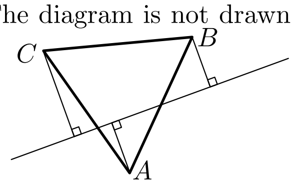
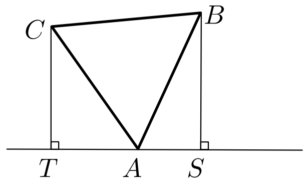
Solutions to February 2009 Problems

Problem 1. Vertex A of an equilateral triangle lies on one side of a certain line, and is at distance 2 from that line. Vertices B and C of the triangle are on the other side of the line, and, respectively, at distances 2 and 3 from that line. What is the length of a side of the triangle? (The diagram is not drawn to scale.)



Solution. Shift the line parallel to itself until it goes through A. Introduce coordinate axes so that the shifted line becomes the x-axis, with A the origin. Let S be the point where the perpendicular from B to the x-axis meets that axis, and let T be the corresponding point for C.



We have $BS = 4$ and $CT = 5$. Let the x-coordinates of S and T be, respectively, s and t. Then B has coordinates $(s, 4)$, and C has coordinates $(t, 5)$. Let each side of the equilateral triangle have length a. Then, by the Pythagorean Theorem, we have the equations

$$s^2 + 16 = t^2 + 25 = (s - t)^2 + 1 = a^2.$$

There is nothing interesting about solving the equations. From $s^2+16 = t^2+25$, we get $s^2 - t^2 = 9$. From $t^2 + 25 = (s - t)^2 + 1$ we obtain $s^2 - 2st = 24$. Thus $t = (s^2 - 24)/2s$. Substitute for t in $s^2 - t^2 = 9$. We get

$$s^2 - \frac{(s^2 - 24)^2}{4s^2} = 9,$$

which after a little simplification becomes $s^4 + 4s^2 - 192 = 0$. Solve. It turns out that $s^2 = 12$, and therefore $a^2 = 28$. So the required side length is $\sqrt{28}$.

Problem 2. We have an unlimited number of 1 gram weights, 10 gram weights, 100 gram weights, and 1000 gram weights. How many different combinations of weights is it possible to put in one pan of a pan balance so as to balance a 2009 gram kitten in the other pan?

Solution. The main practical problem is keeping the kitten in the pan. But we can only deal with the mathematics. The number of 1000 gram weights could be 2, 1, or 0. If we use 2, then there is only 1 way to finish the job. Next we examine separately the number of combinations where the number of 1000 gram weights is 1, and where the number of 1000 gram weights is 0.

Getting to 1009 using only 1's, 10's, and 100's: The number of ways is the same as the number of ways to get to 1000. The number of 100's used is 0, 1, 2, ..., or 10. That leaves 1000, 990, 980, ..., 10, 0 to be done with 10's and 1's. Examine them in reverse order. There is 1 way to do 0 with 10's and 1's. There are 2 ways to do 10, 3 ways to do 20, and so on. Finally, there are 101 ways to do 1000. We get a total of

$$1 + 2 + 3 + \cdots + 100 + 101.$$

In the usual way, we find that this sum is 5151.

Getting to 2009 using only 1's, 10's, and 100's: Here the number of 100's used is 0 to 20, leaving 2000, 1990, ..., 0 to be done with 10's and 1's. The number of ways is therefore $201 + 200 + \cdots + 1$. This is 20301.

Add up. The number of combinations is $1 + 5151 + 20301$, which is 25453.

Another Way. We looked first at the number of 1000's. We could look first at the number of 1's used, which is 9, 19, 29, ..., 2009. That leaves 2000, 1990, 1980, ..., 0 to be done with 1000's, 100's, and 10's. This problem is equivalent to doing 200, 199, 198, ..., 0 with 100's, 10's, and 1's. Continue, again dealing with 1's first. The details are a bit messier than the 1000's first approach.

Problem 3. Show that if $x \ge 1$, then

$$\sqrt{x+2} - \sqrt{x} < \frac{1}{\sqrt{x}} < \sqrt{x+1} - \sqrt{x-1}.$$

Solution. There are two inequalities to prove. While the proofs could be to some degree combined, it is clearer to deal with them separately.

First we show that $\sqrt{x+2} - \sqrt{x} < 1/\sqrt{x}$ for all positive x. Since $\sqrt{x}$ is positive, we get an equivalent inequality by multiplying both sides by $\sqrt{x}$. So we want to prove that $\sqrt{x^2+2x} - x < 1$, or equivalently that $\sqrt{x^2+2x} < x+1$. Since both sides are positive, we get an equivalent inequality by squaring both sides. So we want to prove that $x^2 + 2x < x^2 + 2x + 1$, which is obviously true.

Another Way. Although the manipulations were easy, they hide somewhat what is going on. Note that

$$\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 = x + 2 + \frac{1}{x} > x + 2.$$

Thus $\sqrt{x} + 1/\sqrt{x}$ is bigger than $\sqrt{x+2}$ (though not by much, if x is large). It follows that $1/\sqrt{x}$ is less than $\sqrt{x+2} - \sqrt{x}$.

Another Way. The next argument is overly elaborate. Note that

$$\sqrt{x+2}-\sqrt{x}=\frac{(\sqrt{x+2}-\sqrt{x})(\sqrt{x+2}+\sqrt{x})}{\sqrt{x+2}+\sqrt{x}}=\frac{2}{\sqrt{x+2}+\sqrt{x}}.$$

(This step of "rationalizing the numerator" is an often useful tool.) To prove our inequality, it is therefore enough to prove that $(\sqrt{x+2}+\sqrt{x})/2>\sqrt{x}$. (We took reciprocals; note the reversal in the direction of the inequality.) This last inequality is obvious, since $\sqrt{x+2}>\sqrt{x}$.

Next we prove that $1/\sqrt{x}<\sqrt{x+1}-\sqrt{x-1}$ when $x\ge 1$. We will use the "rationalizing the numerator" trick already used in the first part. Note that

$$\sqrt{x+1}-\sqrt{x-1}=\frac{(\sqrt{x+1}-\sqrt{x-1})(\sqrt{x+1}+\sqrt{x-1})}{\sqrt{x+1}+\sqrt{x-1}}=\frac{2}{\sqrt{x+1}+\sqrt{x-1}}.$$

So to prove the desired inequality, it is enough to show that $\sqrt{x}>(\sqrt{x+1}+\sqrt{x-1})/2$, or equivalently, (multiply both sides by 2, and then square) that $4x>2x+2\sqrt{x^2-1}$. But this is obvious, since $x>\sqrt{x^2-1}$.

Another Way. We want to show that $1/\sqrt{x}<\sqrt{x+1}-\sqrt{x-1}$, or equivalently (multiply both sides by $\sqrt{x}$) that $1<\sqrt{x^2+x}-\sqrt{x^2-x}$. So we want to show that $1+\sqrt{x^2-x}<\sqrt{x^2+x}$. Equivalently (square both sides, there is no problem, since both are positive) we want to show that $1+x^2-x+2\sqrt{x^2-x}<x^2+x$. So we would like to show that $2\sqrt{x^2-x}<2x-1$, or equivalently that $4x^2-4x<4x^2-4x+1$. This is obvious.

Comment. It is not hard to verify that $\sqrt{x+5/4}-\sqrt{x-3/4}\le 1/\sqrt{x}$ when $x\ge 1$. This is a better approximation to $1/\sqrt{x}$ than the $\sqrt{x+2}-\sqrt{x}$ of the problem.

Problem 4. (a) Show that if the positive integer n is a multiple of 3, then 7^n-6^n is a multiple of 127. (b) Show that if n is not a multiple of 3, then 7^n-6^n is not a multiple of 127.

Solution. (a) Let $n=3k$. Then

$$7^n-6^n=7^{3k}-6^{3k}=(7^3)^k-(6^3)^k.$$

Recall the identity

$$x^m-y^m=(x-y)(x^{m-1}+x^{m-2}y+x^{m-3}y^2+\cdots+xy^{m-2}+y^m).$$

In this identity, put $m=k$, $x=7^3$, and $y=6^3$. We conclude that 7^3-6^3 divides $7^{3k}-6^{3k}$. But $7^3-6^3=127$.

(b) If n is not divisible by 3, there are two possibilities: (i) n leaves a remainder of 1 on division by 3, meaning that $n=3k+1$ for some integer k or (ii) $n=3k+2$ for some integer k.

Case (i): We could examine separately the remainders when 7^{3k+1} and 6^{3k+1} are divided by 127. Since 127 is not small, this approach would be painful. Instead, we use a little trick, variants of which are useful surprisingly often. Note that $6^{3k+1} = (7)6^{3k} - 6^{3k}$, and therefore

$$7^{3k+1} - 6^{3k+1} = 7(7^{3k} - 6^{3k}) + 6^{3k}.$$

By the result of part (a), 127 divides $7(7^{3k} - 6^{3k})$. So if 127 divided $7^{3k+1} - 6^{3k+1}$, then 127 would divide 6^{3k}. But it obviously does not, since the only primes that divide 6^{3k} are 2 and 3.

Case (ii): Basically the same trick works.

$$7^{3k+2} - 6^{3k+2} = 49(7^{3k} - 6^{3k}) + (13)6^{3k}.$$

The only prime divisors of $(13)6^{3k}$ are 13, 2, and 3, so it is not divisible by 127.