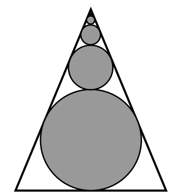
Solutions to November 2010 Problems

Problem 1. The base of an isosceles triangle is 20, and the equal sides are both 26. There are infinitely many shaded circles. The largest is inscribed in the triangle, and each of the others is tangent to two sides of the triangle and to the circle below it. What is the sum of the perimeters of *all* the circles?



Solution. The triangle is a close relative of something familiar. Drop a perpendicular from the top vertex to the base. This divides the isosceles triangle into two right-angled triangles, each with hypotenuse 26 and base 10. By the Pythagorean Theorem, the remaining side of each right triangle has length $\sqrt{26^2 - 10^2}$. This turns out to be 24. So now, or earlier, we recognize that each right triangle is the familiar 5-12–13 triangle, scaled by a factor of 2. The height of our isosceles triangle (with respect to the base of length 20) is 24.

Let r_1, r_2, r_3, and so on be the radii of our circles, in decreasing order of size. Then the sum of the perimeters of all the circles is the infinite "sum"

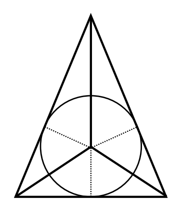
$$2\pi r_1 + 2\pi r_2 + 2\pi r_3 + \cdots + 2\pi r_n + \cdots .$$

This infinite sum is

$$\pi(2r_1 + 2r_2 + 2r_3 + \cdots + 2r_n + \cdots).$$

A glance at the diagram shows that $2r_1 + 2r_2 + \cdots$ is just the altitude of our isosceles triangle, namely 24. It follows that the sum of the perimeters is 24π.

Another Way. The first solution was very simple, very quick. We now do more work: we calculate explicitly the radius of the first circle, of the second, and so on, then calculate the perimeters, then add up.



In the picture, we removed all the circles except the first one. Join the centre of this circle to the vertices. This splits out triangle into three triangles. Draw (dotted lines) line segments from the centre of the circle to the points of tangency. Look first at the bottom triangle. It has area $20r_1/2$.

The two "side" triangles each have area $26r_1/2$, so the combined area of the three triangles that our big triangle has been split into is $20r_1/2 + 26r_1/2 + 26r_1/2$. But this sum is equal to the area of the big triangle, which is $(20)(24)/2$. Thus $72r_1/2 = (20)(24)/2$. Solve for r_1. Simplifying a bit, we get $r_1 = 20/3$.

Comment. In general, suppose that a triangle has area A and perimeter p. Let r be the radius of the incircle of the triangle. Essentially the same argument as the one above shows that

$$r = \frac{2A}{p}.$$

Now we show how to calculate r_2, r_3, and so on. Draw a line parallel to the base, and tangent to the biggest circle at the top of that circle. The triangle "above" it is similar to the given big triangle. The height of this new triangle is $24 - 40/3$, or more simply $32/3$. So the ratio of the new height to the old is $(32/3)/24$, or more simply $4/9$. Thus the new triangle is the big one with all dimensions scaled by the factor $4/9$.

It follows that $r_2 = (4/9)r_1$. The same scaling argument shows that $r_3 = (4/9)r_2$, $r_4 = (4/9)r_3$, and so on. Thus in general $r_n = (4/9)^{n-1}r_1$. So the perimeter of the n-th circle is $2\pi(20/3)(4/9)^{n-1}$. Adding up, we find that the sum of all the perimeters is

$$(2\pi)(20/3)(1 + x + x^2 + x^3 + \cdots),$$

where $x = 4/9$. It is a standard and fairly easily proved fact that (if $|x| < 1$), then the sum of the infinite geometric series $1 + x + x^2 + x^3 + \cdots$ is given by

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}.$$

Thus the sum of the perimeters is $(2\pi)(20/3)/(1 - 4/9)$. This simplifies to 24π.

Comment. There is a variant of the above argument that bypasses the formula for the sum of an infinite series (or, to take another point of view, shows how to sum the infinite series). Let the sum of all the perimeters be S. Then S is the perimeter of the biggest circle plus the sum of the perimeters of the rest of the circles. But the second triangle, obtained by drawing the line parallel to the base and tangent to the big circle at the top of that circle, is the big triangle scaled by $4/9$, so the sum of the perimeters of the rest of the circles is $(4/9)S$. It follows that

$$S = (2\pi)\left(\frac{20}{3}\right) + \left(\frac{4}{9}\right)S.$$

Solve for S. We get $S = 24\pi$.

The first argument was much simpler. But if we want instead the sum of the areas of the circles, nothing as simple as the first solution is available, but a minor adaptation of the second solution does the job.

Problem 2. Define the sequence A_0, A_1, A_2, and so on by $A_0 = A_1 = 1$, and $A_n = 2A_{n-1} + A_{n-2}$ for $n \geq 2$. Let $x = 1/3$. Calculate

$$A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \cdots + A_n x^n + \cdots.$$

Manipulate "infinite sums" freely, assume they behave algebraically like finite sums.

Solution. The calculation imitates the standard way to sum the infinite geometric series $1 + x + x^2 + \cdots$. Recall that if we let that sum be $G(x)$, then $G(x) - xG(x) = 1$ (almost everything cancels). It follows that $G(x) = 1/(1 - x)$. So let our sum be $S(x)$. Multiply $S(x)$ by $2x$, and subtract from the expression for $S(x)$. Gathering like powers of x together, we obtain

$$S(x) - 2xS(x) = A_0 + (A_1 - 2A_0)x + (A_2 - 2A_1)x^2 + (A_3 - 2A_2)x^3 + (A_4 - 2A_3)x^4 + \cdots .$$

If $n \ge 2$, then $A_n - 2A_{n-1} = A_{n-2}$. Using this, we find that

$$S(x) - 2xS(x) = 1 - x + A_0x^2 + A_1x^3 + A_2x^4 + \cdots = 1 - x + x^2S(x).$$

A little manipulation now gives $S(x) = (1 - x)/(1 - 2x - x^2)$. When $x = 1/3$, this is equal to 3.

Comment. Things are somewhat more messy looking if from the beginning we work with 1/3 rather than x. This increases the probability of error, and more importantly makes it more likely that a nice structural pattern will be missed. Quite often in problems, even when specific numbers are mentioned, it can be useful to replace them by letters. Any "algebra" will look much neater, and one may get a general result. Working with specific numbers from the beginning may be necessary, but it should be postponed if possible. In particular, premature use of the calculator can hide vital structural information.

As instructed, we operated "formally" on the series, ignoring issues of convergence. It turns out that our series converges if $|x| < \sqrt{2} - 1$, which (no accident!) is one the roots of the equation $1 - 2x - x^2 = 0$. That root is roughly 0.4142, and 1/3 is safely smaller. A proof that there is convergence at $x = 1/3$ is not hard. It is enough to show (say by induction) that $A_n < 0.35^n$ if n is large enough.

Problem 3. Without using calculus, find the slope of the tangent to the hyperbola $xy = 1$ at the point $(1/3, 3)$. Ideally, find two entirely different approaches.

Solution. Draw a picture of the curve (not shown here). The curve is a hyperbola, symmetrical about the line with equation $y = x$. The full geometry will be clearer if we also draw the third quadrant part of the curve.

The tangent line is not vertical, so it has equation of the shape $y - 3 = m(x - 1/3)$, where m is the slope. To find the x-coordinates of the points where this line meets the hyperbola, substitute $3 + m(x - 1/3)$ for y in the equation $xy = 1$, and simplify. We arrive at the equation

$$mx^2 - (m/3 - 3)x - 1 = 0.$$

It is geometrically clear that the slope is negative, and in particular non-zero. The product of the roots of the equation is $-1/m$. Since one of the roots is 1/3, the other is $-3/m$. (We could also work with the sum of the roots, that is slightly more complicated.)

Since m is negative, there are two distinct positive roots, unless, of course, 1/3 and $-3/m$ coincide, in which case there is only one root. But the picture shows that in the case of tangency, there is only one root. Thus $-3/m = 1/3$, and $m = -9$.

A slightly different way of doing the same thing is to refer to the Quadratic Formula. The "$b^2 - 4ac$" part of that formula is called the *discriminant.* The discriminant is $(m/3 - 3)^2 + 4m$, which simplifies to $(1/9)(m + 9)^2$. This discriminant is always non-negative, and it is positive unless $m = -9$. So there are two distinct real roots unless $m = -9$. A scan of $xy = 1$ shows that unequal roots means non-tangency, so we have tangency precisely if $m = -9$. The discriminant approach is somewhat more complicated than the earlier approach through the product (or sum) of the roots. but the discriminant *is* important, so maybe it is a good idea to use it even though it complicates things. But then again, the sum and product stuff about the roots is arguably even more important than the discriminant.

Essentially the same argument can be used to compute the slope of the tangent line to $xy = k$ at the point $(a, k/a)$.

Another Way. Let $s = 3x$ and let $t = y/3$. Then equation $xy = 1$ becomes $st = 1$, and the point $x = 1/3$, $y = 3$ becomes $s = 1$, $t = 1$. Now let's think about the geometry. Draw two graphs, of the curves $xy = 1$ and $st = 1$. The graph of $st = 16$ is just the graph of $xy = 1$, scaled in the horizontal direction by a stretching factor of 3 and in the vertical direction by a stretching factor of 1/3.

The tangent line to $xy = 1$ at $(1/3, 3)$ is transformed by the stretching into the tangent line to $st = 1$ at the point $(1, 1)$. Now we take advantage of symmetry, which was the whole point of the game. The tangent line to $st = 1$ at $(1, 1)$ obviously has slope -1. Now *Transform Back*, by scaling in the horizontal direction by the factor 1/3 and in the vertical direction by a factor of 3. The tangent line to $st = 1$ at $(1, 1)$ is transformed back into the tangent line to $xy = 1$ at $(1/3, 3)$. The scaling by 1/3 in the x-direction multiplies slopes by 3, and the scaling by 3 in the y-direction again multiplies slopes by 3. This is easy to verify algebraically, and is geometrically obvious. So our slope is -9.

Comment. We have used a particular example of a general technique often called "Transform, Solve, Transform Back." Many techniques, both elementary and not so elementary, fall under this rubric. As a familiar example, suppose that we are interested in the curve $y = x^2 + 4x - 17$. Rewrite as $y = (x + 2)^2 - 21$. Let $t = x + 2$. We are looking at $y = t^2 - 21$, which has pleasant symmetry about $t = 0$. Equivalently, move the curve to the left by the amount 2. We arrive at $y = x^2 - 21$. It is obvious where this curve crosses the x-axis. Transform back to solve $x^2 + 4x - 17 = 0$. Techniques of linear algebra such as diagonalization are important because the transformed problem is often easy to solve.

Problem 4. Beth has a biased loonie that lands heads with probability p, and tails with probability $1 - p$. Alicia tosses the coin repeatedly, and keeps a running count of heads and tails. If the number of heads is *ever* greater than the number of tails, Alicia wins the game (and the coin). What is the probability that Alicia wins the game? There are sharp differences between the cases $p < 1/2$ and $p \geq 1/2$.

Solution. Let x be the probability that Alicia wins the game. Winning for Alicia can happen in two ways: (1) the first toss is a head or (2) the first toss is a tail. The first toss is a head with probability p. If the first toss is a head, the game is over, Alicia has won.

If the first toss is a tail (probability $1 - p$) then in order to win, Alicia must at some time draw even, and then at some later time get ahead. The probability of at some time drawing even is x, for drawing even when she is "1 down" is the same problem as getting 1 head ahead when they are tied, and that probability is x. And *given* that Alicia has drawn even, the probability Alicia ultimately wins is x. We have obtained the equation

$$x = p + (1 - p)(x)(x).$$

If $p = 1$, there is no issue, Alicia wins with probability 1. If $p \neq 1$, look at the quadratic equation above. In standard form, it is $(1 - p)x^2 - x + p = 0$. Solve. The simplest way is to observe that 1 is a root. But the product of the roots is $p/(1 - p)$, so the other root is $p/(1 - p)$.

Our equation has two roots. Which one is the answer? If $p > 1/2$, the root $p/(1-p)$ is greater than 1, so cannot be a probability. Thus $x = 1$. If $p = 1/2$, the two roots are equal, so again $x = 1$.

The case $p < 1/2$ needs some argument. In that case, $p/(1-p)$ is between 0 and 1. So is it the answer? We must eliminate the other root $x = 1$ as a possibility. If $p < 1/2$, then it is intuitively reasonable (and even true) that after n tosses, where n is large, the number of heads obtained, divided by n, is very likely to be close to p. Thus if n is large, then with positive probability we never return to a situation where the number of heads is equal to the number of tails. That means that there is a positive probability that after the first toss, we never have equality of heads and tails. That eliminates the root $x = 1$.

One might note that $p/(1-p) = 1/2$ if and only if $p = 1$. So the game is a fair one if the loonie lands heads with probability 1/3.

Comment. The probability that A and B are tied after $2n$ coin flips is $\binom{2n}{n}p^n(1-p)^n$. We estimate $\binom{2n}{n}$. It is not hard to verify that

$$\binom{2n}{n} = \frac{(2n)(2n-1)}{n^2}\binom{2n-2}{n-1} < 4\binom{2n-2}{n-1}.$$

From this we can conclude that $\binom{2n}{n} < 4^n$ if $n > 0$.

Suppose that $p \neq 1/2$. By completing the square, we can show that $p(1-p) < 1/4$. The probability that there is a tie at the $2m$-th toss *or beyond* is less than

$$\sum_{k=m}^{\infty} \frac{2k}{k} p^k(1-p)^k.$$

Let $t = 4p(1-p)$. If $p \neq 1/2$, then $t < 1$, so the above sum is less than

$$\sum_{k=m}^{\infty} t^k.$$

This can be made < 1 (indeed arbitrarily close to 0) by taking m large enough. It follows that that if $p \neq 1/2$ and m is large, the probability that the contestants are not tied after $2m$ tosses or beyond is greater than 0. We had argued earlier that this is intuitively very reasonable, but wanted to show that with some effort one can produce a formal proof.

There is a huge literature on questions related to the "Alicia" problem. One can find an explicit formula for the probability that Alice first draws ahead on the n-th toss, but that approach takes much more work than the solution described above. Terms to search for include "random walk," "Reflection Principle," and "Catalan number." Random walks, beside being mathematically beautiful, have applications in many branches of science.

http://www.pims.math.ca/education/math_problems/
http://www.math.ubc.ca/~adler/problems/