Solutions to December 2011 Problems

Problem 1. How many factors of 111^{2011} have last digit 1?

Solution. Note that $111 = 3 \cdot 37$. The factors of 111^{2011} are the numbers of the shape $3^m 37^n$ where $0 \le m \le 2011$ and $0 \le n \le 2011$.

Starting from 3^0, the powers of 3 have last digit 1, 3, 9, 7, 1, 3, ... It is clear that once we get to last digit 1, the game begins again, so the last digits cycle with cycle length 4. Similarly, the powers of 7 have last digit 1, 7, 9, 3, 1, 7, ...

In going from $k = 0$ to $k = 2011$, there are 503 full cycles, with no leftovers. That is convenient, and means that a problem of this type is structurally simplest when the power of 111 is 1 less than a multiple of 4.

We get last digit 1 in several ways: (i) Each of 3^m and 37^n provides a 1: (ii) 3^m provides a 3 and 37^n provides a 7; (iii) 3^m provides a 9 and so does 37^n; (iv) 3^m provides a 7 and 3^n provides a 7. For each of the possibilities, there are 503 choices for m, and for each such choice, there are 503 choices for n, for a total of 503^2. Thus the full number of possibilities is $4 \cdot 503^2$. This happens to be 1012036. Note that the numerical answer is less informative than the more structural $4 \cdot 503^2$.

Problem 2. Find (with proof) all values taken on by $\dfrac{x}{x^2 - 4x + 3}$ as x ranges over the real numbers.

Solution. We could get a reasonably good idea of the range of values by having a graphing utility plot $y = \dfrac{x}{x^2 - 4x + 3}$. Or else more directly we ask for what y there is an x such that $y = \dfrac{x}{x^2 - 4x + 3}$, or, more or less equivalently$yx^2 - (4y + 1)x + 3y = 0$. Imagine solving for x. If $y = 0$, we get $x = 0$. For $y \ne 0$, we can use the Quadratic Formula. There is an x satisfying the equation precisely if the discriminant $(4y + 1)^2 - 12y \ge 0$. This is the case when $y \ge (-2 + \sqrt{3})/2$ and when $y \le (-2 - \sqrt{3})/2$.

Another Way. Flip our equation over. We get

$$\frac{1}{y} = x + \frac{3}{x} - 4.$$

Examine $x + \dfrac{3}{x}$ for positive x, say for $x = u^2$. Since $(u - \sqrt{3}/u)^2 \ge 0$, we get that $u^2 + 3/u^2 \ge 2\sqrt{3}$, and it is easy to check that every value $\ge 2\sqrt{3}$ is achievable. Similarly, by choosing x negative we can have $x + 3/x$ take on any value $\le -2\sqrt{3}$.

It follows that $1/y$ can take on all values $\ge 2\sqrt{3} - 4$ and all values $\le -2\sqrt{3} - 4$. From this by manipulation we can find the bounds on y of the first solution.

Problem 3. Let $P(n)$ be the product

$$\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{2^3}\right)\cdots\left(1 - \frac{1}{2^n}\right).$$

Find, with proof, a number $c > 0$ such that $P(n) > c$ for all n. Please note that the c need not be the largest c that will work, nor anywhere near it. Wolfram Alpha says that $P(n)$ is always bigger than 0.288788.

Solution. Let $f(n)$ be the product up to the term $1 - \dfrac{1}{2^n}$. We show by induction that we can take $c = 1/4$. More precisely,

$$f(n) \geq \frac{1}{4} + \frac{1}{2^{n+1}}.$$

The result is true at $n = 1$. For the induction step, note that

$$f(m+1) = f(m)\left(1 - \frac{1}{2^{m+1}}\right) \geq \left(\frac{1}{4} + \frac{1}{2^{m+1}}\right)\left(1 - \frac{1}{2^{m+1}}\right).$$

Expand the product on the right. We get

$$\frac{1}{4} + \frac{1}{2^{m+1}} - \frac{1}{4} \cdot \frac{1}{2^{m+1}} - \frac{1}{2^{2m+2}}. \qquad (*)$$

Rewrite $\dfrac{1}{4} \cdot \dfrac{1}{2^{m+1}}$ as $\dfrac{1}{2^{m+2}} - \dfrac{1}{2^{m+3}}$. Then $(*)$ becomes

$$\frac{1}{4} + \frac{1}{2^{m+2}} + \frac{1}{2^{m+3}} - \frac{1}{2^{2m+2}}.$$

The term $\dfrac{1}{2^{m+3}} - \dfrac{1}{2^{2m+2}}$ is non-negative. Thus $f(m+1) \geq \dfrac{1}{4} + \dfrac{1}{2^{m+2}}$, which completes the induction step.

Comment. The solution uses fancier machinery than necessary. I do not have the time right now to devise a simpler proof.

Problem 4. Find all triples (x, y, z) of non-negative integers such that $x \leq y \leq z$ and $xyz = 2(x + y + z)$. Proof is needed that the list is complete.

Solution. There is the obvious solution $x = y = z = 0$. All other non-negative solutions must have x, y, and z positive. We do some experimentation, gathering evidence.

Could we have $x = 1$? Then $yz = 2(1 + y + z)$. But $yz - 2 - 2y - 2z = (y - 2)(z - 2) - 6$. So we need $(y - 2)(z - 2) = 6$. Since $y \leq z$, this gives the possibilities $y - 2 = 1$, $z - 2 = 6$ (meaning $y = 3$, $z = 8$) and $y - 2 = 2$, $z - 3 = 3$, meaning that $y = 4$ and $z = 5$.

Could we have $x = 2$? Then $2yz = 2(2 + y + z)$. Divide by 2. We get $yz = 2 + y + z$, which can be rewritten as $(y - 1)(z - 1) = 3$. That gives $y = 2$ and $z = 4$.

One could continue, seeing what happens with $x = 3$. But suppose that $x \geq 3$. Then $y \geq 3$, so $xyz \geq 9z$. But $2(x + y + z) \leq 6z$, so we cannot have equality.