Solutions to February 2006 Problems

Problem 1. What are the first (leftmost) three digits in the decimal expansion of 2^{2006}? (Only a basic scientific calculator should be used.)

Solution. If I use the "x^y" key, my calculator displays an – E – and insists on being reset. The culprit is *overflow*: the calculator can not handle such a large number, even in scientific notation.

The calculator program bundled with Microsoft Windows, however, works just fine, and says that the leftmost three digits are 7, 3, and 4. This program also has limitations: it can not handle 2^{202006} directly. The Microsoft Windows calculator is not a "simple scientific calculator," but some cheating is OK, as long as we then solve the problem with a less powerful tool.

We can find the answer with an ordinary calculator if we avoid overflow. There are many ways to do this. Mr. Hank Duan of Pinetree noted that $2006 = 34 \times 59$, so $2^{2006} = (2^{34})^{59}$. The calculator gives $2^{34} \approx 1.7179869 \times 10^{10}$, so the first few digits of 2^{2006} are the same as the first few digits of $(1.7179869)^{59}$. These are easy to find with an ordinary calculator.

Another way to avoid overflow is to recall the useful fact that 2^{10} is nearly equal to 1000. More precisely, $2^{10} = 1.024 \times 10^3$. It follows that

$$2^{2000} = (2^{10})^{200} = 1.024^{200} \times 10^{600}.$$

A scientific calculator has no difficulty with 1.024^{200}: this is roughly 114.81307. Thus

$$2^{2006} = 2^6 \times 2^{2000} \approx 7348.03645 \times 10^{600}.$$

(The number *rounded* to three significant figures is 7.35×10^{603}, but that's not what the problem asked for.)

There are other powers of 2 that are nearly equal to a power of 10, but they are all much bigger than 2^{10}. For instance, $2^{196} \approx 1.004 \times 10^{59}$. This example was not found by experimentation: we used a beautiful subject called *continued fractions.*

We can avoid overflow in many other ways. For example, there is no overflow problem in finding 2^{100}, at least with my calculator: I get 1.2676506×10^{30}. If I now try to raise this to the 20-th power to get to 2^{2000}, there is overflow. But we can help the calculator by noting that we want $(1.2676506)^{20} \times 10^{600}$.

The calculator says that the answer is $114.8130691 \times 10^{600}$. The last few digits should not be taken seriously, since it was taking the 20-th power of an eight significant figure approximation, and gave a ten significant figure answer. Multiply by 2^6 to get the final answer.

Comment. We freely used the "x^y" key of the scientific calculator. Maybe all we have is a grocery bill calculator. These can not handle scientific notation, and do not have an "x^y" key. But they can always multiply, and usually have an "x^2" key.

A more serious concern, when we want to do high precision integer arithmetic by computer, is that we do not want to use "real number" exponentiation to do integer exponentiation. High precision real number exponentiation routines are very slow, and not always reliable.

It is not difficult to use a grocery bill calculator to find the answer. We can quickly get to high powers of any number by repeated squaring. Overflow is managed by dividing every so often by an appropriate power of 10, and keeping track of these powers.

Another Way. The next approach requires a lot more machinery. It was used by Owen Ren of Magee Secondary and Kevin Oh of Mennonite Educational Institute.

Definition. Let x be positive, and let b be a positive number other than 1. Then $\log_b x$ is the number t such that $b^t = x$. The number $\log_b x$ is called the *logarithm* of x to the base b.

Let $b = 10$. Traditionally, and on scientific calculators, $\log_{10} x$ is usually called $\log x$.

By the above definition, $\log 2$ is the number t such that $10^t = 2$. More simply, $10^{\log 2} = 2$. It follows that

$$2^{2006} = (10^{\log 2})^{2006} = 10^{2006 \log 2}.$$

The calculator says that $\log 2$ is approximately equal to 0.301029995, and that therefore $2006 \log 2 \approx 603.8661719$.

If we now try to use the calculator to find $10^{2006 \log 2}$, there is a little problem, since the answer is much too large. But there is an easy work-around. We have

$$10^{2006 \log 2} = 10^{2006 \log 2 - 603} \times 10^{603} \approx 7.348036434 \times 10^{603}.$$

It follows that the required leftmost three digits are 7, 3, and 4.

We need to have some faith in the calculator. There could be round-off errors, and subtle inaccuracies in the calculator's program. For example, the Microsoft Windows scientific calculator gives, to 10 significant figures, $7.348036450 \times 10^{603}$, so does not agree with the simple calculator in the eighth decimal place.

Comment. For more than 350 years, logarithms to the base 10 were a basic tool for scientific computation, and came up quite early in the secondary school curriculum. Logarithms—nowadays more commonly to the base e, the "ln" button on the calculator—are still an important theoretical tool. But, with the advent of electronic calculators, they became obsolete for computation.

We sketch briefly the old main use of logarithms. Adding or subtracting even complicated numbers "by hand" is quick. Multiplication by hand is less pleasant, division harder still, and exponentiation quite hideous. We show how multiplication can be simplified by using logarithms.

Suppose we want to find the product xy, where x and y are positive. By multiplying or dividing x and y by appropriate powers of 10, we can reduce the problem to finding xy with $1 \leq x < 10$ and $1 \leq y < 10$. Note that $x = 10^{\log x}$, and $y = 10^{\log y}$. So

$$xy = 10^{\log x} 10^{\log y} = 10^{\log x + \log y},$$

and therefore

$$\log(xy) = \log x + \log y.$$

Look up $\log x$ and $\log y$ in a table of logarithms, and add (by hand). We now have $\log(xy)$. Then use the table of logarithms "backwards" to find xy. Up to the 1970s, all students in the sciences owned a book of tables that included a table of logarithms and other important tables, such as tables of sines and cosines. We also carried around a slide rule, which was an analog device for (approximate) multiplication, and was based on logarithms. Engineering students carried their slide rules in hip hholster. Mathematics students did not think this was cool.

There is controversy as to who discovered logarithms. Probably it was Bürgi, but Napier got to a similar idea independently, and was first to publish (1614). Not much later, Briggs and others made logarithms more computationally useful by publishing tables of what are now called logarithms to the base 10.

Logarithms were almost immediately adopted by astronomers. At the time, these were the only scientists who did complex calculations. A little-known fact is that up to the seventeenth century most astronomers used a version of the Babylonian base 60 arithmetic! The availability of tables of logarithms to the base 10 was one reason for the triumph of base 10 notation in scientific calculations. Base 10 notation, at least for integers, had been standard for a couple of centuries in commercial work.

Problem 2. On March 21, Alfonso and Beti did a science experiment on the coast of Ecuador. Alfonso was standing 50 meters above sea level, and Beti was on a cliff 50 meters above Alfonso.

The sun was setting due west over the Pacific. At the moment Alfonso saw the sun dip below the horizon, he signalled Beti, who found with a stopwatch that the sun dipped below *her* horizon 22.5 seconds after it dipped below Alfonso's. Find the radius of the Earth. Hint: December 2005.

Solution. Assume that Earth is a perfect sphere of radius R. By using the Pythagorean Theorem, it is not hard to show that for someone at height h above sea level the distance to the horizon is $\sqrt{2Rh + h^2}$.

The details of the argument are given in the solution of Problem 4, December 2005. There it is also pointed out that when h is "small," the (line of sight) distance to the horizon is approximately $\sqrt{2Rh}$, and that line of sight distance and distance along the surface of the Earth are the same for all practical purposes.

Let R be measured in kilometers. For convenience, let a be the height of Alfonso above sea level, b the height of Beti, and t the elapsed time shown by the stopwatch.

The distance to Alfonso's horizon is about $\sqrt{2Ra}$, while the distance to Beti's is $\sqrt{2Rb}$. The Earth goes through one revolution every 24 hours, that is, every

D seconds, where $D = (24)(3600)$. Thus points at the Equator travel at $2\pi R/D$ kilometers per second. It took Earth t seconds to travel $\sqrt{2bR} - \sqrt{2aR}$, so

$$\frac{(\sqrt{2Rb} - \sqrt{2Ra})D}{2\pi R} \approx t$$

and therefore

$$R \approx \frac{(\sqrt{2b} - \sqrt{2a})^2 D^2}{4\pi^2 t^2}.$$

Calculate. To the nearest kilometer, the result is 6408.

Comment. The answer is suspiciously close to the true (equatorial) radius of 6378.39 km. Alfonso and Beti didn't feel like doing all that climbing, so they obtained the 22.5 seconds by working backwards from the known value of R. Improving or inventing experimental data so they will fit theory is an ever-present temptation. Even great scientists have not been immune.

The horizon method has built-in inaccuracies. Sunset is hard to time exactly—Alfonso and Beti would need to practice to make sure that their definitions of sunset match. A timing error of half a second translates in this case to about a 5% change in the estimate for the radius of the Earth. The reliability of the method is much improved if Beti climbs higher.

The *true* time of sunset is hard to determine, mainly because of diffraction by Earth's atmosphere. However, the diffraction error affects both observers almost equally, so they still can get a good estimate of the *difference* between sunset times. It is remarkable that a ballpark estimate of R can be obtained in this low-tech way.

We used the estimate $\sqrt{2Rh}$ instead of $\sqrt{2Rh + h^2}$ for the distance to the horizon of an observer at height h above the surface of the Earth. We did this because we can then solve simply for R in terms of t. If we use $\sqrt{2Rh + h^2}$, we end up with an equation for R that is very unpleasant to solve exactly. However, we can find excellent numerical approximations by using techniques from *numerical analysis,* or more quickly by using the "Solve" button that can be found on many fancy calculators. If we do this, we get a value for R that is, to four significant figures, the same as the 6408 estimate found earlier.

In principle, the calculations with $\sqrt{2Rh}$ and $\sqrt{2Rh + h^2}$ are both "wrong," since we should be calculating the distance to the horizon as measured along the surface of the Earth. This distance is actually $R\cos^{-1}(R/(R + h))$. Here $\cos^{-1}$ is the inverse cosine function (radian mode) on the calculator. The right equation for R turns out to be

$$\frac{\cos^{-1}(R/(R+b)) - \cos^{-1}(R/(R+a))}{2\pi} = \frac{t}{D}.$$

Put in our values of a, b, t, and D, and use a numerical method (or the "Solve" button) to solve for R. Again we get, to four significant figures, 6408.

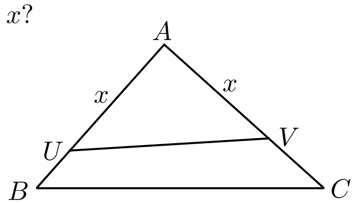
More informally, let us compare the distance to the horizon as given by the correct line of sight formula $\sqrt{2Rh + h^2}$, the approximate line of sight formula

$\sqrt{2Rh}$, and the correct surface distance formula $R\cos^{-1}(R/(R+h))$. Take say $R = 6400$ and $h = 0.1$. To six significant figures, the results are

$$\sqrt{2Rh+h^2} \approx 35.7772, \quad \sqrt{2Rh} \approx 35.7771, \quad R\cos^{-1}(R/(R+h)) \approx 35.7769.$$

Note that these are all equal to 5 significant figures.

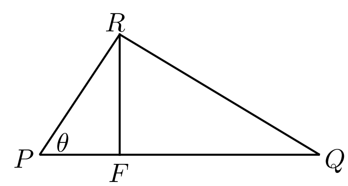
Problem 3. In $\triangle ABC$, $AB = 8$, $AC = 9$, and $BC = 12$. Points U and V are chosen on segments AB and AC with $AU = AV = x$. Suppose that UV cuts $\triangle ABC$ into two parts of equal area. What is x?



Solution. We can get a quick solution by appealing to the following useful fact, which we more or less prove.

Lemma. Let PQR be a triangle, and suppose that $\angle RPQ = \theta$. Then the area of $\triangle PQR$ is $(1/2)|PQ||PR|\sin\theta$. (Here $|XY|$ means the length of the segment XY. Most of the time, we use the notation XY for a segment, for its length, and for the entire line through X and Y. The ambiguity is usually harmless.)

Proof. In the picture below, F is the foot of the perpendicular from R to the line PQ.



By the definition of the sine function, we have $|RF|/|PR| = \sin\theta$, so $|RF| = |PR|\sin\theta$. Now use the "one-half of base times height" formula to conclude that $\triangle PQR$ has area $(1/2)|PQ||PR|\sin\theta$.

The proof is incomplete because θ was drawn as acute. For obtuse θ the picture and argument need to be modified somewhat. □

We return to our problem. Let $\angle CAB = \theta$. Then the area of $\triangle ABC$ is $(1/2)((9)(8)\sin\theta$, while the area of $\triangle VAU$ is $(1/2)x^2\sin\theta$. Thus

$$(1/2)x^2\sin\theta = (1/2)(1/2)(9)(8)\sin\theta.$$

Since $\sin\theta \neq 0$, we conclude that $x^2 = 36$, and therefore $x = 6$.

Another Way. Draw line segment BV. View $\triangle ABC$ as having base AC, and $\triangle ABV$ as having base AV. Then the two triangles have the same height. It follows that the area of $\triangle ABV$ is $x/9$ times the area of $\triangle ABC$.

Now view $\triangle ABV$ as having base AB, and $\triangle AUV$ as having base AU. Then the two triangles have the same height. It follows that the area of $\triangle AUV$ is $x/8$ times the area of $\triangle ABV$.

By going from $\triangle ABC$ to $\triangle ABV$ to $\triangle AUV$, we find that the area of $\triangle AUV$ is $(x/8)(x/9)$ times the area of $\triangle ABC$. Thus $(x/8)(x/9) = 1/2$, and therefore $x = 6$.

Comment. Another efficient similarity argument was found by Grace Xu of Magee. And an argument that uses Heron's formula was found by Owen Ren of Magee. These have not (yet?) been added to the solutions.

Problem 4. Let $\mathcal{P}$ be a convex polygon with n sides. How many acute angles can $\mathcal{P}$ have?

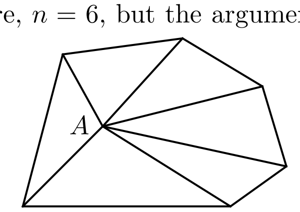
Solution. The question is somewhat ambiguous. We interpret it as follows. For any n, determine all possible values of k such that there is a convex n-gon with exactly k acute angles. The answer depends on n, but not much.

Note that $n \geq 3$. First let $n = 3$. It is easy to make a triangle that has 3 acute angles, and also easy to make one with 2. Since the angles of a triangle (have measures that) add up to $180°$, a triangle can not have 2 or 3 non-acute angles, so for $n = 3$ the possible numbers of acute angles are 2 and 3.

For $n > 3$, we use the following result.

Lemma (Sum of Angles)**.** Let $\mathcal{P}$ be a convex n-gon. Then the (interior) angles of $\mathcal{P}$ (have measures that) add up to $180n - 360$ degrees.

Proof. Let A be a point in the interior of $\mathcal{P}$. Join A in all possible ways to the vertices of $\mathcal{P}$ as in the picture below. In the picture, $n = 6$, but the argument is general.



The lines from A divide the n-gon into n triangles. The sum of the angles of these triangles is $180n$ degrees. But the sum of these angles is the sum of the interior angles of the n-gon, together with the sum of the angles "at" the point A. This last sum is $360°$. It follows that the sum of the angles of the n-gon is $180n - 360$ degrees. □

Comment. Suppose there is a point A in the interior of the n-gon $\mathcal{P}$ from which every point B on the boundary of $\mathcal{P}$ is *visible.* This means that every point in the line segment AB, except for B itself, is in the interior of the polygon.

More informally, think of the polygon as the walls of a possibly weirdly-shaped art gallery such that you can position a single guard so that she can see every bit of every wall. In any such n-gon, the sum of the interior angles of the n-gon is $180n - 360$ degrees. The proof is the same as in the convex case.

If $\mathcal{P}$ is convex, then all points on the boundary are visible from *any* point in the interior, you can post a guard anywhere. But it is not hard to draw a non-convex n-gon for which there *is* an interior point from which all boundary points are visible.

We now use the Sum of Angles Lemma to show that a convex n-gon can not have more than 3 acute angles. For suppose that there are k acute angles, and therefore $n-k$ non-acute angles.

Each acute angle has measure less than $90°$, and each non-acute angle has measure less than $180°$, so the sum of the angles is less than

$$90k + 180(n-k).$$

But the sum is $180n - 360$, and therefore

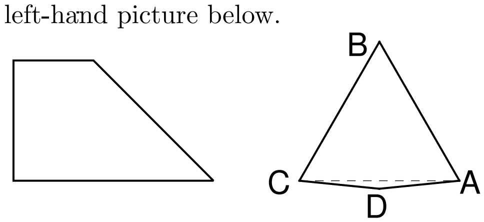
$$180n - 360 < 90k + 180(n-k).$$

A little manipulation changes this inequality to $90k < 360$, that is, to $k < 4$.

More informally, we are aiming for an angle sum of $180n - 360$. If there are 4 or more acute angles, we have already "lost" more than $360°$, and even if all the other angles are almost $180°$, we can never catch up. Note that if $n = 4$, we can have 4 non-obtuse angles, namely 4 right angles. If $n > 4$, we can not even have 4 non-obtuse angles.

Are we finished? No. We have shown that there can never be more than 3 acute angles. But can there be 0? Can there be 1? Can there be 2? Can there be 3? All submitted solutions were incomplete on this point.

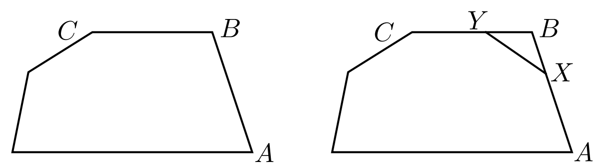
We first deal with quadrilaterals. We can certainly make a quadrilateral with 0 acute angles—a square will do. What about a quadrilateral with 1 acute angle? An example is given in the left-hand picture below.



Next we make a quadrilateral with 2 acute angles. That is easy, use for example any rhombus that is not a square.

Finally, we deal with 3 acute angles. Start for example with an equilateral triangle ABC, as in the right-hand diagram above. Then pick a point D slightly below the midpoint of CA, and form the quadrilateral $ABCD$. This quadrilateral has a $60°$ angle, two angles a bit bigger than $60°$, and one obtuse angle.

Next we show how starting with an n-gon that has exactly k acute angles, *and* at least one non-acute angle, we can make an $(n+1)$-gon with exactly k acute angles. Look for example at the left-hand figure below, which has 2 acute angles.



Pick one of the non-acute angles, in the picture the one at B. Now choose points X and Y on the line segments BA and BC, with $BX = BY$, and each of them "small." With a razor knife, cut along the line XY and throw away the little triangle. We have increased the number of vertices of the polygon by 1, thrown away a non-acute angle, but added two more. The number of acute angles is unchanged.

We can do this kind of surgery arbitrarily often. So starting with a quadrilateral with exactly k acute angles, we can produce an n-gon with k acute angles, for any $n > 4$. It follows that for any $n \geq 4$, there is a convex n-gon with exactly k acute angles, where $k = 0$, 1, 2, or 3.

Problem 5. A calculator is defective: the only operation keys that work are the "$-$" key and the "$1/x$" key. Show how to use this calculator to find (a) the square of a number; (b) the product of two numbers. (It is OK if your procedures do not work for very simple numbers such as 0 or 1.)

Solution. (a) Since $x + y = x - (0 - y)$, we can use the broken calculator to add, so from now we can assume that the + key also works. With the broken calculator compute

$$\frac{1}{t-1} - \frac{1}{t}, \qquad \text{that is,} \qquad \frac{1}{t^2 - t}$$

(unless $t = 0$ or $t = 1$, in which case we should not need a calculator to find t^2). Then use the "$1/x$" key to find $t^2 + t$, and subtract t to get t^2. There are a number of other approaches.

(b) Use the identity $(x + y)^2 - (x - y)^2 = 4xy$. By part (a), we can square with the broken calculator, so we can find $4xy$. If $xy \neq 0$, get rid of the 4 by inverting, adding the result to itself twice, and inverting again.

We can save many steps by using a variant of the idea of part (a). Note that $1/(t - 2) - 1/(t + 2) = 4/(t^2 - 4)$. Let $f(t) = (t^2 - 4)/4$. Then $f(t)$ is easy to compute, and $xy = f(x + y) - f(x - y)$.

Comment. The first multiplication procedure breaks down if $x + y$ or $x - y$ is equal to 0 or 1, for we then run into a "division by 0" problem. In these cases we can compute xy in other ways. For example if $x - y = 1$ then $xy = x^2 + x$, which is easy to compute with the broken calculator. The more efficient variant multiplication procedure can also fail, but again there is an easy fix.

It is hard to imagine a practical application for part (a), but ideas like those of part (b) have many uses, even in applied subjects such as physics. To describe applications would take too long. Instead we describe an almost forgotten item from the history of calculation.

In the bad old days before calculators, adding by hand was reasonably fast, but multiplication took a while. So to multiply people used slide rules, or tables of logarithms.

They also multiplied by using a *quarter-squares* table, that is, a table that list $t^2/4$, say for t from 0.0000 to 2.0000. If x and y are between 0 and 1, calculate $x+y$ and $x-y$, look up $(x+y)^2/4$ and $(x-y)^2/4$ in the table and subtract! If x and y aren't in the right range, shift decimal points until they are, multiply as before, then shift back.

They also multiplied using a *half-squares* table. Here the relevant identity is

$$xy = \frac{(x+y)^2}{2} - \left(\frac{x^2}{2} + \frac{y^2}{2}\right).$$

To multiply with a quarter-squares table takes two table look-ups, an addition, and two subtractions. With half-squares there are three table look-ups, two additions, and a subtraction. Half-squares were more popular. Nobody likes subtraction.

Problem 6. Alicia has 4 dollars and Beth has 2 dollars. A neutral third party tosses a fair coin. If the result is "head," Beth gives Alicia 1 dollar, and if the result is "tail," Alicia gives Beth 1 dollar. The coin tossing continues until one of the two players is bankrupt.

(a) What is the probability that it is Beth who ultimately goes bankrupt?

(b) What about if at the start Alicia has \$40 and Beth has \$20?

Solution. (a) We introduce some notation that is overly fancy for part (a), but is good preparation for part (b). Throughout the gambling, the combined fortune of Alicia and Beth is \$6. So if we specify how much money Alicia has at a certain time, then we also know how much Beth has.

If at a certain time Alicia has n dollars, we will say that she is in state n. The possible states for Alicia at any time are 0 (Alicia has gone bankrupt), 1, 2, 3, 4, 5, and 6 (Beth is bankrupt).

For $n = 0, 1, 2, \dots, 6$, let p_n be the probability that Beth will ultimately go bankrupt given that Alicia now has n dollars. For brevity, we write "Alicia wins" instead of "Beth ultimately goes bankrupt." Our problem is to find p_4, the probability that Alicia wins given that Alicia now has 4 dollars.

A few of the p_n are easy to find. For example, it is clear that $p_0 = 0$ (if Alicia is down to nothing, the probability that Alicia wins is 0: the game is over, Alicia has lost). Similarly, $p_6 = 1$.

It is maybe a little less obvious that $p_3 = 1/2$. If Alicia is in state 3, then Alicia and Beth have the same amount of money. Thus since the coin is fair, Alicia and Beth are equally likely to be wiped out ultimately. We are only looking at games in which *somebody* goes bankrupt. Thus, given that somebody goes bankrupt, the probability is 1/2 that it is Beth.

(Actually, with probability 1 *somebody* will be wiped out. This is almost obvious, since any medium long run of heads or tails wipes out one of our contestants. A full formal proof that with probability 1 somebody is wiped out will follow from the solution of part (b).)

Suppose that Alicia starts in state 4. With probability 1/2, the first toss is head, in which case Alicia goes into state 5, and the probability that Alicia (ultimately) wins is p_5. And with probability 1/2, the first toss is tail. Then Alicia goes into state 3, and the probability that Alicia wins is p_3, that is, 1/2. We conclude that

$$p_4 = (1/2)p_5 + (1/2)(1/2). \tag{1}$$

Now suppose that Alicia is in state 5. Then with probability 1/2, the next toss is head, Alicia moves into state 6, she has won, the probability that Alicia wins is 1. And with probability 1/2, the next toss is tail, Alicia moves into state 4, and the probability Alicia wins is p_4. It follows that

$$p_5 = (1/2)(1) + (1/2)p_4. \tag{2}$$

We have obtained two equations in the two unknowns p_4 and p_5. Substituting for p_5 in the first equation, we get

$$p_4 = (1/2)((1/2)(1) + (1/2)p_4) + (1/2)(1/2).$$

Simplify. We get $(3/4)p_4 = 1/2$, so $p_4 = 2/3$: the probability that Beth ultimately goes bankrupt if Alicia has 4 dollars and Beth has 2 dollars is 2/3.

There is a more compact version of the same argument. If the first toss is tail (probability 1/2) the probability that Alicia wins is 1/2. If the first toss is head, either the next toss is head, and Alicia has won, or the next toss is tail, in which case Alicia is back to state 4, and has probability p_4 of winning. Thus

$$p_4 = (1/2)(1/2) + (1/2)(1/2)(1) + (1/2)(1/2)p_4.$$

Now solve for p_4.

Another Way. Imagine that a warning bell sounds the *first* time (if ever) Alicia is down to 2 dollars. Because Alicia's initial capital is \$4, halfway between 2 and 6, with probability 1/2 Alicia wins without hearing a warning bell, and with probability 1/2 Alicia hears a bell before the game is over. (If she hears the bell, she still has a chance of winning.)

If Alicia hears the bell, then Beth has 4 dollars, so by symmetry Beth has probability p_4 of winning, meaning that Alicia's probability of winning is $1 - p_4$. So with probability 1/2, Alicia wins without hearing the bell, and with probability $(1/2)(1-p_4)$ Alicia hears the warning bell but wins. It follows that

$$p_4 = 1/2 + (1/2)(1 - p_4).$$

Again, we have obtained a linear equation for p_4.

Another Way. We could look at all "paths" that lead to a win by Alicia, calculate the probability of each path, and add up. That is not an easy task, since there are infinitely many such paths. An example of a long path that leads to a win by Alicia is the sequence HT repeated a million times, followed by HH.

A full analysis along these lines is awfully messy, so we take advantage of the fact that if Alicia and Beth each have 3 dollars, then the probability that Alicia wins is, by symmetry, equal to 1/2.

So T, HTT, HTHTT, HTHTHTT, and so on all give probability 1/2 that Alicia wins. Their contributions to Alicia's probability of winning is

$$(1/2)\left[\frac{1}{2}+\frac{1}{8}+\frac{1}{32}+\frac{1}{128}+\cdots\right].$$

In addition, HH, HTHH, HTHTHH, HTHTHTHH, and so on lead to an Alicia win. They contribute a probability of

$$\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256}+\cdots.$$

The two infinite geometric progressions can be summed by a standard formula. Each has sum 1/3, so the probability Alicia wins is 2/3.

Another Way. Many probability problems are difficult or impossible to solve "exactly." A common way of getting information in such cases is to do a *simulation.* Informally, we play the game a few thousand times, and record the proportion of times that Alicia wins.

In principle, this is insufficient. We have seen that Alicia wins with probability 2/3. Suppose that we play the game say 3000 times. Let A be the number of times that Alicia wins. Even though Alicia's probability of winning a game is 2/3, it is *possible* that $A/3000$ is not even close to 2/3.

However, it is *very* unlikely that $A/3000$ is far from 2/3. (We can in fact make a good estimate of how unlikely it is. The mathematics behind this estimate is the same as the mathematics that a polling company uses to declare that a result is correct within 3 percentage points 19 times out of 20.) A 3000 game simulation should give a reliable estimate of p_4.

In practice, the simulation is not done with actual coins. Instead, a computer program produces (pseudo)-random sequences of heads and/or tails. This can even be done with a calculator, preferably a programmable one. Many calculators have (pseudo)-random number generators.

Simulations of this type are of great practical importance. The procedure is often called a *Monte Carlo* method, after the famous casino. Monte Carlo methods first came into prominence in World War II, when they were used to improve anti-submarine warfare techniques. Later they were used by American scientists to solve design problems around the hydrogen bomb. Mathematics is useful for killing.

(b) We look into what happens if Alicia starts with \$40 and Beth starts with \$20. These numbers are fairly large, so we might as well solve a more general problem.

Suppose that Alicia and Beth between them have N dollars, and Alicia start with a dollars, and so Beth starts with $N-a$. In part (a) we examined the case $N=6$, $a=4$.

At any stage of the game, Alicia can have 0, 1, 2, ..., or N dollars. If at a certain time she has n dollars, we will like before say that she is in state n. The possible states are thus 0, 1, 2, ..., N.

Let p_n be the probability that if Alicia is now in state n, she will (ultimately) win the game, that is, end up in state N, with all the money. It is easy to see that $p_0 = 0$ (if Alicia has 0 dollars, Beth has won), and that $p_N = 1$. The p_n for $0 < n < N$ are more difficult.

Imagine that Alicia is in state n, where $0 < n < N$. With probability $1/2$, the next coin toss is head, Alicia wins a dollar, and so has $n+1$ dollars, is now in state $n+1$. If that happens, her probability of winning the game is now p_{n+1}.

But with probability $1/2$, the next toss is a tail, Alicia loses a dollar, so goes into state $n-1$. Her probability of winning the game is now p_{n-1}. We have therefore proved the following result.

Lemma (The Fundamental Recurrence)**.** If $0 < n < N$, then

$$p_n = (1/2)p_{n+1} + (1/2)p_{n-1}. \tag{3}$$

We can think of the Fundamental Recurrence as a large collection of linear equations, to be solved for the variables p_1, p_2, …, p_{n-1}. But it turns out that the problem is less complicated than it sounds.

Fractions can be confusing, so let us multiply both sides by 2 We get $2p_n = p_{n+1} + p_{n-1}$, which can be rewritten as

$$p_{n+1} - p_n = p_n - p_{n-1}. \tag{4}$$

We conclude that "differences" are constant, that is, the sequence p_0, p_1, p_2, …, p_N is an *arithmetic progression.* This arithmetic progression has first term 0 and last term 1.

From the fact that the arithmetic progression has first term 0, we conclude that $p_n = dn$ for some d. But then from the fact that $p_N = 1$, we conclude that $dN = 1$, and therefore $d = 1/N$. We have therefore shown that

$$p_n = \frac{n}{N}. \tag{5}$$

In particular, if Alicia starts with \$40 and Beth starts with \$20, then the probability that Alicia wins is 40/60, that is, 2/3.

Comment. 1. The key trick for dealing with the Fundamental Recurrence was to rewrite it as $p_{n+1} - p_n = p_n - p_{n-1}$. Another way of viewing this trick is that if we let $d_n = p_{n+1} - p_n$, then $d_n = d_{n-1}$. So here *differences* behave very nicely. It was pointed out in the solution to Problem 6, January 2006, that this is often the case.

2. By our formula, if Alicia starts with a, and Beth with b, then Alicia wins with probability $a/(a+b)$. The same formula shows that Beth wins with probability $b/(a+b)$. The sum of these probabilities is $(a+b)/(a+b)$, that is, 1. That was intuitively obvious, for surely the probability that the game goes on forever is 0. But now we have a formal proof.

3. The answer in part (b) is the same as the answer of part (a): the only thing that matters is the *ratio* of the initial fortune of Alicia to the combined fortune. Is it "obvious" that this should be the case? Maybe, but I do not (yet?) see it.

We could look at a more general problem. Suppose that Alicia and Beth are using a biased coin that has probability $h \neq 1/2$ of landing head. By an analysis that is nearly the same as the one given above, we arrive at the fundamental recurrence

$$p_n = hp_{n+1} + (1-h)p_{n-1}.$$

With a little ingenuity, we can use this recurrence to find a general formula for p_n. (Try!)

Take for example a coin that lands head with probability 0.45, and tail with probability 0.55, meaning that it is somewhat biased against Alicia. It turns out that Alicia has probability about 0.528 of winning if she starts with \$4 and Beth starts with \$2. But if Alicia starts with \$40 and Beth starts with \$20, the probability that Alicia wins is a miserable 0.018. So in the case of the biased coin, it is definitely not only the ratio of the fortunes that matters.