Solutions to December 2010 Problems

Problem 1. A square is cut up into 169 squares. At least 168 of these are 1×1 squares. What are the possible side lengths of the 169-th square?

Solution. Imagine that an $a \times a$ square has been cut up into 169 squares, of which at least 168 are 1×1, and the 169-th is say $b \times b$. It is probably obvious that a and b must be integers. By comparing areas, we can see that

$$a^2 - b^2 = 168.$$

Now we have a number-theoretic problem. What are the positive integer solution of $a^2 - b^2 = 168$? This equation can be rewritten as

$$(a + b)(a - b) = 168.$$

So $a + b$ and $a - b$ must be positive integer divisors of 168. In symbols, $a + b = m$ and $a - b = n$, where m and n are positive integers, and $mn = 168$.

Are there any further restrictions on m and n? Look at the system of equations

$$a + b = m, \qquad a - b = n.$$

This has the solutions $a = (m + n)/2$, $b = (m - n)/2$. In order for $m + n$ (and $m - n$) to be divisible by 2, m and n must be both odd or both even. And in order for b to be positive, we need $m > n$. Since $mn = 168$, we cannot have m and n both odd, so they must be both even. Thus we have obtained a recipe for finding all positive integers a, b such that $a^2 - b^2 = 168$: (i) Express 168 as a product mn of two even integers, with $m > n$; (ii) Let $a = (m + n)/2$, $b = (m - n)/2$.

In order not to lose any possibilities, we express 168 as a product of prime powers: $168 = 2^3 3^1 7^1$. We want m and n to be both even, say $m = 2m'$, $n = 2n'$. So we want $m'n' = 42 = 2^1 3^1 7^1$. To make sure we don't miss any possibilities, note that $2^1 3^1 7^1$ has $(1 + 1)(1 + 1)(1 + 1) = 8$ positive divisors. So there are 4 pairs (m', n') such that $m'n' = 42$ and $m' > n'$.

Maybe $m' = 42$, $n' = 1$. That gives $a = 43$, $b = 41$. Maybe $m' = 21$, $n' = 2$. That gives $a = 23$, $b = 19$. Maybe $m' = 14$, $n' = 3$. That gives $a = 17$, $b = 11$. Finally, maybe $m' = 7$, $n' = 6$. That gives $a = 13$, $b = 1$. So the possible values of a, the side of the 169-th square, are 41, 19, 11, and 1.

But we cannot really be sure that these 4 answers are indeed *geometrically* possible. We know that nothing else is possible, since we need positive integer solutions of $a^2 - b^2 = 168$, and we certainly have found all of those. But now we must construct for each of the 4 possibilities above a big $a \times a$ square, and a decomposition of that big square into an $a \times a$ square and 168 1×1 squares. The last case, $a = 13$, $b = 1$ is the easiest. Make a 13×13 square. It is obvious how to decompose this into 169 1×1 squares! The other 3 answers are also geometrically realizable. We deal for example with $a = 43$, $b = 41$. Make a white 43×43 square, and colour the 41×41 square at its upper right-hand corner red. The remaining white "ell" can easily be divided into 1681×1 squares.

Problem 2. Let $f(x) = (x^2+x+1)/(x^2-x+1)$. What are the possible values of $f(x)$, as x travels over the real numbers? A graphing program or calculator may lead to a plausible conjecture, but proof is needed that the conjecture is correct. No calculus, please!

Solution. A graphing program shows that the curve $y = f(x)$ seems to reach a maximum when $x = 1$, or nearly 1, and a minimum when $x = -1$, or nearly -1. If that is true, the maximum value taken on by $f(x)$ is 3, and the minimum is 1/3. And it looks as if every value between 1/3 and 3 is taken on by $f(x)$. So at the practical level, the graphing program settles things. But let's use algebra for confirmation.

Note that top and bottom are always positive, and indeed $\geq 3/4$, since $x^2-x+1 = (x-1/2)^2+3/4$, and $x^2+x+1 = (x+1/2)^2+3/4$. Note also that $f(-x) = (x^2+x+1)/(x^2-x+1) = 1/f(x)$. So if we know everything about $f(x)$ for $x \geq 0$, we know everything about $f(x)$ for $x \leq 0$. (We actually will not need this information, but identifying symmetries is a useful automatic reflex.)

We try to identify the numbers k such that $f(x) = k$ for some x. So we are trying to solve the equation

$$\frac{x^2+x+1}{x^2-x+1} = k, \quad \text{or equivalently} \quad (k-1)x^2 - (k+1)x + (k-1) = 0.$$

If $k = 1$, the (only) solution is given by $x = 0$. So now assume that $k \neq 1$. By the Quadratic Formula, the equation has solution(s)

$$x = \frac{(k+1) \pm \sqrt{-(3k^2-10k+3)}}{2(k-1)}.$$

The expression above is real precisely if $3k^2 - 10k + 3 \leq 0$. But

$$3k^2 - 10k + 3 = (3k-1)(k-3),$$

and $(3k-1)(k-3) \leq 0$ if and only if $1/3 \leq k \leq 3$. This shows that $f(x)$ takes on precisely all values in the interval $1/3 \leq y \leq 3$.

Comment. There are many other approaches. We look for example for the maximum value of $f(x)$ for positive x. It is easy to see that $f(x) = 1 + 2x/(x^2-x+1)$. So we want to maximize $2x/(x^2-x+1)$, where x is positive. Equivalently, we want to *minimize* $(x^2-x+1)/2x$. So we want to minimize $(1/2)(x-1+x)$.

The minimum value of $x+1/x$, with x positive, is 2, and is taken on at $x = 1$. This is a standard and useful fact. The easiest proof is to note that $x + 1/x = (\sqrt{x} - 1/\sqrt{x})^2 + 2$. But $(\sqrt{x} - 1/\sqrt{x})^2$ is non-negative, and is 0 precisely if $x = 1$.

Putting things together, we find that the minimum value of $(1/2)(x-1+x)$ is 1/2), so the maximum value of $f(x)$, for positive x, is $1 + 1/(1/2)$, that is, 3.

Another approach is simply to show directly that $f(x) \leq 3$. So we want to show that $(x^2+x+1)/(x^2-x+1) \leq 3$, or equivalently that $2x^2-4x+2 \geq 0$. But this is obvious, since we are looking at $2(x-1)^2$.

Problem 3. Imagine tossing a fair coin over and over again, until you get three consecutive heads. What is the mean of the total number of tosses required?

Solution. One approach might start as follows. The total number of tosses required could be 3, or 4, or 5, or 6, and so on. Let p_n be the probability that the first occurrence of three consecutive heads happens on the n-th toss. Then the average (mean) number of tosses required is the infinite sum

$$3p_3 + 4p_4 + 5p_5 + \cdots + np_n + \cdots .$$

There are two difficulties with this approach. Most importantly, obtaining an explicit expression for p_n is not easy. And even if we do obtain such an expression, calculating the infinite sum is not a trivial task. But the following idea works nicely for this problem, and many similar problems.

Let the required average number of tosses be A. Suppose that the first toss is a tail (T, probability 1/2). If that happens, then we have "wasted" a toss, the wait for the first 3 consecutive heads starts all over again, and the average total number of tosses required is $1 + A$.

Suppose that the first toss is H, and the second toss is T (probability 1/4). Then we have wasted 2 tosses, and the average total number of tosses until the first 3 consecutive heads is $2 + A$.

Suppose that the first two tosses are H, and the next toss is T (probability 1/8). Crushing disappointment, after our hopes have been built up. We have wasted 3 tosses, and the average wait until the first 3 consecutive heads is $3 + A$.

Finally, suppose that the first three tosses are H, H, and H (probability 1/8). Then the game is over early, and in this case that average (and actual) wait until the first 3 consecutive heads is 3.

These are the only possibilities. It follows that

$$A = \left(\frac{1}{2}\right)(1 + A) + \left(\frac{1}{4}\right)(2 + A) + \left(\frac{1}{8}\right)(3 + A) + \left(\frac{1}{8}\right)(3).$$

The equation above is a linear equation in A, and is easy to solve. We get $A = 14$.

Comment. We look at a simpler problem, the average waiting time until the first 2 consecutive heads. An argument much like the one above, but a little simpler, shows that this average waiting time is 6. And of course the average waiting time until the first 2 consecutive T is also 6. What is the average waiting time until the first occurrence of the pattern TH? A similar calculation shows that this average waiting time is 4. This surprises many people. Since H and T are, on any one toss, equally likely, as are HH, HT, TH, and HT on two tosses, it seems reasonable to think that the average waiting time until the first TH should be the same as the average waiting time until the first HH. But it isn't, it is substantially shorter, another instance of the fact that intuitions about probabilities are not necessarily accurate! Suppose that Alphonse and Beti play the following game. They toss a coin repeatedly. If the pattern HH occurs before the pattern TH, Alphonse wins. Otherwise, Beti wins. It seems at first sight reasonable to think that Alphonse and Beti have equal probabilities of winning. But on a little thought, this is obviously false. Unless the first two tosses are HH (probability 1/4), it is *certain* that TH will occur before HH, so the probability that Alphonse wins is 1/4, and the probability that Beti wins is 3/4.

Problem 4. Let S be any infinite set of positive integers such that no element of S has a prime divisor greater than 7. Show that S has an infinite subset C such that if a and b are in C, then a divides b or b divides a.

Solution. Suppose first that for every number m in S, there is a number n in S such that $n \neq m$ and m divides n. Then it is easy to find a set C with the desired property. Let c_0 be any element of S, say the smallest. Let c_1 be the smallest element of S such that $c_1 \neq c_0$ and c_0 divides c_1. Similarly, let c_2 be the smallest element of S such that $c_2 \neq c_1$ and c_1 divides c_2, and so on. Then the set $C = \{c_0, c_1, c_2, \dots\}$ has the required property.

The other possibility is that there is a number m in S such that no element of S other than m is divisible by m. So if $m = 2^a 3^b 5^c 7^d$, then for any element $n \neq m$ of S, the highest power of 2 that divides n is less than 2^a, or the highest power of 3 that divides n is less than 3^b, or the highest power of 5 that divides n is less than 5^c, or the highest power of 7 that divides n is less than 7^d.

Now let S_2 be the set of elements n of S such that the highest power of 2 that divides n is less or equal to 2^a. Similarly, let S_3 be the set of elements n of S such that the highest power of 3 that divides n is less than or equal to 3^b. Define S_5 and S_7 analogously.

Note that S is the union of the sets S_2, S_3, S_5, and S_7 (the sets will overlap, certainly at m, and possibly a lot more).

The union S of our four sets is infinite. Thus at least one of the four sets is infinite. Without loss of generality we may assume that S_7 is infinite. Note that for every number n in S_7, the highest power of 7 that divides n is less than or equal to 7^d.

Now let $S_{7,0}$ be the numbers n in S_7 such that the highest power of 7 that divides n is 7^0. Similarly, let $S_{7,1}$ be the numbers n in S_7 such that the highest power of 7 that divides n is 7^1. In general, for $i \le d$, let $S_{7,i}$ be the numbers n in S_7 such that the highest power of 7 that divides n is 7^i.

The union of the $S_{7,i}$, where $0 \le i \le d$, is S_7. So since S_7 is infinite, at least one of the $S_{7,i}$ is infinite, say $S_{7,k}$. This means that every element of $S_{7,k}$ is of the shape $7^k t$ where the prime divisors of t are taken from $\{2, 3, 5\}$.

Let T be the set of all t that arise in expressing numbers in $S_{7,k}$ as $7^k t$. Then T is an infinite set such that any element of T has no prime divisors other than possibly 2 and/or 3 and/or 5.

Now we can repeat the argument. If for every number m in T, there is a number n in T such that $n \ne m$ and m divides n, we can easily find a set C with the desired property. Otherwise, define T_2, T_3, and T_5 in analogy to how S_2, S_3, S_5 were defined. At least one of the T_j, say T_5, is infinite. After a while we reduce the problem to showing that if we have an infinite set U all of whose elements are divisible by only the primes 2 and/or 3, then U has an infinite subset such that for any two elements of U, one of them is divisible by the other.

Repeating the argument one more time, we see that all we need to do is to show that if V is an infinite set of powers of (say) 2, then there is an infinite subset of V such that for any two elements of the subset, one of them is divisible by the other. But of course V itself has that property.

Comment. Essentially the same argument works if P is *any* finite set of primes, and S is an infinite set such that for any n in S, no prime other than a prime in P divides n. The proof is by induction on q, the number of primes in P. The result is obvious if $q = 1$. Suppose that the result is true for $q = e$. We show that the result is true for $q = e + 1$. If for every m in S, there is $n \ne m$ in S such that m divides n, the result is obvious. Otherwise, proceed exactly as we did in producing T from S.