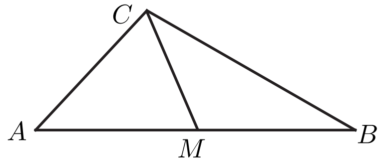
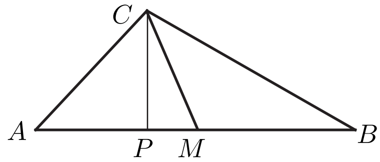
Solutions to December 2008 Problems

Problem 1. In the figure below, $AB = 4$, $BC = 3$, $AC = 2$, and M is the midpoint of the line segment AB. What is the length of the median CM?



Solution. Drop a perpendicular from C to AB, meeting AB at P. Let the height CP be h, let $AP = u$ and $PB = v$.



Note that $u + v = 4$. By the Pythagorean Theorem,

$$u^2 + h^2 = 4 \qquad \text{and} \qquad v^2 + h^2 = 9.$$

From these two equations, we obtain $v^2 - u^2 = 5$. But $(v^2 - u^2)/(v + u) = v - u$, and therefore $v - u = 5/4$. Since $v + u = 4$, we conclude that $v = 21/8$ (and $u = 11/8$). From this it follows that $h^2 = 1335/64$. Since $PM = 21/8 - 2$, we can calculate CM by using the Pythagorean Theorem. We obtain $CM = \sqrt{10}/2$.

Another Way. We use the *Cosine Law,* a frequently useful generalization of the Pythagorean Theorem. We state the Cosine Law in general. In $\triangle ABC$, let $c = AB$, $b = CA$, and $a = BC$. Let $\angle ACB = \theta$. Then

$$c^2 = a^2 + b^2 - 2ab \cos \theta.$$

Let $\angle AMC = \phi$, let $\angle BMC = \psi$, and let $x = CM$. Using the Cosine Law on $\triangle AMC$, and noting that $AM = 2$, we obtain

$$4 = x^2 + 4 - 4x \cos \phi. \qquad (1)$$

Now look at $\triangle BMC$. Note that $\psi = 180^\circ - \phi$, and therefore $\cos \psi = -\cos \phi$. Thus, using the Cosine Law on $\triangle BMC$, we obtain

$$9 = x^2 + 4 + 4x \cos \phi. \qquad (2)$$

From Equations 1 and 2, it follows that

$$13 = 2x^2 + 8,$$

and therefore $x = \sqrt{5/2}$.

Comment. Either argument can be used to find the lengths of the medians of any triangle, given the sides. Without much effort, both methods can be adapted to find CM, where M divides side AB in *any* specified ratio.

Problem 2. Prove, using only "high school" ideas, that for any positive integer n,

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots + \frac{1}{n^2} < \frac{5}{3}.$$

(It turns out that the infinite sum $1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$ is equal to $\pi^2/6$. This was proved (with some gaps) by Euler in the 18th century. The desired inequality then follows from standard estimates of π. But that approach takes us well beyond high school ideas.)

Solution. One can imagine finding an explicit "algebraic" expression for the sum of the problem, and then manipulating it to prove the inequality. That kind of approach is often (and in this case), overly ambitious. But we can get good estimates without finding an explicit expression.

For concreteness, we work with specific numbers instead of letters. So for example we will get a handle on the size of

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots + \frac{1}{177^2}.$$

The idea is to add up explicitly the first few terms of the sequence, and to show that the rest of the terms (the "tail") don't add up to much. As an example, we will find an estimate for the tail

$$\frac{1}{13^2} + \frac{1}{14^2} + \frac{1}{15^2} + \cdots + \frac{1}{177^2}. \qquad (3)$$

Let k be an integer in the interval from 13 to 177, inclusive. Note that

$$\frac{1}{k^2} < \frac{1}{(k-1)(k)}$$

So in particular,

$$\frac{1}{13^2} < \frac{1}{(12)(13)}, \qquad \frac{1}{14^2} < \frac{1}{(13)(14)}, \qquad \frac{1}{15^2} < \frac{1}{(14)(15)},$$

and so on. It is easy to verify that in general

$$\frac{1}{(k-1)(k)} = \frac{1}{k-1} - \frac{1}{k}.$$

Thus Sum 3 above is less than

$$\left(\frac{1}{12}-\frac{1}{13}\right)+\left(\frac{1}{13}-\frac{1}{14}\right)+\cdots+\left(\frac{1}{176}-\frac{1}{177}\right).$$

Add up, noting the wholesale cancellations. We find that Sum 3 is less than 1/12. Precisely the same argument shows that if $k > 1$, then

$$\frac{1}{k^2}+\frac{1}{(k+1)^2}+\cdots+\frac{1}{n^2}<\frac{1}{k-1}. \tag{4}$$

We return to the original problem. Note that

$$1+\frac{1}{2^2}+\cdots+\frac{1}{n^2}=\left(1+\frac{1}{2^2}+\frac{1}{(k-1)^2}\right)+\left(\frac{1}{k^2}+\cdots+\frac{1}{n^2}\right).$$

We now exploit the fact that the tail $1/k^2+\cdots+1/n^2$ is less than $1/(k-1)$. A bit of experimentation is useful here. Let $k = 11$. Using a calculator, we find that

$$1+\frac{1}{2^2}+\cdots+\frac{1}{10^2}<1.55.$$

But if $n \geq 11$, then the tail $1/11^2+\cdots+1/n^2$ is less than 0.1. It follows that if $n \geq 11$, then $1+1/2^2+\cdots+1/n^2 < 1.65$. The same result obviously holds if $n < 11$. Since $1.65 < 5/3$, our result follows.

Another Way. Let

$$S(n)=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots+\frac{1}{n^2}.$$

We want to show that $S(n) < 5/3$ for every positive integer n. It is easy to verify that $S(1) < 5/3$. We would like to show that if $S(k)$ is "small" (less than 5/3) for a particular integer k, then the next sum $S(k+1)$ is small. That would do the job, since from the fact that $S(1)$ is small, we could conclude that $S(2)$ is small, but from the fact that $S(2)$ is small we could conclude that $S(3)$ is small, and so on.

This simple induction idea has a fatal flaw: merely knowing that $S(k) < 5/3$ will not allow us to conclude that $S(k+1)$ is small, since after all, $S(k+1)$ is *bigger* than $S(k)$.

The idea can be salvaged, and the argument will ultimately be quite simple, but finding it is a delicate matter. We will show that if $S(k)$ is a little bit less than 5/3, then $S(k+1)$ is a (different) little bit less than 5/3. What little bit should we use? This requires some experimentation.

It turns out that a reasonably good choice is to attempt to prove that $S(n) < \frac{5}{3}-\frac{1}{n}$ for all n. Unfortunately, this inequality is not quite true! Note for example that it fails at $n = 1$, and at $n = 2$. But a fairly easy calculation shows that the inequality holds at $n = 5$.

Suppose that we know that for a particular k, $S(k) \le \frac{5}{3} - \frac{1}{k}$. We would like to show that the same sort of inequality holds for the "next" sum. That is, we would like to show that $S(k+1) < \frac{5}{3} - \frac{1}{k+1}$. We have

$$S(k+1) = S(k) + \frac{1}{(k+1)^2}.$$

Thus, from our assumed inequality for $S(k)$, we have

$$S(k+1) \le \frac{5}{3} - \frac{1}{k} + \frac{1}{(k+1)^2} = \frac{5}{3} - \left(\frac{1}{k} - \frac{1}{(k+1)^2}\right).$$

But

$$\frac{1}{k} - \frac{1}{(k+1)^2} = \frac{k^2+k+1}{k(k+1)^2} > \frac{k^2+k}{k(k+1)^2} = \frac{1}{k+1}$$

and so indeed, if $S(k) < \frac{5}{3} - \frac{1}{k}$, then $S(k+1) < \frac{5}{3} - \frac{1}{k+1}$. This completes the proof.

Problem 3. Find a simple expression for

$$\binom{100}{1} + 2\binom{100}{2} + 3\binom{100}{3} + \cdots + 99\binom{100}{99} + 100\binom{100}{100}.$$

(Here, $\binom{n}{r}$ denotes the number of ways of choosing r objects from n objects. On scientific calculators, $\binom{n}{r}$ is usually denoted by ${}_n\mathrm{C}_r$.)

Solution. We use more or less the same idea as the one used to sum an arithmetic sequence, sometimes called (wildly incorrectly) the "Gauss trick."

Let's forget temporarily about the last term $100\binom{100}{100}$, which is simply 100. Let

$$T = 1\binom{100}{1} + 2\binom{100}{2} + \cdots + 98\binom{100}{98} + 99\binom{100}{99}.$$

So our desired sum is just $T + 100$. Now write the sum for T "backwards."

$$T = 99\binom{100}{99} + 98\binom{100}{98} + \cdots + 2\binom{100}{2} + 1\binom{100}{1}.$$

Add corresponding terms in our two sums. We will need to use the maybe familiar fact that in general, $\binom{n}{r} = \binom{n}{n-r}$. This fact can be quickly derived by using the general formula $\binom{n}{x} = \frac{n!}{x!(n-x)!}$, and doing a little calculation.

But there is a much nicer way of deriving the fact that $\binom{n}{r} = \binom{n}{n-r}$. By definition, $\binom{n}{r}$ is the number of ways of *choosing* r people from a group of n people. Imagine for example that a teacher wants to choose r students from a group of n students, say to give a prize to. There are $\binom{n}{r}$ ways for her to do this. But she can do the choosing in another way, by pointing to the $n-r$ people who will *not* get the prize. There are $\binom{n}{n-r}$ ways of doing the second task, so $\binom{n}{r} = \binom{n}{n-r}$.

We go back to our problem. By adding, we obtain

$$2T = \left(1\binom{100}{1} + 99\binom{100}{99}\right) + \left(2\binom{100}{2} + 98\binom{100}{98}\right) + \cdots + \left(99\binom{100}{99} + 1\binom{100}{1}\right).$$

But $\binom{100}{99} = \binom{100}{1}$, $\binom{100}{98} = \binom{100}{2}$, and so on. Thus

$$2T = 100\binom{100}{1} + 100\binom{100}{2} + \cdots + 100\binom{100}{99}.$$

Already this sum looks a little simpler than our previous sum, it is just 100 times the sum $\binom{100}{1} + \binom{100}{2} + \cdots \binom{100}{99}$.

To find the latter sum, it is better to look at the sum S, where

$$S = \binom{100}{0} + \binom{100}{1} + \cdots + \binom{100}{99} + \binom{100}{100}.$$

It turns out that $S = 2^{100}$. There are many ways to prove this. The "best" way, probably, is to note that S is the number of ways of choosing 0 objects from 100, plus the number of ways of choosing 1 object from 100, plus the number of ways of choosing 2 objects from 100, and so on. So S is the number of *subsets* of a set of 100 objects. How many ways are there of choosing "some" (possibly 0) objects from a set of 100 objects? Line up the 100 objects in a row. Now under each object, put a Y (for yes) if the object is to be chosen, and a N (for no) if the object is not to be chosen. There are then just as many ways to choose some (possibly 0) objects as there are words of length 100 over the alphabet $\{\mathrm{Y}, \mathrm{N}\}$. There are 2^{100} such words.

Now put everything together. We have $2T = 200 + 100(2^{100} - 2)$, and therefore $T = (50)2^{100}$.

Comment. Instead, we could have noted that the original desired sum is equal to U, where

$$U = 0\binom{100}{0} + 1\binom{100}{1} + 2\binom{100}{2} + \cdots + 99\binom{100}{99} + 100\binom{100}{100}.$$

(The first term is of course 0, but putting it in makes things look nicer.) Now use the writing backwards trick. After a short while we find that

$$2U = (100)2^{100}$$

Exactly the same argument shows that in general

$$\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = (n)2^{n-1}.$$

Another Way. When $1 \le k \le n$, we have the identity

$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

The most straightforward way to prove this identity is by calculation. Note that

$$k\binom{n}{k} = k\frac{n!}{k!(n-k)!} = n\frac{(n-1)!}{(k-1)!(n-k)!} = n\binom{n-1}{k-1},$$

since $n - k = (n-1) - (k-1)$.

Thus our sum is equal to

$$100\left[\binom{99}{0} + \binom{99}{1} + \cdots + \binom{99}{99}\right].$$

By the argument used in the first solution, this expression is equal to $(100)2^{99}$.

Another Way. We have a group of 100 people, and we want to choose a committee, together with a designated chair of the committee. (Altogether, the committee could have 1 person on it, or 2, or 3, and so on up to 100.) We will count the number of such committees in *two different ways.*

A committee of k people can be chosen in $\binom{100}{k}$ different ways. Once we have chosen the committee, the chair can be chosen in k different ways. So the number of chaired committees with k people is equal to $k\binom{100}{k}$. It follows that the total number of chaired committees is equal to

$$1\binom{100}{1} + 2\binom{100}{2} + \cdots + 98\binom{100}{98} + 99\binom{100}{99}.$$

Or else we could choose the chair first, and then choose the rest of the committee. The chair can be chosen in 100 ways. For each such way, we can fill out the committee by choosing *any subset* of the remaining 99 people to join the chair on the committee. A set of 99 people has 2^{99} subsets. It follows that the total number of chaired committees is $(100)2^{99}$.

We have counted the number of chaired committees in two different (and correct!) ways. It follows that our two answers must be the same. Thus

$$1\binom{100}{1} + 2\binom{100}{2} + \cdots + 98\binom{100}{98} + 99\binom{100}{99} = (100)2^{99}.$$

Problem 4. (a) Is there a non-constant polynomial $P(x)$ such that whenever a is rational, $P(a)$ is irrational?

(b) Is there a non-constant polynomial $P(x)$ such that whenever a is irrational, $P(a)$ is rational?

Solution. (a) The answer is *yes.* The simplest example of such a polynomial is given by $P(x) = \sqrt{2} + x$. It is a standard fact that $\sqrt{2}$ is irrational, that is, $\sqrt{2}$ is not a ratio of two integers.

Now let a be rational. Suppose that $P(a)$ is rational. Let $P(a)$ be the rational number b. Then $\sqrt{2} + a = b$, and therefore $\sqrt{2} = b - a$. But if a and b are any rational numbers, then $b - a$ is also rational. So if $P(a)$ is rational for some rational number a, it follows that $\sqrt{2}$ is rational, which is not the case. Thus $P(a)$ must be irrational for every rational a.

(b) The answer is *no*. To prove this, we must in principle work quite a bit harder than in part (a). In part (a), we only needed to find *one* example of a polynomial. But for part (b), we must show that *no* non-constant polynomial $P(x)$ will "work."

Call a polynomial $P(x)$ *special* if $P(a)$ is rational for every irrational a. We want to show that every special polynomial is constant.

Let $P(x)$ be a special polynomial, and let $Q(x) = P(x+1) - P(x)$. We prove first that $Q(x)$ is special. We need to show that for any irrational a, $Q(a)$ is rational. This is easy. Note that if a is irrational, then $a + 1$ is irrational. So since $P(x)$ is special, $P(a+1)$ must be rational. But since $P(x)$ is special, $P(a)$ is also rational. It follows that the difference $P(a + 1) - P(a)$ is rational.

Suppose that $P(x)$ has degree d, where $d \geq 1$. Thus

$$P(x) = Kx^d + \text{terms of lower degree,}$$

where K is a non-zero constant. A quick calculation shows that

$$P(x + 1) - P(x) = dKx^{d-1} + \text{terms of lower degree.}$$

Thus if $P(x)$ has degree $d \geq 1$, then $Q(x)$ has degree $d - 1$.

Now imagine for example that there is a special polynomial $P(x)$ of degree 4. Then the polynomial $Q(x) = P(x + 1) - P(x)$ is special of degree 3. So if there is a special polynomial of degree 4, then there is a special polynomial of degree 3. But then by the same argument there is a special polynomial of degree 2. But then by the same argument there is a special polynomial of degree 1. This argument generalizes easily: If there is a special polynomial of degree $d \geq 2$, there must be a special polynomial of degree 1.

We finish by showing that there *cannot* be a special polynomial of degree 1. For let $P(x) = Kx+L$, where K and L are constant, and $K \neq 0$. We show that $P(x)$ cannot be special. Suppose to the contrary that $P(x)$ is special. Then $P(x+1)-P(x)$ is special. But $P(x+1)-P(x) = K$. Thus K is rational. Since $P(x)$ is special, $P(\sqrt{2})$ is rational. Thus $K\sqrt{2}+L$ is rational, say $K\sqrt{2}+L = M$. For the same reason, $P(2\sqrt{2})$ is rational, say $2K\sqrt{2} + L = N$. A little algebra now shows that $L = 2M - N$, so L is rational.

So if $Kx + L$ is special, where $K \neq 0$, then K and L are rational. That's impossible, since in that case $K\sqrt{2} + L$ is irrational.

http://www.pims.math.ca/education/math_problems/
http://www.math.ubc.ca/~adler/problems/