increases.

Problem 3. In quadrilateral $ABCD$, $AB=AC=BD$ and $BC=CD=DA$. Find AB/BC.



Solution. The argument has two parts. First we do an angle-chase to identify the angles in the figure. Then comes a simple similarity argument. The proof could be shortened by taking advantage of the almost obvious symmetries. Let I be the intersection of the two diagonals, and let x, y be the measures of angles DAC and DBC.



Because $\triangle CDA$ is isosceles, $\angle ACD = x$. Similarly, $\angle CDB = y$. Because lines CD and BA are parallel, we have then $\angle CAB = \angle ACD = x$ and $\angle ABD = y$.

Because $\triangle ABD$ is isosceles, $\angle BDA = 2x$ and $\angle BCA = 2y$. And since $\angle AID = \angle CID$, we have $3x = 3y$, so $x = y$. It follows that the sum of the angles of our trapezoid is $10x$. This sum is $360°$, and therefore $x = 36°$. Now it is easy to see that $\angle AID$ is $180° - 108°$, so it is $72°$.

The result of this angle chase is that $\triangle DAI$ is similar to $\triangle ABD$. Let $BA = s$ and $CD = t$. We wish to find s/t. Note that $AI = BI = s$. It follows that $ID = s - t$. By the similarity of $\triangle DAI$ and $\triangle ABD$, we have then

$$\frac{s}{t} = \frac{t}{s-t}.$$

From the above equation, we obtain $s^2 - st - t^2 = 0$. It is clear that $t \neq 0$. Divide both sides by t^2, and let $w = s/t$. We obtain the equation $w^2 - w - 1 = 0$. It follows that $w = (1 + \sqrt{5})/2$.

Comment 1. For fun the diagram above contains the "missing" part of the figure of the problem. Take two line segments of length CD, and put them to form $36°$ angles with the line AB. It is easy to verify that we have constructed a regular pentagon. So this problem was a disguised version of the standard fact that the ratio of the diagonal of a regular pentagon to the side of the pentagon is the "golden number" $(1 + \sqrt{5})/2$, variously also called τ or ϕ, which comes up so often both in "puzzle" mathematics and more serious mathematics.

Comment 2. Let $AD = 1$. Drop a perpendicular from D to the line AC, say meeting AC at M. Then $AM = (1 + \sqrt{5})/4$. Since $AD = 1$, it follows that the cosine of $\angle DAM$ is $(1 + \sqrt{5})/4$. So we have found a simple explicit expression for the cosine of a $36°$ angle. We can proceed to find explicit expressions for the trigonometric functions evaluated at the $36°$ and $72°$ angles.

We can also use the results of this problem to construct a regular pentagon with straightedge and compass. Start with a line segment CD, of length that we will call 1 unit. It is easy to construct a line segment of length $(1+\sqrt{5})/2$ units. The $\sqrt{5}$ part for example can be obtained by making a right angled triangle with legs 1 and 2; then the hypotenuse has the right size.

Now with center D, draw a circle of radius 1. With center D draw a circle of radius $(1+\sqrt{5})/2$. Where they meet is what is called A in the diagram (they also meet on the other side of CD, but that is not needed). Now that we have A, the rest of the regular pentagon can easily be constructed. (There are cleverer straightedge and compass constructions of the regular pentagon, dating back to Euclid and undoubtedly earlier.)

Problem 4. At the beginning, bowl A contains 2 cups of sugar, and nothing else, and bowl B contains 1 cup of flour, and nothing else.

Xavier transfers 1 cup of stuff from A to B, mixes the contents of B thoroughly, then transfers 1 cup of stuff from B to A, and mixes the contents of A thoroughly. That's the end of the first *cycle.* Xavier then transfers 1 cup of stuff from A to B, mixes the contents of B thoroughly, transfers one cup of stuff from B to A, and mixes thoroughly. That's the end of the second cycle. Xavier continues. (a) How much flour is in bowl A at the end of the 4-th cycle? (b) Find a simple expression for the amount of flour in bowl A at the end of the n-th cycle, and prove that your expression is correct.

Solution. (a) We can make an explicit computation. At the beginning (time 0), there are 2 cups of sugar in A, and 0 cups of flour; there are 0 cups of sugar in B, and 1 cup of flour.

After the first transfer (at time 1) there is 1 cup of sugar in A, and 0 of flour; there is 1 cup of sugar in B, and 1 of flour.

Now look at the second transfer; here 1 cup of a mixture that is half sugar and half flour is transferred to A, so 1/2 cup of sugar and 1/2 of flour is transferred. The net result is 3/2 of sugar and 1/2 of flour in A, and 1/2 of sugar and 1/2 of flour in B. We have finished cycle 1.

Comment 3. It is easy to make computational errors. There are a couple of checks possible. For any n, look at the situation after n transfers. If n is odd, the total in A is 1 and the total in B is 2. If n is even, the total in A is 2 and the total in B is 1.

Also, after an even number n of transfers, the amount of flour in A is equal to the amount of sugar in B. This observation is the subject of an old puzzle. Suppose that we have 2 cups, A and B, and initially A has tea and B has milk (the amounts are irrelevant). We transfer one teaspoonful of tea from A to B, mix thoroughly (or sloppily, it does not matter), then transfer one teaspoonful of liquid from B to A. Question: Which is bigger now, the amount of milk in A or the amount of tea in B? The answer is that the amounts are equal. Since the volume of stuff in A has not changed, and neither has the volume of stuff in B, whatever milk is now in A must be balanced by an equal amount of tea in B.

More subtly, look at the situation after an odd number of transfers. It is not hard to argue that the amount of sugar in A will be equal to the amount of flour in B. The remark is true after 1 transfer, because of the choice of initial amounts. Thereafter, it must remain true even if mixing is not thorough.

Now we deal with cycle 2. The contents of A are 3/2 cups of sugar, and 1/2 of flour, so 1 cup of the stuff in A has 3/4 cups of sugar and 1/4 of flour. After the third transfer, there will remain 3/4 of sugar, 1/4 of flour in A, and 5/4 of sugar, 3/4 of flour in B. On the fourth transfer, we are transferring 5/8 of sugar and 3/8 of flour from B to A. The net result is 11/8 sugar, 5/8 flour in A, and 5/8 sugar, 3/8 flour in B. Continue. The computations do not change in character. We summarize the results in a table. The first row (labelled T) represents time. In the next two rows, the first entry in any pair is the amount of sugar present, the second is the amount of flour.

T	0	1	2	3	4	5	6	7	8
A	$2, 0$	$1, 0$	$\frac{3}{2}, \frac{1}{2}$	$\frac{3}{4}, \frac{1}{4}$	$\frac{11}{8}, \frac{5}{8}$	$\frac{11}{16}, \frac{5}{16}$	$\frac{43}{32}, \frac{21}{32}$	$\frac{43}{64}, \frac{21}{64}$	$\frac{171}{128}, \frac{85}{128}$
B	$0, 1$	$1, 1$	$\frac{1}{2}, \frac{1}{2}$	$\frac{5}{4}, \frac{3}{4}$	$\frac{5}{8}, \frac{3}{8}$	$\frac{21}{16}, \frac{11}{16}$	$\frac{21}{32}, \frac{11}{32}$	$\frac{85}{64}, \frac{43}{64}$	$\frac{85}{128}, \frac{43}{128}$

At the end of the fourth cycle, there are 85/128 cups of flour in bowl A.

Another Way. We have gone to more trouble than necessary—we need only follow the flour. Let $f(n)$ be the amount of flour in bowl A after n cycles. Then the amount of flour in bowl B is $1 - f(n)$. Half of $f(n)$ is transferred to bowl B, but half of that comes back, together with half of $1 - f(n)$. It follows that

$$f(n+1) = \frac{f(n)}{2} + \frac{f(n)}{4} + \frac{1 - f(n)}{2} = \frac{f(n)}{4} + \frac{1}{2}.$$

Now we can compute. We have $f(0) = 0$ and therefore $f(1) = \frac{1}{2}$ and therefore $f(2) = \frac{5}{8}$ and therefore $f(3) = \frac{21}{32}$ and therefore $f(4) = \frac{85}{128}$.

(b) We notice that the denominators at time 2, 4, 6, and 8 are 2, 8, 32, and 128 (there may be an anomaly at time 0). It is reasonable to conjecture that the denominator after n cycles (at time $2n$) is 2^{2n-1}. The numerators are more tricky. Look at the amount of flour in bowl A at times 2, 4, 6, 8. The numerators are 1, 5, 21, and 85. Maybe we think of multiplying these by 3, getting 3, 15, 63, 255. These are very recognizable numbers, $2^2 - 1$, $2^4 - 1$, $2^6 - 1$, and $2^8 - 1$. It is reasonable now to conjecture that the numerators at time $2n$ are $(2^{2n} - 1)/3$. Thus we conjecture, in the notation of the second argument for part (a), that

$$f(n) = \frac{2^{2n} - 1}{3 \cdot 2^{2n-1}}.$$

(Note that this formula also gives the correct answer at $n = 0$.) How shall we *prove* the formula always holds? The normal approach is to do it by induction, and that is what our argument will amount to, but we use slightly unusual notation. Let

$$g(n) = \frac{2^{2n} - 1}{3 \cdot 2^{2n-1}}, \quad \text{with} \quad g(0) = 0.$$

We prove that $f(n) = g(n)$ for all n by showing that $g(n)$ satisfies the same recurrence equation as $f(n)$. Note that

$$\begin{aligned} \frac{g(n)}{4} + \frac{1}{2} &= \frac{2^{2n} - 1}{3 \cdot 2^{2n+1}} + \frac{1}{2} \\ &= \frac{2^{2n} - 1}{3 \cdot 2^{2n+1}} + \frac{3 \cdot 2^{2n}}{3 \cdot 2^{2n+1}} \\ &= \frac{2^{2n+2} - 1}{3 \cdot 2^{2n+1}} = g(n+1). \end{aligned}$$

So g satisfies the same recurrence equation as f, and they are both 0 at 0, so they are equal everywhere.

Another Way. We start again from $f(n+1) = f(n)/4 + 1/2$. We want to get rid of the unpleasant constant term $1/2$. The idea is to let $f(m) = h(m) + c$. Then our recurrence becomes $h(n+1) + c = h(n)/4 + c/4 + 1/2$. Choose c so that $c = c/4 + 1/2$. That is satisfied if $c = 2/3$. So our recurrence becomes

$h(n+1) = h(n)/4$. Note that $f(0) = 0$, and therefore $h(0) = -2/3$. Each time that n is incremented by 1, $h(n)$ is divided by 4. It follows that

$$h(n) = -\frac{2}{3 \cdot 2^{2n}} = -\frac{1}{3 \cdot 2^{2n-1}}.$$

It follows that

$$f(n) = \frac{2}{3} - \frac{1}{3 \cdot 2^{2n-1}}.$$

A short calculation shows that this expression for $f(n)$ is equivalent to the one obtained earlier. This expression is in some sense "better," since it makes it clear that as n gets large (and it doesn't have to get *too* large), $f(n)$ approaches 2/3. This is physically obvious. At the beginning, we had a total of 2 cups of sugar and 1 of flour. Our transfer process mixes the sugar and flour. At the end of a cycle, there are 2 cups of stuff in bowl A. In the long run, about 1/3 of that, namely 2/3 of a cup, should be flour.

Comment 4. We might notice during our calculations that $f(n)$ is about 2/3, in fact a bit under 2/3. This, as was pointed out earlier, also makes physical sense. We might therefore want to define a new function $h(n)$ by $f(n) = 2/3 - h(n)$. (In the argument above, we did that because of algebraic considerations.) The recurrence for $h(n)$, as we saw, turns out to be very simple, and it is easy to get an explicit expression for $h(n)$, and therefore for $f(n)$.

Comment 5. The arithmetic (and the typing) were more unpleasant than necessary because of the presence of 3 in the denominator. That can be fixed. Invent a new unit of measure, the *small cup* (s-cup), which is 2/3 of a regular cup. Then we start with 3 s-cup of sugar in bowl A, and 3/2 s-cup of flour in bowl B. Each transfer is 3/2 s-cup.

Then in s-cup, the amount of flour in bowl A after n cycles turns out to be $(2^{2n} - 1)/2^{2n}$, or more simply $1 - 1/2^{2n}$. The arithmetic in the verification that the formula is correct becomes significantly clearer. At the end, we can get the answer in ordinary cups by multiplying by 2/3.

Problem 5. Alicia types all the numbers from 1 to 10^5 inclusive, and Beti types all the numbers from 1 to 10^6, inclusive. (a) Show that the total number of digits that Alicia types is equal to the total number of 0's that Beti types. (b) Generalize, with 10^k and 10^{k+1} in lieu of 10^5 and 10^6.

Solution. (a) It is not hard to find how many digits Alicia types. There are 9 1-digit numbers; typing them involves typing 9 digits. The 2-digit numbers run from 10 to 99. There are 90 of them, for a total of $2 \cdot 90$ digits typed. The 900 3-digit numbers use up $3 \cdot 900$ digits, and so on. At the end comes the single 6 digit number 10^5. The total number of digits typed by Alicia is therefore

$$(9 + 2 \cdot 90 + 3 \cdot 900 + 4 \cdot 9000 + 5 \cdot 90000) + 6.$$

At this stage there is no striking need to "simplify," indeed there is good reason not to. For one thing, it would take some work. Perhaps more seriously, computing a decimal expansion typically hides structure. We will do the calculating at the end, if calculation proves necessary.

We will now count the number of 0's Beti types. Again it seems reasonable to deal with 1-digit numbers, then 2-digit numbers, and so on. In typing the numbers 1 to 9, there are no 0's typed. For the numbers 10 to 99, there are 9 0's typed (for 10, 20, ..., 90). Next we deal with the numbers 100 to 999. For these, there are some choices in how to count the total number of occurrences of 0.

But there is a best choice. There are no occurrences of 0 as the leftmost digit. We count the number of occurrences of 0 in the next position, moving rightward. How many numbers give rise to a 0 there? The first digit such a number can be filled in 9 ways, and for every such way, the last digit can be filled in 10 ways, for a total of 90. We now count in the same way the number of occurrences of 0 in the rightmost position. How many numbers give rise to a 0 there? The first digit of such a number can be filled in 9 ways, and for each such way, the second digit can be filled in 10 ways, for a total of 90. It follows that the number of occurrences of 0 from 100 to 999 is $2 \cdot 90$.

We deal next with the 4-digit numbers 1000 to 9999. There are no occurrences of 0 in the leftmost position. We deal with the number of occurrences of 0 in the next position. For every such occurrence, the rest of the digits of the number can be filled in $9 \cdot 10 \cdot 10$ ways, for a total of 900. Similarly, there are 900 occurrences of 0 as the third digit from the left, and 900 occurrences as the fourth digit from the left, for a total of $3 \cdot 900$.

The work with the 5-digit numbers is essentially the same. The 0's can occur in 4 positions only. If you put a 0 in the leftmost allowed position, the rest of the digits can be chosen in 9000 ways, so there are 9000 occurrences of 0 in that position. Similarly, there are 9000 occurrences of 0 in the next allowed position, and so on for a total of 4 allowed positions, and a total of $4 \cdot 9000$ 0's. Essentially the same argument shows that there are $5 \cdot 90000$ occurrences of 0 among 6-digit numbers. And finally, the 7-digit number 10^6 has 6 0's. It follows that the number of occurrences of the digit 0 among the numbers 1 to 10^6 is

$$(9 + 2 \cdot 90 + 3 \cdot 900 + 4 \cdot 9000 + 5 \cdot 90000) + 6.$$

This is precisely the same *expression* as the expression we got for the total number of digits typed when typing the numbers 1 to 10^5.

Comment 6. It is not hard to get a closed form for a sum of the shape $1 + 2r + 3r^2 + \cdots nr^{n-1}$. Let the sum be S. Then $rS = r + 2r^2 + 3r^3 + \cdots + nr^n$ and therefore $S - rS = 1 + r + r^2 + \cdots + r^{n-1} - nr^n$. The right-hand side, apart from the nr^n term, is a geometric progression, which we can sum in the usual way, and then we are almost finished.

(b) There is nothing serious left to do. The method used in part (a) to deal with the case $k = 5$ works just as easily in general.

First note that the number of digits in 10^k, namely k, is equal to the number of 0's in 10^{k+1}. For the rest, observe that in part (a), for example, the number of digits in the numbers 100 to 999, namely $3 \cdot 900$, was the same as the number of 0's in the numbers 1000 to 9999. We show that the total number of digits in the numbers in the interval $[10^{n-1}, 10^n)$ is always equal to the total number of 0's in the numbers in $[10^n, 10^{n+1})$. (The notation $[a, b)$ means the numbers from a to b, including a but not b.)

The numbers in $[10^{n-1}, 10^n)$ are the n-digit numbers. The first digit of such a number can be chosen in 9 ways. For every such choice, there are 10^{n-1} ways to choose the next $n-1$ digits, so there are $9 \cdot 10^{n-1}$ such numbers. (Alternately, there are $10^n - 10^{n-1}$ numbers in our interval, and $10^n - 10^{n-1} = (10-1)10^{n-1} = 9 \cdot 10^{n-1}$.) Each of these $9 \cdot 10^{n-1}$ has n digits, giving a total of $n \cdot 9 \cdot 10^{n-1}$ digits.

It remains to count the number of 0's in the numbers in $[10^n, 10^{n+1})$. The numbers in this interval are all the $n+1$-digit numbers. There cannot be a 0 in the leftmost position. We now count the number of occurrences of 0 in the second position from the left. Given that there is a 0 in that position, the leftmost position can be filled in 9 ways, and for each such way the remaining $n-1$ positions can be filled in 10^{n-1} ways, for a total of $9 \cdot 10^{n-1}$. Exactly the same argument shows that there are $9 \cdot 10^{n-1}$ occurrences of 0 in the third position from the left. Continue, through all the n non-leftmost positions. We get a total of $n \cdot 9 \cdot 10^{n-1}$ 0's. This completes the argument.

Comment 7. The *occurrences* of digits in the numbers in $[10^{n-1}, 10^n)$ can be matched in a one-to-one way with the occurrences of 0's in the numbers in $[10^n, 10^{n+1})$. We illustrate the idea with occurrences of digits in $[100, 999)$ and occurrences of 0's in $[1000, 9999]$. Take the *first* digit of 347. Match that with the 0 in the second place from the left of the number 3047. The *second* digit of 347 gets matched with the 0 in the third place from the left of the number 3407, and the third digit of 347 gets matched with the 0 in the fourth place from the left of 3470. This works just as well with a number like 107. The three occurrences of digits get matched with the occurrences of 0 in the second place from the left, third place from the left, and fourth place from the left, in the numbers 1007, 1007, 1070. This kind of showing that two counts match by providing an explicit one-to-one correpsondence between two sets is sometimes called a *combinatorial proof,* and is currently quite fashionable.

http://www.pims.math.ca/education/math_problems/
http://www.math.ubc.ca/~adler/problems/