Solutions to November 2008 Problems

Problem 1. Let D be the 10-element set consisting of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. How many ordered pairs (A, B) are there such that the union of the sets A and B is equal to D? One such ordered pair has $A = \{0, 1, 2, 5, 6, 8, 9\}$ and $B = \{0, 1, 3, 4, 7, 9\}$.

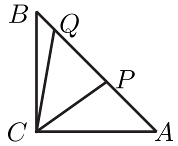
Solution. Let (A, B) be an ordered pair of subsets of D such that the union of A and B is D. Arrange our 10 digits in a row, say in their natural order. Below each digit, write the letter X if the digit is in A but not in B, write the letter Y if the digit is in B but not in A, and write the letter Z if the digit is in both A and B. In this way we obtain a 10-letter word over the alphabet $\{textX, \text{Y}, \text{Z}\}$. Every ordered pair (A, B) of the type we are considering gives rise to a 10-letter word, and every 10-letter word arises from precisely one of our ordered pairs. Thus there are exactly as many ordered pairs as there are 10-letter words over a 3-letter alphabet. The number of such 10-letter words is 3^{10}.

Another Way. Let D_n be a set with n elements. We count the number of ordered pairs (A, B) such that the union of A and B is D_n. Let $f(n)$ be the number of such ordered pairs. We will find an expression for $f(n + 1)$ in terms of $f(n)$.

We can think of D_{n+1} as being obtained by adding an element d to an n-element set D_n. Look at an ordered pair (A, B) of elements of D_n such that the union of A and B is D_n. From such an ordered pair, we can obtain an ordered pair (A', B') whose union is D_{n+1} in exactly 3 ways: (i) add d to A but not to B; (ii) add d to B but not to A; (iii) add d to both A and B. Note that all ordered pairs (A', B') whose union is D_{n+1} can be obtained in this way. It follows that $f(n + 1) = 3f(n)$.

It is obvious that $f(0) = 1$. Thus $f(1) = 3$, $f(2) = 3^2$, and so on, and in general $f(n) = 3^n$.

Problem 2. Let ABC be an isosceles triangle which is right-angled at C. Let P and Q be points on the hypotenuse AB, with P and Q coming in the order shown in the picture below. Suppose that $\angle QCP$ has measure 45°. Show that $(AP)^2 + (BQ)^2 = (PQ)^2$.



Solution. Drop a perpendicular from C to the line AB. Let this perpendicular meet AB at M (by symmetry, M is the midpoint of AB). By scaling if necessary, we may assume that $CM = 1$. (It is harmless to let $CM = k$. The argument below works with minor modification, it is becomes a little harder to type.)

Let $\alpha = \angle PCM$ and let $\beta = \angle QCM$. Then $\alpha + \beta = \pi/4$. (We are using radian notation, for no particularly good reason. If you prefer, you can use degree notation, in which case $\alpha + \beta = 45°$.)

Note that $PM = \tan\alpha$, and $MQ = \tan\beta$. It follows that $PQ = \tan\alpha + \tan\beta$, $AP = 1 - \tan\alpha$, and $BQ = 1 - \tan\beta$. So we want to prove that

$$(\tan\alpha + \tan\beta)^2 = (1 - \tan\alpha)^2 + (1 - \tan\beta)^2.$$

Expand the above expressions. A little cancellation shows that we want to show that

$$2\tan\alpha\tan\beta = 2 - 2\tan\alpha - 2\tan\beta,$$

or equivalently that

$$\tan\alpha\tan\beta = 1 - \tan\alpha - \tan\beta.$$

But this ought to be almost familiar. Note that by the addition laws for sines and cosines, we have

$$\tan(x + y) = \frac{\sin(x + y)}{\cos(x + y)} = \frac{\sin x\cos y + \cos x\sin y}{\cos x\cos y - \sin x\sin y}.$$

Divide "top" and "bottom" of the right-hand side of the above equation by $\cos x\cos y$. We obtain the useful identity

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x\tan y}. \tag{1}$$

In particular, let $x = \alpha$ and $y = \beta$. Then since $\alpha + \beta = \pi/4$, it follows that $\tan(\alpha + \beta) = 1$. Thus from Identity 1, it follows that

$$1 = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta},$$

which is essentially what we wanted to show.

Comment 1. It would be nice to find an argument which is more purely geometric—surely there is one!

Problem 3. Alphonse and Beti are mathematicians who collaborate in separating people from their money. Alphonse is blindfolded. The mark (whose name is Mark) is asked to remove 5 cards from a standard 52-card deck, and hand them back to Beti. Beti gives one of the 5 cards to Mark (keeping 4). Mark puts that card back in the deck, and shuffles thoroughly. Beti then arranges the remaining 4 cards in a neat face-up row. Beti gives 10 to 1 odds that Alphonse can find the card that Mark had put in the deck. Alphonse takes off the blindfold, looks at the 4 cards in the row Beti made, and identifies the missing card. How can this be done?

Solution. Beti could convey much information through the placement of the row of cards (far, near, to her left, her right). There are many other things Beti could do, such as use code words in chatter, mark cards with a thumbnail, scratch her nose, that sort of thing. I will consider all these not within the spirit of the question. We now turn to what *can* be done.

In a primitive way, the cards left on the table give some information—the card that was put back is not one of them! Also, information can be conveyed by the *choice* of card that Beti puts back in the deck. Beti can also convey a considerable amount of information by *ordering* the cards when she arranges them in a "neat face-up row." This we *will* allow.

Take some natural ranking of cards, whatever feels comfortable. Among the 4 cards that are left on the table, let A be the lowest ranking, B the next lowest, and so on up to D. Then the order in which the cards are left can be thought of as a 4-letter word over the alphabet $\{A, B, C, D\}$. These 4! words identify the permutation Beti used, and can be ordered alphabetically. For example, the permutation $CBAD$ is word #15 in the alphabetic ordering, so can be identified with the number 15. If we used a 28 card deck instead of a standard deck, the permutation that Beti generated would be enough to identify the missing card, since there are 4! permutations, and 24 missing cards. And if Mark chose 6 cards instead of 5, the 5! permutations of the 5 cards left on the table would be much more than enough to identify the missing card. However, 24 is short, but not far short (indeed only 1 bit short), of the 48 items we need to distinguish between.

Ultimately, we may want to think numerically, so we can set up some natural correspondence between cards and the numbers from 1 to 52. One way is to think of any ♣ as being lower than any ♢, which is lower than any ♡, which is lower than any ♠. Within suits cards are ordered by value, with Aces low. For example the ♡ Queen is card 36. Any other correspondence will do the job, and you may prefer to put all Aces before all 2's before all 3's and so on. But I think that is harder to compute with.

Imagine the 52 numbers arranged in a circle, like the numbers on a clock, with 52 at the "top," followed clockwise by 1, 2, 3, and so on. The 5 cards picked by Mark occupy 5 positions on that circle. Let A_1 be the smallest of the 5 numbers (cards), and let the other numbers be A_2, A_3, A_4, A_5, in order. Then A_2 is the nearest number to A_1, going clockwise from A_1, and A_5 is the nearest number to A_1, going counterclockwise.

Beti looks at the gap as one goes clockwise from A_5 to A_2. This is the number of clocks numbers strictly between A_5 and A_2, going clockwise from A_5. If this number is less than or equal to 24, then Beti puts A_1 back in the deck. If the clockwise gap from A_5 to A_2 is greater than 24, then Beti puts A_3 back in the deck.

Now Alphonse takes off the blindfold. Let B_1, B_2, B_3, and B_4 be the numbers, in order, that he sees. If the clockwise gap from B_4 to B_1 is between 1 and 24, then Alphonse knows that the gap was created by Beti. So Alphonse knows that the card that was put back lies in the clockwise interval from B_4 to B_1. That cuts down the range of uncertainty to at most 24 cards.

If the clockwise gap from B_4 to B_1 is greater than 24, then the gap was not created by Beti, so $B_1 = A_1$, $B_2 = A_2$, and $B_4 = A_5$. The card put back in the deck must have come from the clockwise gap from B_2 to B_3. Since the clockwise gap from B_4 to B_2 is greater than 24, the number of unoccupied spaces there is greater than or equal to 24. Thus the clockwise gap from B_2 to B_3 is less than or equal to 24, since the total number of unoccupied spaces is 48.

In either case, there remain 24 or fewer candidates for the card that was put back. Beti can use the 24 possible permutations of the remaining 4 cards to signal to Alphonse which card was put back.

Comment 2. The algorithm loosely described above is painfully slow to carry out, it is more a "proof of concept" than a practical tool. The convoluted way in which the range of uncertainty was cut down to 24 is a reflection, not of the difficulty of the problem, but of the fact that the person creating the algorithm is a mathematician, not a card player. It might be interesting to devise a more practical algorithm.

Problem 4. Find all ordered triples (a, b, c) of positive integers such that b divides $2a + 1$, c divides $2b + 1$, and a divides $2c + 1$. (A correct list is not enough: one must show that the list is complete.)

Solution. We can use the (partial) symmetry to simplify the problem. Assume that neither b nor c is smaller than a, that is, that $a \leq b$ and $a \leq c$. Note that since $2a + 1$, $2b + 1$, and $2c + 1$ are all odd, our conditions force b, c, and a to be odd.

Since b divides $2a + 1$, and $b \geq a$, it follows that $b = a$ or $b = 2a + 1$ (recall that b must be odd.)

If $b = a$, we get that a divides $2a + 1$, c divides $2a + 1$, and a divides $2c + 1$. From the fact that a divides $2a+1$, we can conclude immediately (since a divides $2a$) that a divides 1, so $a = 1$. Thus c divides 3, giving $c = 1$ or $c = 3$. And it is automatic in either case that a divides $2c + 1$. This gives us the ordered triples $(1, 1, 1)$ and $(1, 1, 3)$. For the original problem, we have reached the solutions $(1, 1, 1)$, $(1, 1, 3)$, $(1, 3, 1)$, and $(3, 1, 1)$.

Now suppose that $b = 2a + 1$. Since c divides $2b + 1$, it follows that c divides $4a + 3$. Since $a \geq 1$, we have that $4a + 3 \leq 7a$. Dividing a number which is less than or equal to $7a$ by something greater than 7 puts us below a. Thus $c = (4a + 3)/1$, $c = (4a + 3)/3$, $c = (4a + 3)/5$, or $c = (4a + 3)/7$.

We first deal with the case $c = (4a + 3)/1$. Since a divides $2c + 1$, we get that a divides $8a + 7$. It follows that $a = 1$ or $a = 7$. The case $a = 1$ gives $b = 3$, $c = 7$. This gives the triple $(1, 3, 7)$. For the orginal problem, we also get the triples $(7, 1, 3)$ and $(3, 7, 1)$. The case $a = 7$ gives $b = 15$, $c = 31$. This gives the triple $(7, 15, 31)$, and, for the original problem, the triples $(31, 7, 15)$ and $(15, 31, 15)$.

Next we look at the case $c = (4a + 3)/3$. Since c is an integer, a must be a multiple of 3, say $a = 3a'$. Thus $c = 4a' + 1$. Since a divides $2c + 1$, it follows that $3a'$ divides $8a' + 3$. Since a' divides $8a'$, it follows that a' divides 3. If $a' = 1$, then $3a'$ does not divide $8a' + 3$. If $a' = 3$, everything works fine, we get $a = 9$, $b = 19$, $c = 13$, so we obtain the triple $(9, 19, 13)$, and for the original problem also the triples $(13, 9, 19)$ and $(19, 13, 9)$.

Next we look at the case $c = (4a + 3)/5$. Since $a \leq c$, we get that $a \leq (4a + 3)/5$, which yields $a \leq 3$. If $a = 1$, then $(4a + 3)/5$ is not an integer. If $a = 3$, we get $c = 3$. But this does not work, since a does not divide $2c + 1$.

Finally, we look at the case $c = (4a + 3)/7$. Since $a \leq c$, we find that $a = 1$. This gives the triple $(1, 3, 1)$, which we had already found.