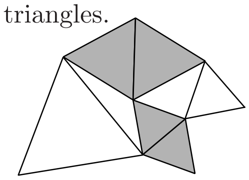
Solutions to February 2010 Problems

Problem 1. Every point on the circumference of a circle is coloured red or blue. Show that there is an isosceles triangle whose vertices lie on the circle and are all of the same colour.

Solution. Let A, B, C, D, and E be, in counterclockwise order, the vertices of a regular pentagon inscribed in the circle. We could examine all colourings of these vertices (there are not many), and show that for each colouring there is a monochromatic isosceles triangle. But we can save some time by taking advantage of symmetry. It is useful to make a sketch in order to follow the argument.

At least 3 of the vertices are of the same colour, say blue. Without loss of generality we may assume that A is blue. If B and E are blue, we are finished, for then $\triangle ABE$ is isosceles, with all vertices blue. If B and E are both red, then C and D must be blue, and $\triangle ACD$ is a blue isosceles triangle. Finally, suppose that B and E are of opposite colours. Without loss of generality B is blue and E is red. But then at least one of C or D is blue, since blue is the "majority" colour. If C is blue, then $\triangle ABC$ is blue and isosceles. If D is blue, then $\triangle ABD$ is blue and isosceles.

Problem 2. The picture shows six equilateral triangles, and two triangles that don't look equilateral. Two pairs of congruent equilateral triangles are joined to form the shaded rhombi. Show that the sum of the areas of the rhombi is equal to the sum of the areas of the unshaded equilateral triangles.



Solution. Let a be the side of one of the rhombi, and b the side of the other. Let P be the point where the two rhombi meet. There are two unshaded triangles that have common vertex P. In the picture, one of them, say $\mathcal{S}$, has an acute angle at P, and the other (call it $\mathcal{T}$) has an obtuse angle at P. Let θ be the angle at P of triangle $\mathcal{S}$, and let ϕ be the angle at P of triangle $\mathcal{T}$.

The angles at P of the shaded triangles add up to 180°. It follows that $\phi = 180^\circ - \theta$.

Now let x be the side of $\mathcal{S}$ which is opposite to P. By the Cosine Law, we have

$$x^2 = a^2 + b^2 - 2ab\cos\theta.$$

Let y be the side of $\mathcal{T}$ which is opposite to P. By the Cosine Law, and using the fact that $\cos\phi = -\cos\theta$, we have

$$y^2 = a^2 + b^2 + 2ab\cos\theta.$$

Add. We conclude that $x^2 + y^2 = 2a^2 + 2b^2$.

Now we use the fact that the area of an equilateral triangle is equal to a constant k times the square of the length of the side. So the area of one of the unshaded equilateral triangles is kx^2, and the area of the other one is ky^2. Also, the two rhombi have area $2ka^2$ and $2kb^2$. Our result now follows from the fact that $x^2 + y^2 = 2a^2 + 2b^2$.

Comment. It so happens that if area is measured in square units, then $k = \sqrt{3}/4$, but that is of no importance here. Indeed we could choose our unit of area so that $k = 1$. It is actually not uncommon in the real world for area to be measured in something other than unit^2, where *unit* is some standard unit of length. For example, in the traditional Imperial and US customary systems, an acre is 4840 square yards, so it is the area of a square whose side is the irrational number $22\sqrt{10}$. But for obvious reasons $22\sqrt{10}$ yards is not a standard unit of length!

Problem 3. For any real number x, let $\lfloor x \rfloor$ be the greatest integer which is less than or equal to x. For example, $\lfloor \pi \rfloor = 3$. Show that if n is a non-negative integer, then

$$\left\lfloor \sqrt{n} + \sqrt{n+1} \right\rfloor = \left\lfloor \sqrt{4n+2} \right\rfloor.$$

Solution. It is useful (but not necessary) to deal with the case $n = 0$ separately. When $n = 0$, the left hand side is 1, and the right-hand side is $\lfloor \sqrt{2} \rfloor$, which is clearly 1. (The integer 0 is a little peculiar, for instance it is the only integer n such that both n and $n+1$ are perfect squares. Indeed the integer 0 is so peculiar that for millenia it was not thought of as a number.) From here on we assume that $n > 0$.

A little later it will be useful to know that

$$\left\lfloor \sqrt{4n+2} \right\rfloor = \left\lfloor \sqrt{4n+1} \right\rfloor.$$

This is easy, for $4n+2$ cannot be a perfect square (it is divisible by 2 but not by 4). So there is no integer in the interval $\sqrt{4n+1} < u \le \sqrt{4n+2}$.

Let $x = \sqrt{n} + \sqrt{n+1}$. Then

$$x^2 = n + 2\sqrt{n}\sqrt{n+1} + (n+1) = 2n + 1 + \sqrt{4n^2 + 4n}.$$

But if $n > 0$ we have $4n^2 < 4n^2 + 4n < 4n^2 + 4n + 1$ and therefore $2n < \sqrt{4n^2 + 4n} < 2n + 1$. it follows that

$$4n + 1 < x^2 < 4n + 2, \quad \text{and therefore} \quad \sqrt{4n+1} < x < \sqrt{4n+2}.$$

Since $x < \sqrt{4n+2}$, we can conclude that

$$\lfloor x \rfloor \le \left\lfloor \sqrt{4n+2} \right\rfloor.$$

Since $x > \sqrt{4n+1}$, we can conclude that

$$\lfloor x \rfloor \geq \left\lfloor \sqrt{4n+1} \right\rfloor = \left\lfloor \sqrt{4n+2} \right\rfloor,$$

which completes the proof.

Comment. It is not hard to show that $4n+3$ can never be a perfect square. So, in the statement of the problem, $4n+2$ could have been replaced by $4n+3$, or by $4n+1$.

Problem 4. For any positive integer n, define $n!!$ by

$$n!! = \begin{cases} (n)(n-2)(n-4)\cdots(1) & \text{if } n \text{ is odd and} \\ (n)(n-2)(n-4)\cdots(2) & \text{if } n \text{ is even.} \end{cases}$$

Show that $2009!! + 2010!!$ is divisible by 2011.

Solution. Things would be easy if 2011 were composite. Suppose for example that 2011 happened to be equal to 43 times 47 (it isn't). It is easy to find terms divisible by 43 and by 47 in each of the products 2009!! and 2010!!, so it is clear that $43 \cdot 47$ divides our sum. However, 2011 is prime, so we need to take another approach.

Comment. The fact that 2011 is prime can be discovered by testing, for every prime $p \leq \sqrt{2011}$, whether p divides 2011. More simply, we can for example ask WolframAlpha™, which can also produce solutions to most high school math homework problems.

Our problem is most easily handled by using *congruences*. Briefly, if m is a positive integer, we say that the integer a is congruent to the integer b modulo m if m divides the difference $a - b$. If a is congruent to b modulo m, we write $a \equiv b \pmod{m}$.

We will need a couple of useful and fairly easily established facts about congruences. Suppose that $s \equiv u \pmod{m}$ and $t \equiv v \pmod{m}$. Then $s + t \equiv u + v \pmod{m}$ and $st \equiv uv \pmod{m}$. The first fact is almost obvious. If m divides $s - u$ and $t - v$, then m clearly divides $(s-u)+(t-v)$, so m divides $(s+t)-(u+v)$, meaning that $s + t \equiv u + v \pmod{m}$. Multiplication is a little trickier. If m divides $s - u$ and $t - v$, then m divides $st - ut$, and also $ut - uv$, so it divides the sum of these, which is $st - uv$.

Let $m = 2011$. Then $2009 \equiv -2 \pmod{m}$, $2007 \equiv 4 \pmod{m}$, $2005 \equiv 6 \pmod{m}$, and so on, until finally $1 \equiv -2010 \pmod{m}$. It follows that

$$2009!! \equiv (-2)(-4)(-6)\cdots(-2008)(-2010) \pmod{m}.$$

In the product on the right, we have 1005 minus signs. So the product on the right is congruent to $(-1)^{1005}2010!!$ modulo m, so it is congruent to $-(2010!!)$ modulo m.

It follows that $2009!! + 2010!!$ is congruent to $-(2010!!) + 2010!!$ modulo m, that is, our sum is congruent to 0 modulo m, which means that m divides our sum.

Comment. One can avoid congruence notation, though it is hard to avoid the basic idea. Note that

$$-2009!! = -(m-2)(m-4)(m-6)\cdots(m-2010).$$

Look at the left-hand side, and imagine multiplying it out. Almost every term in the product has an m "in it." So the remainder when we divde by m is the same as the remainder when $-(-2)(-4)(-6)\cdots(-2010)$ is divided by m. We have an even number of $-$ signs, namely 1006 of them, so the remainder when 2009!! is divided by m is m minus the remainder when 2010!! is divided by m, and the result follows.

Comment. Suppose that n is an *odd* positive integer. The same argument can be used to show that $(2n-1)!! + (2n)!!$ is divisible by $2n+1$. The situation for even n is more complicated. Take for example the case $n = 2$. Then $2n+1 = 5$ and $(2n-1)!! + (2n)!! = 11$, so in this case at least, $2n+1$ does not divide $(2n-1)!! + (2n)!!$.

Comment. The notation $n!!$ is peculiar, since under the definition, $n!!$ is quite different from $(n!)!$, which sounds unreasonable. But the notation is in fact standard, at least for odd n (many people do not define $n!!$ for even n.)