Solutions to December 2006 Problems

Problem 1. Alphonse ran in a cross-country race, running half of the *distance* at 3 minutes per km and half at 3 minutes 10 seconds per km. If he had run half of the *time* at 3 minutes per km, and half at 3 minutes 10 seconds per km, it would have taken him 1 second less to finish the race. How long did Alphonse actually take?

Solution. We solve a somewhat more general problem. Let half the distance be run at a minutes per km, and half at b minutes per km. Suppose that running half the *time* at a minutes per km and half the time at b minutes per km would have saved m minutes. We compute the time t it actually took to run the race. In our problem, $a = 3$, $b = 3 + \frac{10}{60}$, and $m = \frac{1}{60}$.

Why work with a, b, and m instead of the specific numbers of the problem? There are several reasons. We get increased generality. Also, letters are often less complicated to deal with than numbers. Finally, and most importantly, letters may reveal structure that numerical computation hides.

It is natural to use "algebra." We denote certain obviously important quantities by letters, set up equations, and solve them. A more geometric approach would also work.

Let half the distance be h. The time that Alphonse actually took was $ah+bh$, so $t = (a + b)h$. If Alphonse had run half the time at a minutes per km, and half at b minutes per km, he would have saved m minutes. Half of what time? It would have taken him $t - m$ to finish the race, and half of that is $(t - m)/2$.

If we run for time $(t - m)/2$ at a minutes per km, we cover a distance $(t - m)/2a$. Similarly, if we run for time $(t - m)/2$ at b minutes per km, we cover a distance $(t - m)/2a$. The sum of these distances is the actual length of the race course, namely $2h$. We have obtained the equation

$$\frac{t-m}{2a} + \frac{t-m}{2b} = 2h = \frac{2t}{a+b}.$$

Solve for t. We get

$$t\left(\frac{1}{2a} + \frac{1}{2b} - \frac{2}{a+b}\right) = m\left(\frac{1}{2a} + \frac{1}{2b}\right).$$

If we are working numerically, we are finished, for we can calculate the coefficient of t, and the right-hand side, and divide. But since we are working with a and b, we simplify first. Multiply through by $2ab(a + b)$. We obtain

$$t\left(b(a+b) + a(a+b) - 4ab\right) = m\left(b(a+b) + a(a+b)\right),$$

and, after a little more work,

$$t = m\left(\frac{a+b}{a-b}\right)^2.$$

Finally, put $m = 1/60$, $a = 3$, and $b = 3 + \frac{10}{60}$. It turns out that $t = 1369/60$, that is, 22 minutes and 49 seconds.

Problem 2. Imagine calculating the decimal expansion of $(10+\sqrt{101})^{99}$. What are the first two *non-zero* digits after the decimal point?

Solution. An ordinary scientific calculator is of no (direct) use. When I ask mine to evaluate $(10+\sqrt{101})^{99}$, it refuses, since it cannot deal with numbers from 10^{100} on. The calculator bundled with Microsoft Windows stores and displays numbers to limited precision, about 35 digits (with coaxing), so it also cannot handle the problem directly. The excellent (and Canadian) program Maple can do the job, as can Mathematica, and a number of less expensive programs. We will show that with a little thinking, even a cheap scientific calculator can give us the answer.

In many problems involving quantities such as $10+\sqrt{101}$, the related quantity $10-\sqrt{101}$ can be very useful. Imagine expanding $(x+y)^{99}$, where x and y are arbitrary. We get an expression of the type

$$c_0x^{99}+c_1x^{98}y+c_2x^{97}y^2+\cdots+c_{98}xy^{98}+c_{99}y^{99}$$

where the c_i are integers. It is easy to see that $c_0=c_{99}=1$. The other c_i are more complicated to compute. Luckily we don't have to. But in fact there is a simple expression for the c_i. By the Binomial Theorem, $c_i=\binom{99}{i}$, where $\binom{99}{i}$ is the number of ways of *choosing* i objects from a collection of 99 objects. In secondary school, what mathematicians call $\binom{99}{i}$ is often written as ${}_{99}C_i$. There is a relatively simple expression for $\binom{99}{i}$, but we shall not need it here.

Expand $(10+\sqrt{101})^{99}$. We obtain

$$c_0 10^{99}+c_1 10^{98}\sqrt{101}+c_2 10^{97}101+c_3 10^{96}101\sqrt{101}+\cdots+c_{99}101^{49}\sqrt{101}.$$

The fine details of the expansion are unimportant. What matters is that

$$(10+\sqrt{101})^{99}=A+B\sqrt{101}$$

where A and B are integers.

Now expand $10+\sqrt{101})^{99}$. We obtain

$$c_0 10^{99}-c_1 10^{98}\sqrt{101}+c_2 10^{97}101-c_3 10^{96}101\sqrt{101}+\cdots-c_{99}101^{49}\sqrt{101}.$$

Note that $(10+\sqrt{101})^{99}=A-B\sqrt{101}$. Add. We obtain

$$(10+\sqrt{101})^{99}+(10-\sqrt{101})^{99}=2A,$$

so in particular the sum on the left is an integer. We can rewrite the above equation as

$$(10+\sqrt{101})^{99}=2A-(10+\sqrt{101})^{99}.$$

Note that $(10-\sqrt{101})$ is approximately -0.0498756. So $(10+\sqrt{101})^{99}$ is a (very large) integer $2A$ plus a very small positive quantity, namely $(\sqrt{101}-10)^{99}$.

We now need to find the first two non-zero digits after the decimal point of $(\sqrt{101}-10)^{99}$. The calculator bundled with Microsoft Windows has no trouble with this: it says that our number is approximately $1.232944 \times 10^{-129}$. So the required first two non-zero digits are 1 and 2, in that order.

With a cheap scientific calculator, we need to do some additional work. My under 10 dollar TI, though in general very good, cannot handle $(\sqrt{101} - 10)^{99}$. But there is an easy way of working around the limitations.

The calculator says that $(\sqrt{101} - 10)^9$ is approximately 1.9098×10^{-12}. It follows that $(\sqrt{101} - 10)^{99}$ is approximately $(1.9098)^{11} \times 10^{-132}$. The calculator also says that $(1.9098)^{11}$ is approximately 1232.9. So the required first two non-zero digits after the decimal point are 1 and 2.

Problem 3. There are six tickets in a box, with the numbers 1 to 6 written on them. The tickets are taken out of the box one at a time. In how many different ways can this be done, if at any stage of the process the numbers already taken out have to be a *set* of consecutive integers, not necessarily in their natural order? One of these ways is withdrawal in the order 435621, for note that each of $\{4\}$, $\{4, 3\}$, $\{4, 3, 5\}$, $\{4, 3, 5, 6\}$, $\{4, 3, 5, 6, 2\}$, and $\{4, 3, 5, 6, 2, 1\}$ is a set of consecutive integers. It would be best to take an approach that generalizes readily to the numbers from 1 to n.

Solution. We could try to make a complete list. This is not difficult, but has to be done carefully, since it is easy to miss some possibilities. We can save a lot of work by taking advantage of symmetry. It is easy to see that withdrawal in the order $a_1, a_2, \dots, a_6$ satisfies the "set of consecutive integers" rule if and only if the order $7 - a_1, 7 - a_2, \dots, 7 - a_6$ satisfies the rule. This means that we can count all possibilities, where the first number withdrawn is 1, 2, or 3. There will be just as many possibilities where the first number withdrawn is 4, 5, or 6. This simple observation cuts out half of the work. With some care, after a while we get that the total number of ways is 32. We might then *guess* that the answer with n tickets is 2^{n-1}, but that would only be a guess.

Another Way. Let $J(n)$ be the number of ways of doing the job with n tickets. It is easy to see by direct listing that $J(1) = 1$, $J(2) = 2$, $J(3) = 4$, and $J(4) = 8$, which leads to the natural conjecture that $J(n) = 2^{n-1}$ for all n.

Imagine that we have $n + 1$ consecutively numbered tickets in the box. It is easy to see that if they are withdrawn in a "legitimate" order, the last ticket withdrawn must be numbered (a) 1 or (b) $n + 1$. In case (a), the first n tickets must be removed in a legitimate order. This can be done in $J(n)$ ways, so there are $J(n)$ ways of withdrawing the $n + 1$ tickets in legitimate order, with ticket $n + 1$ last. In case (b), the tickets numbered 2 to $n + 1$ must be withdrawn in a legitimate order, and again this can be done in $J(n)$ ways. We have shown that

$$J(n + 1) = 2J(n).$$

Since $J(1) = 1$, and there is doubling each time we add a ticket, it follows that $J(n) = 2^{n-1}$.

Another Way. Imagine the tickets are cards, and that as we withdraw cards one at a time, we form a "hand" with them, with the cards sorted in numerical order. So for example if the first card is 4, the next 3, and the next 5, our sorted hand is first (4), then $(3, 4)$, then $(3, 4, 5)$. As we withdraw tickets, our hand grows. Note that the new card is always placed at the beginning or at the end of the growing hand. Make a movie of the process.

Now play the movie *backwards.* The backwards movie starts with the hand $(1, 2, 3, \ldots, n)$, and at each step a card is removed from the hand and put back in the box. Note that at each step in the backwards movie, the card is removed either from the beginning or the end of the shrinking hand. So at each step in the backwards movie, there are 2 choices (except of course when we are down to 1 card). Thus we have $n - 1$ consecutive steps with 2 choices at each step, for a total of 2^{n-1} ways.

Problem 4. Two circles have the same center; one has radius 2 and the other has radius 3. Points P and Q are chosen independently and at random on the boundary of the outer circle. Find, correct to 3 decimal places, the probability that the line PQ passes through the inner circle.

Solution. We might as well solve a more general problem. Let r be the radius of the small (inner) circle, and let R be the radius of the large circle. By a rotation of a circle about its center, any point on the circle can be carried to any other point. Rotate the big circle about its center (carrying P and Q along) until P at the top of the big circle, as in Figure 1.

The line PQ crosses the small circle precisely if Q lies on the smaller arc that joins the points labelled A and B. The probability that Q lies on arc AB

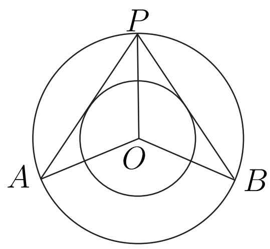

Figure 1: Two Circles

is the ratio of the length of this arc to the circumference of the full circle. This is the same as the ratio of of $\angle AOB$ to a full rotation.

The angle APO is easy to find, for its sine is r/R, where r and R are the radii of the two circles. Thus $\angle APB$ is twice the angle whose sine is r/R. By quoting a general result, or after some angle chasing, we find that $\angle AOB$ is twice $\angle APB$, so $\angle AOB$ is four times the angle whose sine is r/R.

In our case, $r = 2$ and $R = 3$, so $\angle APO$ is the angle (between $0°$ and $90°$) whose sine is $2/3$. We can work in radians, or degrees, or any other angle measure. Because degree measure is familiar, let us work with that. The calculator gives that this angle is about 41.810315 degree. Now to find the probability, multiply by 4, divide by 360. We get about 0.464559, or, correct to 3 decimal places, 0.465. (There is no algebraic "exact" expression for the probability.)

Comment. When this problem was done in a school workshop, the presenter—not a student—argued as follows. If P and Q are chosen at random, the perpendicular distance from O to PQ lies between 0 and 3 (true). The line PQ crosses the circle if this perpendicular distance lies between 0 and 2 (true). "Therefore" the required probability is $2/3$. This is plausible-sounding but wrong. Probability can be tricky, and geometric probability is notoriously so.

Problem 5. Find all integers n such that $2^n + n^2$ is a perfect square, and show that you have found them all.

Solution. First explore a little. Let $f(n) = 2^n + n^2$. It is obvious that n cannot be negative, for if n is a negative integer, then 2^n is not an integer, but n^2 is, so $f(n)$ cannot be an integer, let alone a perfect square.

Note that $f(0)$ is a perfect square. Easy calculations show that $f(n)$ is not a perfect square for $n = 1$, 2, 3, 4, and 5, and that $f(6) = 10^2$. We will show that 0 and 6 are the *only* values of n that work.

We take a brief look at solutions in non-negative integers of the more general equation

$$2^n + x^2 = y^2.$$

This can be rewritten as

$$(y - x)(y + x) = 2^n.$$

For this equation to hold with x and y non-negative integers, we must have

$$y - x = 2^a, \qquad y + x = 2^b$$

where a and b are non-negative integers, $a < b$, and $a + b = n$. This gives the solution

$$x = \frac{2^b - 2^a}{2}, \qquad y = \frac{2^b + 2^a}{2}.$$

Note here that if $a = 0$, then x and y cannot be integers unless $b = 0$, which gives the solution $n = 0$.

So from now on we can assume that $a \geq 1$. It is useful to consider separately the case $a = 1$, because it is simpler than the general case. If $a = 1$, then $b = n - 1$, and

$$x = 2^{n-2} - 1, \qquad y = 2^{n-2} + 1.$$

Note that this is the only situation in which x and y are odd. In all other cases, x and y are even.

Now we go back to our special equation

$$2^n + n^2 = y^2.$$

By the discussion above, if n is an odd solution of this equation, we must have

$$n = 2^{n-2} - 1.$$

We will show that this equation has no odd solutions. We already saw that we cannot have $n = 1$ or $n = 3$. To show that we cannot have n odd and greater than or equal to 5, it is enough to show that $2^{n-2} - 1 > n$ for every $n \geq 5$. We now proceed to show this.

If $n = 5$, then $2^{n-2} - 1 = 7 > 5$, so $2^{n-2} - 1$ is "too big." And the situation gets worse as n increases, because 2^{n-2} increases very rapidly. Suppose that for a given number $k \geq 5$, we have $2^{k-2} - 1 > k$. We will show that the same situation holds for the next number $k + 1$. So we want to show that $2^{k-1} - 1 > k + 1$. This is easy. Note first that $2^{k-1} - 1 > 2(2^{k-2} - 1)$. But $2^{k-2} - 1 > k$, and therefore $2^{k-1} - 1 > 2k$. And clearly $2k > k + 1$ if $k > 1$. We thus conclude that $2^{k-1} - 1 > k + 1$.

This completes the argument for n odd. Now we deal with even n. The general shape of the argument is no harder than the n odd case. We could adapt the previous argument, but it is simpler to proceed in a somewhat different way. Let n be even, say $n = 2m$. We want to solve the equation

$$2^{2m} + 4m^2 = y^2.$$

Let $m > 0$. Note that 2^{2m} is the square of 2^m. The next even perfect square after $(2^m)^2$ is $(2^m + 2)^2$, that is, $2^{2m} + 2^{m+2} + 4$. To show that $2^{2m} + 4m^2$ is not a perfect square, it is enough to show that

$$2^{2m} + 4m^2 < 2^{2m} + 2^{m+2} + 4,$$

or equivalently that $2^{m+2} > 4m^2 - 4$. This inequality does *not* hold for "small" m. But we will show that it does hold for m that are large enough, specifically for $m \geq 4$. If $m = 4$ then $2^{m+2} = 64$, and $4m^2 - 4 = 60$, so the desired inequality does holds at $m = 4$. Now we show that if $k \geq 4$ and the inequality holds at $m = k$, then it holds at $m = k + 1$.

So we want to show that $2^{k+3} > 4(k + 1)^2 - 4$. Note that because the inequality holds at $m = k$, we know that $2^{k+3} = 2(2^{k+2}) > 2(4k^2 - 4)$. So all we need to do is to show that

$$2(4k^2 - 4) > 4(k + 1)^2 - 4.$$

A little algebra shows that the above inequality is equivalent to

$$k^2 - 2k - 2 > 0.$$

Look at the equation $k^2 - 2k - 2 = 0$. This has the roots $k = 1 \pm \sqrt{3}$. In particular, $k^2 - 2k - 2$ is positive from $k = 1 + \sqrt{3}$ on. So we have shown that if $m \geq 4$ then the equation $2^{2m} + 4m^2 = y^2$ cannot have integer solutions. We had already seen that there is a solution for $m = 0$, none for $m = 1$ and $m = 2$, and a solution for $m = 3$. This completes the argument.