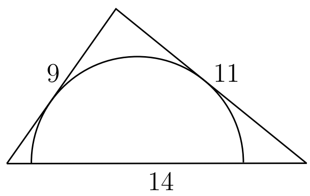
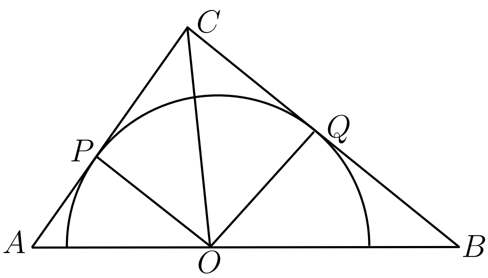
Solutions to November 2009 Problems

Problem 1. A triangle has sides 9, 11, and 14. A semicircle is inscribed in this triangle, with diameter on the side of length 14. Give an exact expression for the radius of the semicircle.



Solution. Join the centre O of the semicircle to the apex of the triangle, and to the points P and Q of tangency—these are almost always good things to do— as in the diagram below.



Let the radius of the semicircle be r. Since OP is perpendicular to AC, and OQ is perpendicular to BC, we have $r = OP = OQ$. Since triangle AOC has base AC and height r, we conclude that $\triangle AOC$ has area $9r/2$. Similarly, $\triangle BOC$ has area $11r/2$. It follows that $9r/2 + 11r/2 = K$, where K is the area of $\triangle ABC$.

To find K, it is most efficient to use *Heron's Formula*, which says that the area of a triangle with sides a, b, and c is

$$\sqrt{s(s-a)(s-b)(s-c)},$$

where s is the semi-perimeter $(a+b+c)/2$. In our case, K turns out to be $12\sqrt{17}$. Thus

$$\frac{9+11}{2}r = 12\sqrt{17}$$

and therefore $r = 6\sqrt{17}/5$.

Another Way. There are other ways to compute the area of $\triangle ABC$. For example, let $\theta = \angle BAC$. By the Cosine Law, we have

$$14^2 = 9^2 + 11^2 - 2(9)(11)\cos\theta.$$

It follows that $\cos\theta = 1/33$, and therefore

$$\sin\theta = \sqrt{1-\cos^2\theta} = \frac{8\sqrt{17}}{33}.$$

The area of the triangle is $(1/2)(AC)(BC)\sin\theta$, which simplifies to $12\sqrt{17}$.

Another Way. Here is a more computationally intensive (and more brute force) approach. It is easy to show that $\triangle OPC$ is congruent to $\triangle OQC$. Thus the line CO bisects $\angle ACB$.

There is a useful result which says that in general, if O is the point on AB such that CO bisects $\angle ACB$, then $AO/BO = AC/BC$. There are clever "Euclidean" ways to prove this result. There is also a natural area argument. Compare $\triangle AOC$ and $\triangle BOC$. They have bases AO and BO, and with respect to these bases they have the same height. Thus the ratio of their areas is AO/BO. But the area of $\triangle AOC$ is $(1/2)(AC)(OC)\sin\psi$, where $\psi = \angle ACO = \angle BCO$. Similarly, the area of $\triangle BOC$ is $(1/2)(AC)(OC)\sin\psi$, so the ratio of their areas is AC/BC. It follows that $AO/BO = AC/BC$.

In our particular case, $AO/BO = 9/11$. Since $AB = 14$, it follows that $AO = (14)(9/20)$. Let $\phi = \angle CAB$. By the Cosine Law, we have

$$11^2 = 9^2 + 14^2 - 2(9)(14)\cos\phi.$$

So now we know $\cos\phi$, it is $156/252$ (which can be simplified a bit). So we know $\sin\phi$. But $\sin\phi = OP/AO$, and now we can compute OP, the radius of the semicircle.

Problem 2. The numbers 1, 2, 3, ..., 2009 are written on index cards, one to a card. The cards are laid out in a row, in some order. Now do the following operation over and over again. Look at the leftmost card: if k is the number written on it, reverse the order of the first k cards, and leave the others where they are. For example, suppose we are looking at the numbers 1 to 9 instead of 1 to 2009, and our initial order is 395672814. Then we next get 593672814 (the first 3 cards have been reversed), and then 763952814 (the first 5 cards have been reversed), and so on.

Show that, after a while, the leftmost card has 1 written on it, so that after a while the order of the cards does not change.

Solution. It is convenient to write n as an abbreviation for 2009. Imagine doing our operation over and over again. At each stage, we are looking at a rearrangement (permutation) of the sequence $(1, 2, 3, \ldots, n)$.

Call these permutations P_1, P_2, P_3, and so on. Let A be the largest number that *ever* appears in the first position, and suppose that this (first) occurs in P_a. Then (except in the trivial case $A = 1$), in the next permutation P_{a+1}, the number A is buried in position A.

Here "buried" is the right word. Imagine for concreteness that the largest number that ever appears in first position is 1968 (a good year), and that this (first) happens on permutation P_{13}. Then in permutation P_{14}, the number 1968 is in position 1968. No number greater than 1968 ever appears in first position. Numbers less than 1968 can only scramble numbers in positions 1 to 1967, and so must leave 1968 buried at position 1968.

Now consider the permutations P_{a+1}, P_{a+2}, $Pa + 3$, and so on. In all of these, A is buried in position A. Let B be the largest number that ever appears in first position of one of these permutations, and suppose that this (first) occurs in permutation P_b. Then (except in the case $B = 1$, but then we are finished), in the next permutation P_{b+1} the number B is buried in position B. The argument is exactly the same as the one for A. For example, if $A = 1968$ and $a = 13$, we are looking at permutations P_{14}, P_{15}, P_{16}, and so on. Suppose that 1939 (a bad year) is the largest number that ever appears in first position of one of these permutations, and that it first occurs (after permutation P_{13}) in permutation P_{17}. (The number 1939 could have been in first position in one of the permutations before P_{13}, but we are only looking beyond P_{13}.) Then from permutation P_{18} on, 1939 is buried in position 1939.

Note that $b > a$, since we are only looking at permutations from P_{a+1} on. Note also that $A > B$. Now consider the permutations P_{b+1}, P_{b+2}, P_{b+3}, and so on. Let C be the largest number that ever appears in first position of one of these permutations, and suppose this (first) happens in permutation P_c. Then (except in the case $C = 1$, but then we are finished), in the next permutation P_{c+1} the number C is buried in position C. Note that $A > B > C$.

Continue. Unless at some stage we have 1 in first position, we obtain an infinite descending sequence $A < B > C > D > \cdots$ of integers, all positive. This is impossible. We conclude that 1 must appear at some time on the leftmost card.

Problem 3. Show that $2^{99} + 3^{99}$ is divisible by 35.

Solution. There is computer software (for example Maple, or Mathematica, or even free software) that will compute $2^{99} + 3^{99}$ exactly, and then compute the remainder when this is divided by 35. But let's see whether we can solve the problem "by hand." For brevity, let $N = 2^{99} + 3^{99}$.

There is a standard approach worth exploring. Note that $35 = (5)(7)$. So to show that N is divisible by 35, it is enough to show that N is divisible by 5 and by 7.

Divisibility by 5 is easy to check because of our familiarity with the decimal system. When $n = 1$, 2, 3, and so on, the rightmost decimal digit of 2^n is 2, 4, 8, 6, 2, 4, 8, 6, ... Similarly, the rightmost decimal digit of 3^n is 3, 9, 7, 1, 3, 9, 7, ... Thus the rightmost decimal digit of $2^n + 3^n$ is 5, 3, 5, 7, 5, 3, 5, 7, ... The rightmost digit of $2^n + 3^n$ is therefore 5 whenever n is odd, and hence in particular $2^n + 3^n$ is a multiple of 5 when $n = 99$.

For divisibility by 7, we do not have the familiar decimal arithmetic to help us (things might get easier once we get well-acquainted with the arithmetic of the seven-fingered Martians). So we make some preliminary general comments.

Let a and m be positive integers. Suppose that we know that the remainder when a^{2009} is divided by m is r. We want to compute the remainder when a^{2010} is divided by m.

Note that $a^{2009} = qm + r$, where q is the "quotient." So $a^{2010} = aqm + ar$. Thus when we divide a^{2010} by m, the part aqm is already divisible by m, so does not affect the remainder. Thus the remainder when a^{2010} is divided by m is the same as the remainder when ar is divided by m.

This has consequences both practical and theoretical. To compute the remainder when a^{n+1} is divided by m, once we know the remainder r when a^n is divided by m, we do not need to calculate a^{n+1}, which might be ludicrously large, we only need to work with the potentially far smaller number r, which is less than 7 if $m = 6$. And since the "next" remainder (for a^{n+1}) depends only on the previous one (the one for a^n), once we spot some cycling in the remainders, that cycling *must* continue. So, at least for some patterns, once we spot the pattern, the fact that it continues is a matter not of *guessing,* but of *knowing.*

The remainders when 2^n is divided by 7, when $n = 0, 1, 2$, and so on are 1, 2, 4, 1, 2, 4, and so on. The remainders when 3^n is divided by 7 are 1, 3, 2, 6, 4, 5, 1, 3, and so on. The first pattern cycles with cycle length 3, and the second with cycle length 6. So when $2^n + 3^n$ is divided by n, the remainders *must repeat* when n is advanced by 6.

These remainders are 2, 5, 6, 0, 6, 2, 2, 5, 5, 0, 6, 2, and so on. So the remainder is 0 precisely when n is 3 more than a multiple of 6, or equivalently an odd multiple of 3.

We conclude that $2^n + 3^n$ is divisible by 5 *and* by 7 (that is, by 35) precisely when n is a multiple of 3 *and* n is an odd multiple of 6. If the second condition is met, so is the first, and therefore $2^n + 3^n$ is divisible by 35 precisely if n is an odd multiple of 3. Finally, note that 99 is an odd multiple of 3. The result follows.

Another Way. Let $x = 2^3$ and $y = 3^3$. Then $2^{99} + 3^{99} = x^{33} + y^{33}$. Note that $x + y = 35$. We will show that in general, if n is an *odd* positive integer, and x and y are integers, then $x + y$ divides $x^n + y^n$. There are various ways to do this. For odd n, we have the general identity

$$x^n + y^n = (x + y)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 - x^{n-3}y^3 + \cdots - xy^{n-2} + y^{n-1}),$$

To verify the identity, multiply out the right-hand side, note how almost every term cancels. So if x and y are integers, then $x^n + y^n$ is $x + y$ times something which is obviously an integer, and the result follows.

Another approach is to note that $x + y$ obviously divides $x^1 + y^1$. Next we show that if $x + y$ divides $x^n + y^n$, then $x + y$ must divide $x^{n+2} + y^{n+2}$. To do this, we can use the following easily verified identity:

$$x^{n+2} + y^{n+2} = (x^{n+1} + y^{n+1})(x + y) - xy(x^n + y^n).$$

The term $(x^{n+1} + y^{n+1})(x + y)$ is divisible by $x + y$, and if $x + y$ divides $x^n + y^n$, then it clearly divides $xy(x^n + y^n)$, so $x + y$ divides the right-hand side, and hence the left.

Problem 4. Find, with proof, the largest possible value of the product $(x_1)(x_2)(x_3)\cdots(x_n)$, as n, and $x_1, x_2, x_3, \ldots, x_n$ range over all positive integers such that

$$x_1 + x_2 + x_3 + \cdots + x_n = 2009.$$

(Note that n is at our disposal, we are free to have as many x_i as we wish, up to 2009.)

Solution. Suppose that a particular n, and $(x_1, x_2, x_3, \ldots, x_n)$ gives a largest product. (There *is* a largest product, for there are only finitely many ways to express n as sum of positive integers.)

None of the x_i can be 1. For suppose for example that $x_n = 1$. Then the product $(x_1)(x_2)\cdots(x_{n-1}+1)$ is greater than $(x_1)(x_2)\cdots(x_n)$.

Next we show that for a "largest product," all the x_i are less than 5. For if x_i is the even number $2k$, then replacing $2k$ by two copies of k does not change the sum, but changes the product by the factor $k^2/2k$, which is obviously greater than 1 if $k > 2$. Similarly, if x_i is the odd number $2k+1$, then replacing $2k+1$ by the two numbers k changes the product by a factor $k(k+1)/(2k+1)$, which is greater than 1 if $k \geq 2$. To check this, we need to show that $k(k+1) > 2k+1$, or equivalently that $k^2 - k - 1 > 0$, if $k \geq 2$.

This is straightforward to show by graphing the curve $y = x^2 - x - 1$. More algebraically, note that $x^2 - x - 1 = (x - 1/2)^2 - 5/4$, so $x^2 - x + 1$ is positive whenever $x > 1/2 + \sqrt{5}/2$ (nice to see the golden ratio crop up again!), and in particular when $x > 1.6181$. Or else note that $k^2 - k - 1 > k^2 - k - 2$, but $k^2 - k - 2 = (k-2)(k+1)$, and $(k-2)(k+1)$ is obviously non-negative if $k \geq 2$. So for a ''largest product," all the x_i are 2, 3, or 4. But if $x_i = 4$, then replacing 4 by two copies of 2 changes neither sum nor product. So we can assume that all the x_i are 2 or 3.

There are many ways to express 2009 as a sum of 2's and 3's. But for a largest product, we cannot have many 2's. Any three 2's can be replaced by two 3's, which changes the product by the factor 9/8. Do this over and over again until the number of 2's is two or less.

For 2009, subtracting 2 gets us to a multiple of 3. So decomposing 2009 as a sum of 670 parts, a 2 plus 669 copies of 3, gives largest product, which is 2×3^{669}.